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Nonlinear tracking control of a dc motor via a boost-converter using linear dynamic output feedback

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***Abstract*— In this contribution a tracking controller for the shaft angular velocity of a dc motor connected to a boost type power converter is designed. This system is not flat and the internal dynamics with respect to the angular velocity are unstable. A linear dynamic output feedback controller is designed which stabilizes the linearized tracking error dynamics about the reference trajectory. To this end, the results in [1], [2] for the design of linear tracking controllers for flat systems are extended to non-flat systems. The differences to the controller design using the state space approach for linear time varying systems are discussed.**

I. Introduction

In [1] and [2] it has been proposed to stabilize a given reference trajectory for a flat nonlinear system using linear dynamic output feedback. To this end, a time varying differential operator representation of the system linearization about the reference trajectory is derived from the nonlinear controller normal form. In this contribution it is at first discussed how a differential operator representation can be derived using methods for linear time varying systems even if the nonlinear system is not flat (Section II). Based on the differential operator representation a stabilizing tracking controller using linear dynamic output feedback is derived and the differences to a standard state space approach for linear time varying systems are discussed (Section III). It is outlined that from the proposed design procedure controllers of very low order result. A further problem that has to be addressed is the fact that in the case of non-flat systems or when the trajectory has not been planned for the flat output the trajectory is not known symbolically. In Section IV it is shown how this problem can be overcome. Finally, the outlined control strategy is applied to the control of the angular velocity of a dc motor which is connected to a boost type dc power converter (Section V). This system is not flat and the internal dynamics with respect to the angular velocity are unstable. It is shown that it is possible with the proposed approach to achieve the rather challenging task of stabilizing the reference trajectory for the angular velocity using pure (dynamic) output feedback. It is furthermore shown how an integral error feedback can be included in the controller design to account e.g. for model errors.

II. Differential operator representation of the linearized system

Consider the following nonlinear single-input system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

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with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}^m$. In [2] it has been shown that if system (1) is flat (see [3], [4]), a derivation of the differential operator representation of the linearized system is possible using the corresponding nonlinear controller normal form. As the investigated dc motor / power converter setup in Section V is not flat, the results in [2] have to be extended to non-flat systems. To this end, it is assumed that a reference trajectory $(x_d(t), u_d(t))$ which results from the design of a feedforward controller signal $u_d(t)$ for system (1) is given. In this case the linearized system about the reference trajectory is given by

$$\Delta\dot{x} = A(t)\Delta x + b(t)\Delta u \tag{3}$$

$$\Delta y = C(t)\Delta x \tag{4}$$

where $\Delta x = x - x_d$, $\Delta u = u - u_d$, $\Delta y = y - y_d$ and

$$A(t) = \frac{\partial(f(x) + g(x)u)}{\partial x}\Big|_{(x_d(t),u_d(t))} \tag{5}$$

$$b(t) = g(x)|_{x_d(t)} \tag{6}$$

$$C(t) = \frac{\partial h(x)}{\partial x}\Big|_{x_d(t)} \tag{7}$$

The controllability matrix of the linearized system (3) is given by

$$Q_c(t) = \begin{bmatrix} p_0 & p_1 & \dots & p_{n-1} \end{bmatrix} \tag{8}$$

(see e.g. [5]) where

$$p_0 = b(t); \quad p_{k+1} = -A(t)p_k + \dot{p}_k \tag{9}$$

System (3) is uniformly controllable on the interval (t_0, t_1) if Q_c is nonsingular for all $t \in (t_0, t_1)$. In [6] it has been outlined that if the linear time varying system (3) is uniformly controllable then it is flat. A flat output of system (3) is given by the last row of the inverse controllability matrix Q_c

$$y_f = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} Q_c^{-1}\Delta x \tag{10}$$

