A Switching Controller for Uncertain Nonlinear Systems

By H.K. Lam, Frank H.F. Leung, and Peter K.S. Tam

Control of nonlinear systems is difficult because no systematic mathematical tools exist to help find necessary and sufficient conditions to guarantee their stability and performance. The problem becomes more complex if some of the parameters of the plant are unknown. By using a Takagi-Sugeno-Kang (TSK) fuzzy plant model [1]-[2], [8], [13], a nonlinear system can be expressed as a weighted sum of some simple subsystems. This model gives a fixed structure to some nonlinear systems and thus facilitates analysis. There are two ways to obtain the fuzzy plant model: 1) by applying identification methods with input-output data from the plant [1]-[2], [8], [13] or 2) directly from the mathematical model of the nonlinear plant.

In recent investigations into the stability of fuzzy systems formed by a fuzzy plant model and a fuzzy controller, several stability conditions have been obtained [4]-[5], [9]-[12], [16]-[19]. A linear controller [14] was also proposed to control the plant represented by the fuzzy plant model. Most of the fuzzy controllers proposed are functions of the grades of membership of the fuzzy plant model. Hence, the membership functions of the fuzzy plant model must be known. This means that the parameters of the nonlinear plant must be known or must be constant when the identification method is used to derive the fuzzy plant model. Practically, the parameters of many nonlinear plants will change during operation (e.g., the load of a dc-dc power converter or the number of passengers on board a train). In these cases, the robustness property of the fuzzy controller is an important concern. Moreover, the investigations [4]-[5], [9]-[12] tackled only a regulation problem such that the controllers drive all the system states to zero. In practice, we may face a nonzero set-point regulation problem or a tracking problem. To tackle these problems, some algorithms integrating fuzzy logic with adaptive control theory [3], [7], [15] or with H^{∞} control theory [6] can be found.

In this article, a switching controller is proposed to control nonlinear plants subject to unknown parameters within known bounds. The nonlinear plant is represented by a fuzzy plant model. This switching controller is able to drive the system states to follow those of a reference model. The switching controller consists of several linear controllers. One of the linear controllers is employed at each moment according to a switching scheme, which is derived based on Lyapunov stability theory.

The remainder of the article is organized as follows. First, we describe a reference model, a fuzzy plant model, and a switching controller. Next, we investigate the system stability of the switching control system. The switching scheme will be derived based on Lyapunov stability theory, and the gains of the switching controllers will be designed. Then, we provide an application example of an inverted pendulum on a cart. Finally, we present our conclusions.

Reference Model, Fuzzy Plant Model, and Switching Controller

The nonlinear plant to be tackled is of the following form:

$$\dot{\mathbf{x}}(t)=\mathbf{A}(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{B}(\mathbf{x}(t))\mathbf{u}(t) \tag{1}$$

where $\mathbf{A}(\mathbf{x}(t))\in\Re^{n\times n}$ and $\mathbf{B}(\mathbf{x}(t))\in\Re^{n\times m}$ are the system matrix and input matrix, respectively, both of which have known structure but may be subject to unknown parameters (all matrices considered herein are real matrices); $\mathbf{x}(t)\in\Re^{n\times 1}$ is the system state vector; and $\mathbf{u}(t)\in\Re^{m\times 1}$ is the input vector. The system of (1) is represented by a fuzzy plant model that expresses the multivariable nonlinear system as a weighted sum of linear systems. A switching controller is to be designed to close the feedback loop of the nonlinear plant based on the fuzzy plant model such that the system states follow those of a reference model.

Reference Model

The reference model is a stable linear system given by

$$\dot{\hat{\mathbf{x}}}(t)=\mathbf{H}_m\hat{\mathbf{x}}(t)+\mathbf{B}_m\mathbf{r}(t), \tag{2}$$

where $\mathbf{H}_m\in\Re^{n\times n}$ is a constant stable system matrix, $\mathbf{B}_m\in\Re^{n\times m}$ is a constant input matrix, $\hat{\mathbf{x}}(t)\in\Re^{n\times 1}$ is the system state vector of this reference model, and $\mathbf{r}(t)\in\Re^{m\times 1}$ is the bounded reference input.

Fuzzy Plant Model

Letting p be the number of fuzzy rules describing the multivariable nonlinear plant of (1), the ith rule is of the following format:

$$\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ and ... and } f_\psi(\mathbf{x}(t)) \text{ is } \mathrm{M}_\psi^i$$
$$\text{THEN } \dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t), \tag{3}$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ in terms of the system states and the unknown parameters of the nonlinear plant, $\alpha=1,2,\ldots,\psi$, $i=1,2,\ldots,p$; ψ is

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a positive integer, and $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known system and input matrices, respectively, of the ith rule subsystem. The system dynamics are described by

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\right), \tag{4}$$

where

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t))=1,\quad w_i(\mathbf{x}(t))\in[0\ \ 1]\ \text{ for all } i, \tag{5}$$

and

$$w_i(\mathbf{x}(t))=\frac{\mu_{M_1^i}(f_1(\mathbf{x}(t)))\times\mu_{M_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{M_\psi^i}(f_\psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{M_1^k}(f_1(\mathbf{x}(t)))\times\mu_{M_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{M_\psi^k}(f_\psi(\mathbf{x}(t)))\right)} \tag{6}$$

are nonlinear functions of the system states and the unknown parameters. (For more details, see [1]-[2], [8], and [13].)

In this article, the fuzzy plant model of (4) is assumed to have the following properties:

$$\overline{\mathbf{A}}_i=\mathbf{A}_i-\mathbf{H}_m=\mathbf{B}_m\mathbf{D}_i,\quad i=1,2,\ldots,p \tag{7}$$

$$\mathbf{B}(\mathbf{x}(t))=\sum_{i=1}^{p} w_i(\mathbf{x}(t))\mathbf{B}_i=\alpha(\mathbf{x}(t))\mathbf{B}_m, \tag{8}$$

where $\mathbf{D}_i\in\Re^{m\times n}$, $i=1,2,\ldots,p$, are constant matrices. $\alpha(\mathbf{x}(t))$ is an unknown nonzero scalar (because $w_i(\mathbf{x}(t))$ is unknown) but with known bounds and sign. It should be noted that because $\alpha(\mathbf{x}(t))\neq 0$ is required, $\mathbf{B}(\mathbf{x}(t))\neq\mathbf{0}$ is assumed.

Switching Controller

A switching controller is employed to control the nonlinear plant of (1). The switching controller consists of some simple subcontrollers that will be switched from one to another to control the system of (1). The switching controller is described by

$$\mathbf{u}(t)=\sum_{j=1}^{p} m_j(\mathbf{x}(t))\left(\mathbf{G}_j\mathbf{x}(t)+\mathbf{r}\right), \tag{9}$$

where $m_j(\mathbf{x}(t))$, $j=1,2,\ldots,p$, takes the value of $-1/\alpha_{\min}$ or $1/\alpha_{\min}$ according to the switching scheme to be discussed later, $\alpha_{\min}$ is the minimum value of $\alpha(\mathbf{x}(t))$, and $\mathbf{G}_j\in\Re^{m\times n}$, $j=1,2,\ldots,p$, are the feedback gains to be designed. It can be seen that (9) is a linear combination of p linear state-feedback controllers. At each moment, one of the linear state-feedback controllers will be chosen to control the nonlinear plant according to the switching scheme.

Stability Analysis and Design of the Switching Controller

In this section, the switching controller will be designed to consider the system stability. The analysis results are summarized by the following lemma.

Lemma 1: The system states of the nonlinear plant of (1) represented by the fuzzy plant model of (4) will follow those of the reference model of (2) if the fuzzy plant model satisfies the conditions of (7) and (8) and the switching controller is designed by choosing

i) $\mathbf{P}\in\Re^{n\times n}>\mathbf{0}$

ii) $m_i=-\dfrac{\operatorname{sgn}\left(\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}(t))\right)}{\operatorname{sgn}(\alpha)\alpha_{\min}}\ i=1,2,\ldots,p$

iii) $\mathbf{Q}=-\left(\mathbf{H}_m^T\mathbf{P}+\mathbf{P}\mathbf{H}_m\right)>\mathbf{0}$

iv) $\mathbf{G}_i=-\mathbf{D}_i, i=1,2,\ldots,p$, where

$$\operatorname{sgn}(z)=\begin{cases}1 & z>0\\ -1 & z\le 0\end{cases},\quad |\alpha|\in[\alpha_{\min}\ \ \alpha_{\max}],\ \text{and}\ \alpha_{\max}>\alpha_{\min}>0.$$

See the Appendix for the proof (it should be noted that $\mathbf{Q}$ is needed to prove stability, not for control design). The design procedure of the linear controller is summarized in the following steps:

- *Step I:* Obtain the fuzzy plant model of a nonlinear plant by means of the methods in [1]-[2], [8], [13] or other suitable ways.
- *Step II:* Choose a reference model in the form of (2).
- *Step III:* Check whether the fuzzy plant model satisfies conditions (7) and (8).
- *Step IV:* Design the switching controller according to conditions i) through iv) of Lemma 1.

Application Example

An application example will be given here to show the design procedure of the switching controller. A cart-pole inverted pendulum system [5] is shown in Fig. 1. A switching controller will be designed for it by following the design procedure given in the previous section.

Step I: The dynamic equation of the cart-pole inverted pendulum system is given by

$$\ddot{\theta}(t)=\frac{g\sin(\theta(t))-aml\dot{\theta}(t)^2\sin(2\theta(t))/2-a\cos(\theta(t))u(t)}{4l/3-aml\cos^2(\theta(t))}, \tag{10}$$

where θ is the angular displacement of the pendulum, $g=9.8$ m/s^2 is the acceleration due to gravity, $m\in[m_{\min}\ \ m_{\max}]=[0.5\ 2]$ kg is the mass of the pendulum, $M\in[M_{\min}\ \ M_{\max}]=[8\ \ 80]$ kg is the mass of the cart, $a=1/(m+M)$, $2l=1$ m is the length of the pendulum, and u is the force applied to the cart.