By the corresponding Brunovský coordinates

$$z = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}^T = \begin{bmatrix} y_f & \dot{y}_f & \dots & y_f^{(n-1)} \end{bmatrix}^T \tag{11}$$

a linear transformation

$$z = T(t)\Delta x \tag{12}$$

is defined in view of (10). In the Brunovský coordinates system (3) is given in time varying controller normal form

$$\dot{z} = A_{cnf}(t)z + b_{cnf}(t)\Delta u \tag{13}$$

The last row of (13) reads

$$\dot{z}_n = a_n^T(t)z + b_n(t)\Delta u \tag{14}$$

Introducing the vector $s(D) = [1, D, \dots, D^{n-1}]^T$ where $D := \frac{d}{dt}$, it follows in view of (11) that

$$z = s(D)y_f, \quad \dot{z}_n = D^n y_f \tag{15}$$

holds. Substituting (15) in (14) and reordering yields

$$\frac{1}{b_n(t)}\left(D^n - a_n^T(t)s(D)\right) y_f = \Delta u \tag{16}$$

With (12) and (15) the output equation (4) can be reformulated

$$\Delta y = C(t)T^{-1}(t)z = C(t)T^{-1}(t)s(D)y_f \tag{17}$$

Equations (16) and (17) establish the time varying differential operator representation (see e.g. [1], [2], [7])

$$n(D,t)y_f = \Delta u \tag{18}$$

$$\Delta y = Z(D,t)y_f \tag{19}$$

with the left polynomial differential operators (see [7])

$$n(D,t) = \frac{1}{b_n(t)}(D^n - a_n^T(t)s(D)) \tag{20}$$

$$Z(D,t) = C(t)T^{-1}(t)s(D) \tag{21}$$

where $Z(D,t)$ is an $(m \times 1)$ vector. Defining $\Gamma_c[n(D,t)]$ as the coefficient of the highest power of D in $n(D,t)$ allows a decomposition of $n(D,t)$ in the highest order term and terms of lower orders

$$n(D,t) = \Gamma_c[n(D,t)]D^n + n_R(D,t) \tag{22}$$

where by comparison with (20)

$$\Gamma_c[n(D,t)] = \frac{1}{b_n(t)} \tag{23}$$

$$n_R(D,t) = -\frac{1}{b_n(t)}a_n^T(t)s(D) \tag{24}$$

If the elements of the matrix $A(t)$ and of the vector $b(t)$ are sufficiently smooth the boundedness of all occuring coefficients is assured and all involved transformations can be shown to be Lyapunov transformations (see [8]). The smoothness of $A(t)$ and $b(t)$ can be assured by suitable trajectory design (see Section V).

III. Design of linear tracking controllers

A. Time varying flat feedback

Setting $\Delta u = 0$ in (3) describes the linearized dynamics of the tracking error Δx in the state space when applying the pure feedforward u_d. These dynamics are time varying and might be too slow or even unstable, depending on system (1). According to [8] the linearized tracking error dynamics (3) can be achieved to be exponentially stable by assigning a stable time invariant dynamics to the corresponding controller normal form (13). This can be achieved by flat feedback. In view of (18) and (22) an additional control action

$$\Delta u = -\Gamma_c[n(D,t)]\bar{n}_R(D)y_f + n_R(D,t)y_f \tag{25}$$

achieves the stable time invariant dynamics

$$(D^n + \bar{n}_R(D))y_f = \bar{n}(D,t)y_f = 0 \tag{26}$$

in view of $\Gamma_c(n(D,t)) \neq 0\, \forall t \in (t_0,t_1)$ when $\bar{n}_R(D,t)$ results from the Hurwitz polynomial

$$\bar{n}(D) = D^n + \bar{a}_{n-1}D^{n-1} + \ldots + \bar{a}_1 D + \bar{a}_0 = D^n + \bar{n}_R(D) \tag{27}$$

Adding $\Gamma_c[n(D,t)]D^n y_f - \Gamma_c[n(D,t)]D^n y_f$ to (25) leads to the more compact formulation

$$\Delta u = (n(D,t) - \tilde{n}(D,t))y_f \tag{28}$$

with the polynomial differential operator

$$\tilde{n}(D,t) = \Gamma_c[n(D,t)]\bar{n}(D) \tag{29}$$

The tracking controller design for the nonlinear system (1) results in the overall control law

$$u = u_d + \Delta u \tag{30}$$

With this control strategy only the feedforward u_d is applied in view of (28) and (10) when system (1) exactly tracks the reference trajectory (x_d, u_d), i.e. $\Delta x = 0$