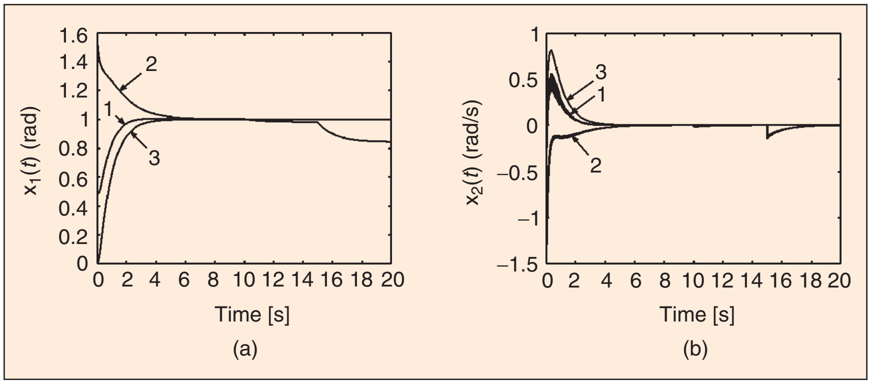

Figure 3. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the switching controller under the initial conditions of* $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45)\ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ *(response 3),* $m=0.5$ *kg,* $M=8$ *kg, and* $r(t)=8$.

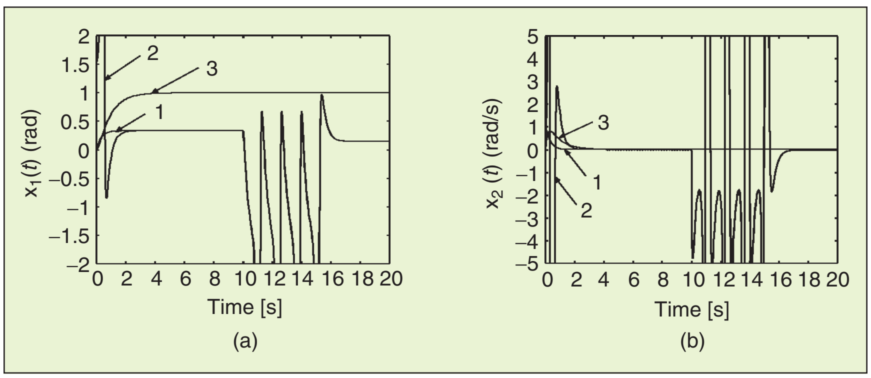

Figure 4. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the linear state-feedback controller under the initial conditions of* $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45)\ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ *(response 3), m = 0.5 kg, M = 8 kg, and* $r(t)=8$.

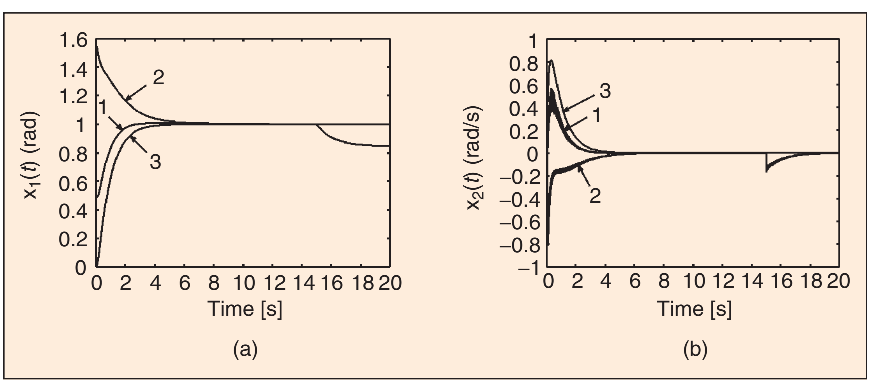

Figure 5. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the switching controller under the initial conditions of* $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45)\ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ *(response 3), m = 0.5 kg, M = 80 kg, and* $r(t)=8$.

and $\mu_{M_1^k}(f_1(\mathbf{x}(t)))=1-\mu_{M_1^1}(f_1(\mathbf{x}(t)))$ for $k=3,4$;

$$\mu_{M_2^k}(f_2(\mathbf{x}(t)))=\frac{-f_2(\mathbf{x}(t))+f_{2_{\max}}}{f_{2_{\max}}-f_{2_{\min}}} \text{ for } k=1,3$$

and $\mu_{M_2^k}(f_2(\mathbf{x}(t)))=1-\mu_{M_2^1}(f_2(\mathbf{x}(t)))$ for $k=2,4$.

Step II: The system matrix and the input vector of the reference model are chosen as follows:

$$\mathbf{H}_m=\begin{bmatrix}0 & 1\\ -8 & -8\end{bmatrix} \tag{14}$$

$$\mathbf{B}_m=\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{15}$$

It can be seen that the reference model is a stable system.

Step III: Conditions i) and ii) of Lemma 1 are satisfied if we choose

$$\mathbf{D}_1=\mathbf{D}_2=\begin{bmatrix}f_{1_{\min}}+8 & 8\end{bmatrix}=[17\ \ 8] \tag{16}$$

$$\mathbf{D}_3=\mathbf{D}_4=\begin{bmatrix}f_{1_{\max}}+8 & 8\end{bmatrix}=[28\ \ 8]. \tag{17}$$

It can be seen that

$$\alpha(\mathbf{x}(t))=f_2(\mathbf{x}(t))<0. \tag{18}$$

Step IV: The switching controller is designed as

$$u(t)=\sum_{j=1}^{4}m_j(\mathbf{x}(t))\left(\mathbf{G}_j\mathbf{x}(t)+\mathbf{r}\right), \tag{19}$$

where

$$m_i=-\frac{\operatorname{sgn}\left(\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+r(t))\right)}{\operatorname{sgn}(\alpha)\alpha_{\min}},$$
$$i=1,2,3,4;$$

$$\mathbf{P}=\begin{bmatrix}1.0625 & 0.0625\\ 0.0625 & 0.0703\end{bmatrix};\ \mathbf{Q}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix};$$

$$\mathbf{G}_i=-\mathbf{D}_i,\ i=1,2,3,4;$$
$$|\alpha|\in[\alpha_{\min}\ \ \alpha_{\max}]=[0.001\ 0.2].$$

We let $\mathbf{x}(t)=[x_1(t)\ \ x_2(t)]^T=[\theta(t)\ \ \dot{\theta}(t)]^T$, $\theta(t)\in[\theta_{\min}\ \ \theta_{\max}]=[-(22\pi/45)\ (22\pi/45)]$, and $\dot{\theta}(t)\in[\dot{\theta}_{\min}\ \ \dot{\theta}_{\max}]=[-5\ \ 5]$. Then the state-space representation of (10) is given by

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)=\begin{bmatrix}0 & 1\\ f_1(\mathbf{x}(t)) & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ f_2(\mathbf{x}(t))\end{bmatrix}u(t). \tag{11}$$

By comparing (11) to (10), we can see that

$$f_1(\mathbf{x}(t))=\frac{g-amlx_2(t)^2\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right)$$

and

$$f_2(\mathbf{x}(t))=-\frac{a\cos(x_1(t))}{4l/3-aml\cos^2(x_1(t))}.$$

The inverted pendulum of (10) can be modeled by a fuzzy plant model having four rules. The ith rule can be written as follows:

$$\begin{aligned}&\text{Rule } i\text{: IF } f_1(\mathbf{x}(t)) \text{ is } \mathrm{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \text{ is } \mathrm{M}_2^i\\ &\text{THEN } \dot{\mathbf{x}}(t)=\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_iu(t) \text{ for } i=1,2,3,4\end{aligned} \tag{12}$$

so that the system dynamical behavior is described by

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{4}w_i(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_iu(t)), \tag{13}$$

where

$$\mathbf{A}_1=\mathbf{A}_2=\begin{bmatrix}0 & 1\\ f_{1_{\min}} & 0\end{bmatrix} \text{ and}$$

$$\mathbf{A}_3=\mathbf{A}_4=\begin{bmatrix}0 & 1\\ f_{1_{\max}} & 0\end{bmatrix};$$

$$\mathbf{B}_1=\mathbf{B}_3=\begin{bmatrix}0\\ f_{2_{\min}}\end{bmatrix} \text{ and } \mathbf{B}_2=\mathbf{B}_4=\begin{bmatrix}0\\ f_{2_{\max}}\end{bmatrix};$$

$$f_{1_{\min}}=9\le f_1(\mathbf{x}(t)) \text{ and } f_{1_{\max}}=20\ge f_2(\mathbf{x}(t));$$

$$f_{2_{\min}}=-0.2\le f_2(\mathbf{x}(t)) \text{ and}$$
$$f_{2_{\max}}=-0.001\ge f_2(\mathbf{x}(t));$$

and

$$w_i=\frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))}{\sum_{l=1}^{4}\left(\mu_{\mathrm{M}_1^l}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^l}(f_2(\mathbf{x}(t)))\right)}.$$

The membership functions, shown in Fig. 2, are given as follows:

$$\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))=\frac{-f_1(\mathbf{x}(t))+f_{1_{\max}}}{f_{1_{\max}}-f_{1_{\min}}} \text{ for } k=1,2$$