B. Dynamic time varying output feedback

If the flat output y_f of the linearized tracking error system (3) and its time derivatives are not available for measurement, the flat feedback (28) can be estimated by a time varying dynamic output feedback. The dynamic output feedback controller is given by (see [1], [2])

$$\Delta(D)\Delta\hat{u} = z_u(D,t)\Delta u + Z_y^T(D,t)\Delta y \tag{31}$$

$$u = u_d + \Delta\hat{u} \tag{32}$$

where $\Delta(D)$ specifies a stable time invariant estimation error dynamics

$$\Delta(D)(\Delta\hat{u} - \Delta u) = 0 \tag{33}$$

The polynomial differential operator $z_u(D,t)$ and the $(m \times 1)$ vector $Z_y(D,t)$ have to be determined from the Diophantine equation

$$z_u(D,t)n(D,t) + Z_y^T(D,t)Z(D,t) = \Delta(D)(n(D,t) - \tilde{n}(D,t)) \tag{34}$$

In [9] conditions for the existence of a solution of (34) are given. The conditions can be used to derive a general form of $z_u(D,t)$ and $Z_y(D,t)$ consisting of polynomials in D with unknown time varying coefficients. These coefficients are determined by a generally under-determined system of linear time varying equations, which is easily derived by inserting the general expressions for $z_u(D,t)$ and $Z_y(D,t)$ into (34) and reordering subsequently. The resulting set of equations is only solvable, if the Sylvester matrix related to the pair $Z(D,t)$ and $n(D,t)$ has full rank (see [9]). Remaining degrees of freedom can be used to meet additional specifications for the controller. This has been illustrated in [2]. As a result, when assuming that the dynamics chosen in (33) is sufficiently fast, the controller (31)–(32) stabilizes the tracking according to the differential equation (26). Even if disturbances must be taken into account, this conclusion remains valid as long as the caused errors $\Delta\hat{u} - \Delta u$ are small.

In view of (33) $\Delta\hat{u}$ is an estimate for the linear functional (28). It has been shown in [10] that a linear functional can be estimated, for certain sensor actuator configurations, with lower order observers than the estimation of all non-measurable states of the system. This is illustrated in Section V-C. The dynamic output feedback (31)–(32) achieves a time invariant dynamics (33) for the error $\Delta\hat{u} - \Delta u$ independently from the inputs of the feedback controller Δu and Δy as long as the initial conditions of the system (1) are not significantly inconsistent with the starting point $x_d(0)$ of the reference trajectory. This is another difference from the standard state space approach where the non-measurable states would be estimated by stabilization of the estimation error dynamics in the coordinates of the time varying observer normal form. As these coordinates are related to the Brunovský coordinates (11) by a time varying transformation, a time varying error dynamics for the control signal results.

IV. Calculation of required time derivatives

For the analysis of the linear time varying system (3) time derivatives of the elements of $A(t)$ and $b(t)$ are needed e.g. to compute the controllability matrix $Q_c(t)$ in (9). Further time derivatives are necessary to compute the coordinates transformation (12) or the solution of the Diophantine equation (34). In case of non-flat systems or when the trajectory is not planned for a flat output of a flat system, the trajectory (x_d, u_d) can, in general, not be determined as a symbolic function of time. Therefore, it is suggested to determine the time derivatives using the prolonged vector field

$$f_{p,\mu} = \begin{bmatrix} f^T + g^T u & \dot{u} & \dots & u^{(\mu)} \end{bmatrix}^T \tag{35}$$

With the prolonged vector field $f_{p,\mu}$ the time derivative of any function of the states x and input u of system (1) can be computed as

$$\begin{aligned} \frac{d}{dt} k(x, u, \dot{u}, \dots, u^{(\nu)}) &= \sum_{i=1}^{n} \frac{\partial k}{\partial x_i}(f_i + g_i u) + \sum_{j=0}^{\mu-1} \frac{\partial k}{\partial u^{(j)}} u^{(j+1)} \\ &= \frac{\partial k}{\partial(x, u, \dot{u}, \dots, u^{(\mu-1)})} f_{p,\mu} \end{aligned} \tag{36}$$

as long as $\nu < \mu$. The derivatives of u can be often computed even if the trajectory (x_d, u_d) has been determined numerically (see Section V).