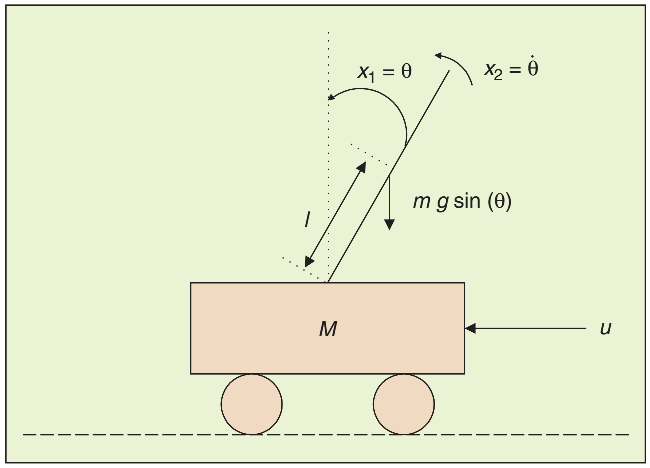

Figure 1. *A cart-pole-type inverted pendulum system.*

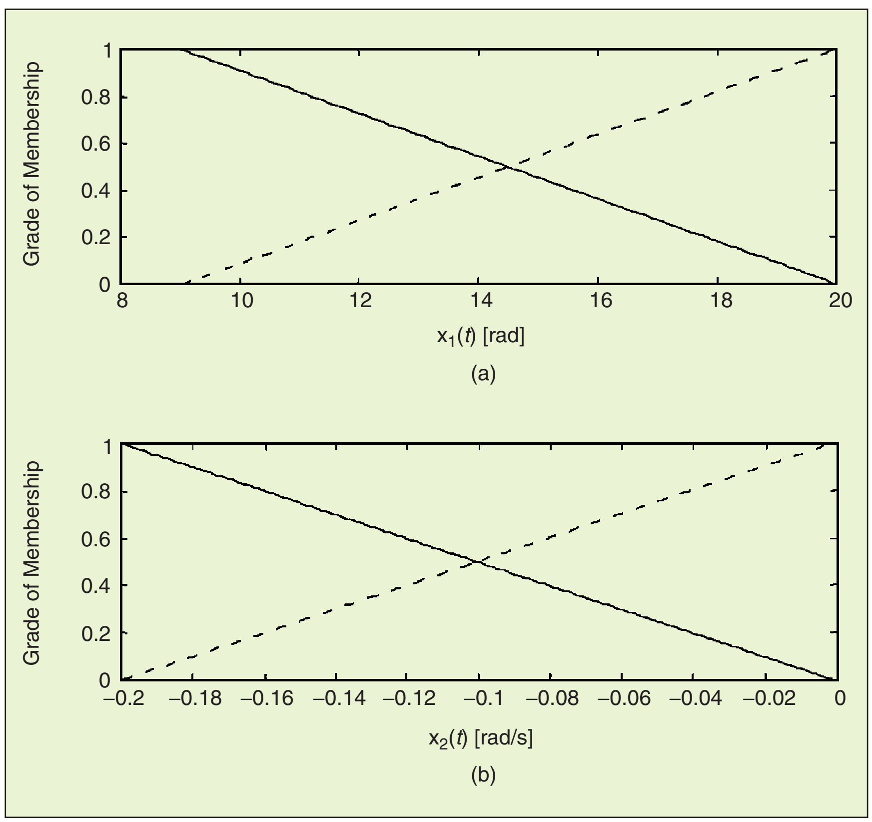

Figure 2. *Membership functions for the fuzzy plant model of the inverted pendulum on a cart.*

To show the merits of the proposed switching controller, we compare the simulation results of the system under the switching controller to those under a linear state-feedback controller. The linear state-feedback controller is designed based on the linearized model of the inverted pendulum around the origin. The dynamics of the linearized model of the inverted pendulum are given by

$$\dot{\mathbf{x}}(t)=\tilde{\mathbf{A}}\mathbf{x}(t)+\tilde{\mathbf{B}}u(t), \tag{20}$$

where

$$\tilde{\mathbf{A}}=\begin{bmatrix} 0 & 1 \\ \dfrac{M_o+m_o}{M_o l}g & 0 \end{bmatrix},\quad \tilde{\mathbf{B}}=\begin{bmatrix} 0 \\ -\dfrac{1}{M_o l} \end{bmatrix},$$

$$m_o=\frac{m_{\min}+m_{\max}}{2},\ \text{and}\ M_o=\frac{M_{\min}+M_{\max}}{2}.$$

The linear state feedback controller output is given by

$$u(t)=\mathbf{G}\mathbf{x}(t)-M_o lr. \tag{21}$$

The feedback gain is set as $\mathbf{G}=[130.0 \quad 32.0]$ such that the dynamics of the closed-loop system are the same as that of the reference model. The simulation results are obtained by using the actual plant of (10) and the switching controller of (19) or the linear state-feedback controller of (21). A disturbance is also injected into the simulated system to reflect the practical situation of the real process. Disturbances are injected into the system states and the control input. The actual system state used by the switching controller in the simulation is $\mathbf{x}(t)+\mathbf{x}_d(t)$. Here, $\mathbf{x}_d(t)$ is the disturbance of the system states defined as $\mathbf{x}_d(t)=[0 \ \ 0]^T$ when $t\le 15$ and $\mathbf{x}_d(t)=[(\pi/20)\ 0]^T$ when $15<t\le 20$. $\mathbf{x}_d(t)$ reaches about 10% of the maximum value of $x_1(t)$. The actual control signal fed to the input of the inverted pendulum is $u(t)+u_d(t)$ in the simulation. $u_d(t)$ is the disturbance defined as $u_d(t)=0$ when $t\le 10$, $u_d(t)=1000$ when $10<t\le 15$, and $u_d(t)=0$ when $15<t\le 20$. $u_d(t)$ reaches about 20% of the maximum value of $u(t)$. Figs. 3 and 4 show the re-

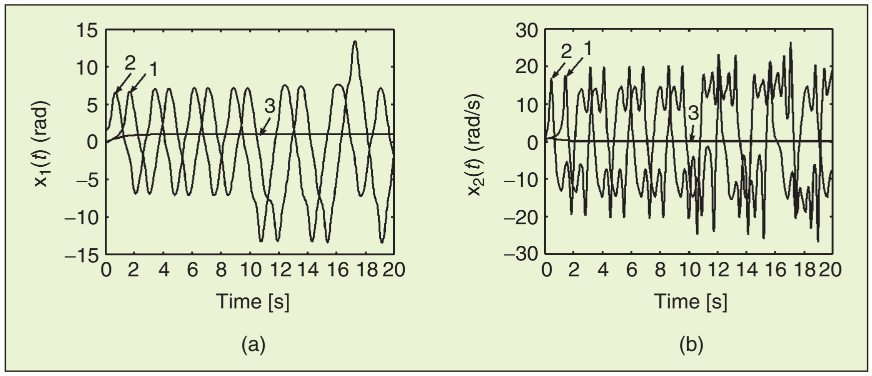

Figure 6. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the linear state-feedback controller under the initial conditions of* $\mathbf{x}(0)=[0.5 \ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45) \ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0 \ \ 0]^T$ *(response 3), m = 0.5 kg, M = 80 kg, and* $r(t)=8$.

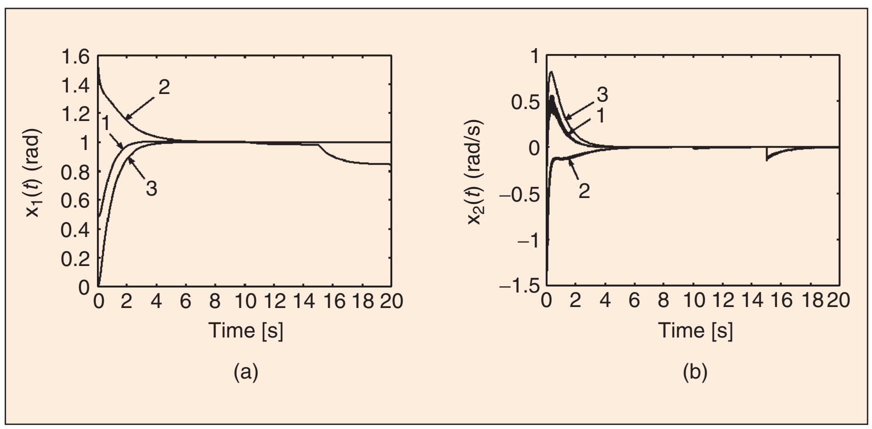

Figure 7. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the switching controller under the initial conditions of* $\mathbf{x}(0)=[0.5 \ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45) \ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0 \ \ 0]^T$ *(response 3), m = 2 kg, M = 8 kg, and* $r(t)=8$.

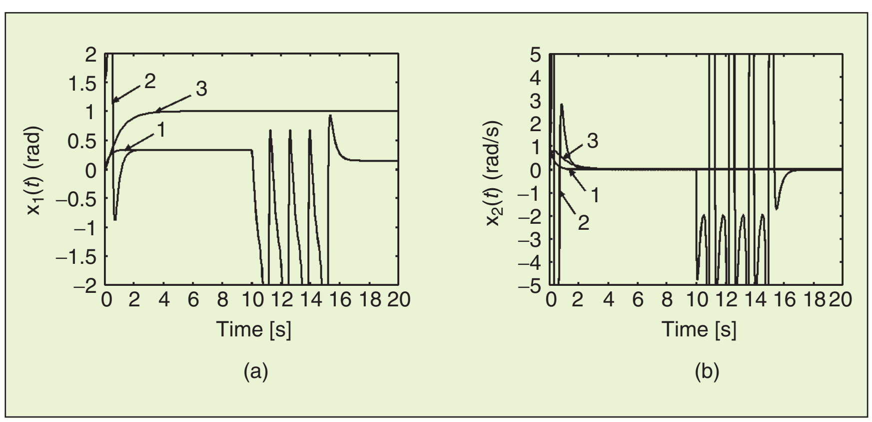

Figure 8. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the linear state-feedback controller under the initial conditions of* $\mathbf{x}(0)=[0.5 \ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45) \ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0 \ \ 0]^T$ *(response 3), m = 2 kg, M = 8 kg, and* $r(t)=8$.

sponses of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ under the initial conditions of $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ (dotted lines), $\mathbf{x}(0)=[(22\pi/45)\,0.5]$ (dashed lines), and $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ (solid lines), with m=0.5 kg, M=8 kg, and $r(t)$=8. Figs. 5 and 6 show the responses for m=0.5 kg and M=80 kg. Figs. 7 and 8 show the responses for m=2 kg and M=8 kg. Figs. 9 and 10 show the responses for m=2 kg and M=80 kg. The simulation results show that the proposed switching controller gives a better performance under different combinations of the values of the uncertain parameters than the linear state-feedback controller.

Conclusion

A switching controller has been designed for nonlinear plants subject to unknown parameters. Under some conditions, this switching controller has the ability to drive the system states to follow those of a reference model. An application example of an inverted pendulum on a cart has been given.

Acknowledgments

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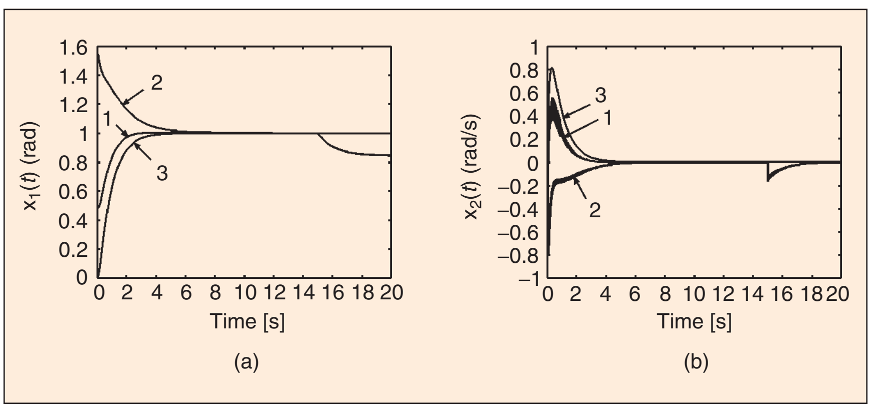

Figure 9. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the switching controller under the initial conditions of* $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45)\ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ *(response 3), m = 2 kg, M = 80 kg, and* $r(t)$=8.

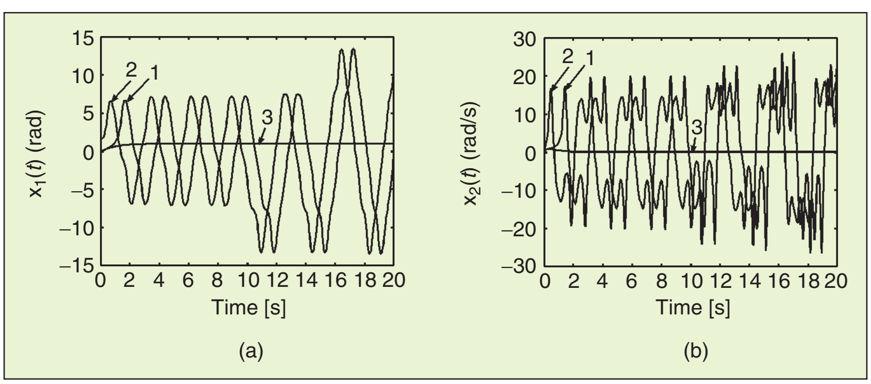

Figure 10. *Responses of* $\mathbf{x}(t)$ *and* $\hat{\mathbf{x}}(t)$ *with the linear state-feedback controller under the initial conditions of* $\mathbf{x}(0)=[0.5\ \ 0.5]^T$ *(response 1),* $\mathbf{x}(0)=[(22\pi/45)\ \ 0.5]^T$ *(response 2),* $\hat{\mathbf{x}}(0)=[0\ \ 0]^T$ *(response 3), m = 2 kg, M = 80 kg, and* $r(t)$=8.

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Appendix

The proof of Lemma 1 is given in this Appendix. From (2) and (4), writing $w_i(\mathbf{x}(t))=w_i$, $m_j(\mathbf{x}(t))$ as m_j, and $\alpha(\mathbf{x}(t))$ as α, and using the property of (5) that $\sum_{i=1}^{p} w_i=1$, we have

$$\begin{aligned}\dot{\mathbf{e}}(t)&=\dot{\mathbf{x}}(t)-\dot{\hat{\mathbf{x}}}(t)=\sum_{i=1}^{p} w_i(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t))-\mathbf{H}_m\hat{\mathbf{x}}(t)-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{x}(t)-\mathbf{H}_m\mathbf{x}(t)-\mathbf{H}_m\hat{\mathbf{x}}(t)+\sum_{i=1}^{p} w_i(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t))\\ &\quad-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)-\mathbf{H}_m\mathbf{x}(t)+\sum_{i=1} w_i(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t))-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\left[(\mathbf{A}_i-\mathbf{H}_m)\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\right]-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\left[(\overline{\mathbf{A}}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{u}(t)\right]-\mathbf{B}_m\mathbf{r}(t),\end{aligned} \tag{A1}$$

where $\overline{\mathbf{A}}_i=\mathbf{A}_i-\mathbf{H}_m$, $i=1,2,\ldots,p$. From (8), (A1) becomes

$$\begin{aligned}\dot{\mathbf{e}}(t)&=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\overline{\mathbf{A}}_i\mathbf{x}(t)+\sum_{i=1}^{p} w_i\mathbf{B}_i\mathbf{u}(t)-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\overline{\mathbf{A}}_i\mathbf{x}(t)+\mathbf{B}(\mathbf{x}(t))\mathbf{u}(t)-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\overline{\mathbf{A}}_i\mathbf{x}(t)+\sum_{i=1}^{p} w_i\mathbf{B}_m\mathbf{G}_i\mathbf{x}(t)\\ &\quad-\sum_{i=1}^{p} w_i\mathbf{B}_m\mathbf{G}_i\mathbf{x}(t)+\mathbf{B}(\mathbf{x}(t))\mathbf{u}(t)-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\left(\overline{\mathbf{A}}_i+\mathbf{B}_m\mathbf{G}_i\right)\mathbf{x}(t)\\ &\quad-\sum_{i=1}^{p} w_i\mathbf{B}_m\mathbf{G}_i\mathbf{x}(t)+\mathbf{B}(\mathbf{x}(t))\mathbf{u}(t)-\mathbf{B}_m\mathbf{r}(t).\end{aligned} \tag{A2}$$

Putting (9) into (A2)

$$\begin{aligned}\dot{\mathbf{e}}(t)&=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\left(\overline{\mathbf{A}}_i+\mathbf{B}_m\mathbf{G}_i\right)\mathbf{x}(t)-\sum_{i=1}^{p} w_i\mathbf{B}_m\mathbf{G}_i\mathbf{x}(t)\\ &\quad+\mathbf{B}(\mathbf{x}(t))\sum_{j=1}^{p} m_j\left(\mathbf{G}_j\mathbf{x}(t)+\mathbf{r}\right)-\mathbf{B}_m\mathbf{r}(t)\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i\left(\overline{\mathbf{A}}_i+\mathbf{B}_m\mathbf{G}_i\right)\mathbf{x}(t)\\ &\quad-\sum_{i=1}^{p} w_i\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}(t))+\sum_{i=1}^{p} m_i\mathbf{B}(\mathbf{x}(t))(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}).\end{aligned} \tag{A3}$$

From (7), (8), and (A3),

$$\begin{aligned}\dot{\mathbf{e}}(t)&=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i(\mathbf{B}_m\mathbf{D}_i+\mathbf{B}_m\mathbf{G}_i)\mathbf{x}(t)\\ &\quad-\sum_{i=1}^{p} w_i\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}(t))+\sum_{i=1}^{p} m_i\alpha\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r})\\ &=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p} w_i(\mathbf{B}_m\mathbf{D}_i+\mathbf{B}_m\mathbf{G}_i)\mathbf{x}(t)\\ &\quad+\sum_{i=1}^{p}(\alpha m_i-w_i)\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}(t)).\end{aligned} \tag{A4}$$

Letting

$$\mathbf{G}_i=-\mathbf{D}_i,\ i=1,2,\ldots,p, \tag{A5}$$

(A4) becomes

$$\dot{\mathbf{e}}(t)=\mathbf{H}_m\mathbf{e}(t)+\sum_{i=1}^{p}(\alpha m_i-w_i)\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t)+\mathbf{r}(t)). \tag{A6}$$

To investigate the stability of (A6), the following Lyapunov function is employed:

$$V=\frac{1}{2}\mathbf{e}(t)^{\mathrm{T}}\mathbf{P}\mathbf{e}(t) \tag{A7}$$

where $\mathbf{P} \in \Re^{n \times n}$ is a constant symmetric positive definite matrix. Differentiating (A7), we have

$$\dot{V} = \frac{1}{2}\left(\dot{\mathbf{e}}(t)^{\mathrm{T}} \mathbf{P}\mathbf{x}(t) + \mathbf{x}(t)^{\mathrm{T}} \mathbf{P}\dot{\mathbf{e}}(t)\right). \tag{A8}$$

Putting (A6) into (A8),

$$\begin{aligned}\dot{V} &= \frac{1}{2}\left(\mathbf{H}_m \mathbf{e}(t) + \sum_{i=1}^{p} (\alpha m_i - w_i)\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right)^T \mathbf{P}\mathbf{e}(t) \\ &\quad + \frac{1}{2}\mathbf{e}(t)^T \mathbf{P}\left(\mathbf{H}_m \mathbf{e}(t) + \sum_{i=1}^{p} (\alpha m_i - w_i)\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right) \\ &= \frac{1}{2}\mathbf{e}(t)^T\left(\mathbf{H}_m^T\mathbf{P} + \mathbf{P}\mathbf{H}_m\right)\mathbf{e}(t) + \sum_{i=1}^{p}(\alpha m_i - w_i) \\ &\quad \mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t)) \\ &= -\frac{1}{2}\mathbf{e}(t)^T\mathbf{Q}\mathbf{e}(t) + \sum_{i=1}^{p}(\alpha m_i - w_i)\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t)),\end{aligned} \tag{A9}$$

where

$$\mathbf{Q} = -\left(\mathbf{H}_m^T\mathbf{P} + \mathbf{P}\mathbf{H}_m\right). \tag{A10}$$

$\mathbf{Q} \in \Re^{n \times n}$ is a constant symmetric positive definite matrix. As α is bounded, we can consider that $|\alpha| \in [\alpha_{\min} \;\; \alpha_{\max}]$, where $\alpha_{\max} > \alpha_{\min} > 0$. We choose m_i, $i = 1, 2, \ldots, p$, as follows:

$$m_i = -\frac{\operatorname{sgn}\left(\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right)}{\operatorname{sgn}(\alpha)\alpha_{\min}}, i = 1, 2, \ldots, p. \tag{A11}$$

From (A9) and (A11),

$$\begin{aligned}\dot{V} &= -\frac{1}{2}\mathbf{e}(t)^T\mathbf{Q}\mathbf{e}(t) \\ &\quad + \sum_{i=1}^{p}\left(-\frac{\alpha \operatorname{sgn}\left(\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right)}{\operatorname{sgn}(\alpha)\alpha_{\min}} - w_i\right) \\ &\quad \times \mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t)) \\ &\le -\frac{1}{2}\mathbf{e}(t)^T\mathbf{Q}\mathbf{e}(t) - \sum_{i=1}^{p}\frac{|\alpha|}{\alpha_{\min}}\left|\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right| \\ &\quad + \sum_{i=1}^{p} w_i\left|\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right| \\ &= -\frac{1}{2}\mathbf{e}(t)^T\mathbf{Q}\mathbf{e}(t) - \sum_{i=1}^{p}\left(\frac{|\alpha|}{\alpha_{\min}} - w_i\right)\left|\mathbf{e}(t)^T\mathbf{P}\mathbf{B}_m(\mathbf{G}_i\mathbf{x}(t) + \mathbf{r}(t))\right|.\end{aligned} \tag{A12}$$

Since $(|\alpha|/\alpha_{\min}) \ge 1 \ge w_i$, $i = 1, 2, \ldots, p$, (A12) becomes

$$\dot{V} \le -\frac{1}{2}\mathbf{e}(t)^{\mathrm{T}}\mathbf{Q}\mathbf{e}(t) \le 0. \tag{A13}$$

Equality holds when $\mathbf{e}(t) = \mathbf{0}$. From (A13), it can be concluded that $\mathbf{e}(t) \to \mathbf{0}$ or equivalently $\mathbf{x}(t) \to \hat{\mathbf{x}}(t)$ as $t \to 0$. This ends the proof.

How Feedback Control Saved My Life–A True Story

Once I filled the trunk of my car with patio stones. Driving over a high bridge on a windy day, I realized that the rear of the car was swaying, so I steered to suppress the motion, only to see it grow in amplitude to a dangerous level. Luckily, I realized that this was a classic case of a nonminimum phase zero with a loop gain that was too high. So I lowered the bandwidth, stabilized the system, and lived to tell the story.

–D.S. Bernstein