V. Application to a boost-converter with motor setup

A. Feedforward controller design

In this section a trajectory for the angular velocity of a dc motor connected to a boost type dc/dc power converter is designed. The system equations of this setup are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{E}{L} \\ -\frac{1}{RC}x_2 - \frac{1}{C}x_3 \\ \frac{1}{L_m}x_2 - \frac{R_m}{L_m}x_3 - \frac{K_e}{L_m}x_4 \\ \frac{K_m}{J}x_3 - \frac{\tau}{J} \end{bmatrix} + \begin{bmatrix} -\frac{1}{L}x_2 \\ \frac{1}{C}x_1 \\ 0 \\ 0 \end{bmatrix} u \tag{37}$$

where x_1 is the converter current, x_2 the capacitor voltage, x_3 the motor current and x_4 the motor shaft angular velocity which is the output to be controlled

$$y_c = h_c(x) = x_4 \tag{38}$$

E in (37) is the constant converter supply voltage and τ is a constant friction torque of the motor. The system parameters have been taken to be $C = 100\,\mu$F, $L = 3\,$mH, $R = 500\,\Omega$, $R_\mathrm{m} = 8.1\,\Omega$, $K_\mathrm{e} = 4.31 \cdot 10^{-2} \frac{\mathrm{V}}{\mathrm{rad/sec}}$, $K_\mathrm{m} = 4.31 \cdot 10^{-2} \frac{\mathrm{V}}{\mathrm{rad/sec}}$, $L_\mathrm{m} = 8.9\,$mH, $J = 7.95 \cdot 10^{-6}\,\mathrm{kgm}^2$, $\tau = 3 \cdot 10^{-3}\,$Nm and $E = 7\,$V. The input is constrained to be in the interval $u \in [0, 1]$ as u is the duty cycle of the PWM input signal to the converter. The parameters are close to those taken in [11] where this power converter with motor setup together with its technical relevance is discussed in detail.

System (37) is not static feedback linearizable (see e.g. [12]) as the distribution $D_2 = \begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix}$ is not involutive. As a consequence, the single-input system (37) is not flat. This is due to the results in [13] as outlined e.g. in [6]. To assign a reference trajectory for the angular velocity it is realized that the output y_c has a well defined relative degree [12] of $r = 3$, i.e.

$$L_g L_f^i h_c(x) = 0,\ i = 0, 1; \quad L_g L_f^2 h_c(x) \neq 0 \tag{39}$$

in the operating region (see the reference trajectory in Figures 1 and 3). An input-output linearizing feedback controller for the output y_c with the new input v is given by

$$u_{io}(x, v) = \frac{1}{L_g L_f^2 h_c(x)} (-L_f^3 h_c(x) + v) \tag{40}$$

Furthermore, the coordinates transformation

$$\begin{bmatrix} \xi^T, \eta \end{bmatrix}^T = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \eta \end{bmatrix} = \begin{bmatrix} y_c \\ \dot{y}_c \\ \ddot{y}_c \\ \eta \end{bmatrix} = \begin{bmatrix} h_c(x) \\ L_f h_c(x) \\ L_f^2 h_c(x) \\ x_1 \end{bmatrix} = \phi(x) \tag{41}$$

can be shown to be nonsingular in the operating region. The internal dynamics are thus given by the dynamics of the converter current, when applying the controller (40), i.e.

$$\dot{\eta} = f_1(x) + g_1(x) u_{io}(x, v)\big|_{x=\phi^{-1}(\xi,\eta)} = q(\eta, \xi, v) \tag{42}$$

where f_1 and g_1 denote the first components of f and g respectively. For the angular velocity a reference trajectory $y_{c,d}$ is specified, which provides a smooth transition of the angular velocity from $y_{c,d}(0) = 200 \frac{\mathrm{rad}}{\mathrm{sec}}$ to $y_{c,d}(t_f) = 350 \frac{\mathrm{rad}}{\mathrm{sec}}$. The reference trajectory is supposed to be a transition between equilibrium points, therefore the boundary conditions $\dot{y}_{c,d}(0) = \ddot{y}_{c,d}(0) = \dot{y}_{c,d}(t_f) = \ddot{y}_{c,d}(t_f) = 0$ and $y_{c,d}^{(3)}(0-) = y_{c,d}^{(3)}(t_f+) = 0$ were incorporated. This corresponds to the equilibrium points $x_d(0) = (115.5\,\mathrm{mA}, 9.18\,\mathrm{V}, 69.6\,\mathrm{mA}, 200\,\frac{\mathrm{rad}}{\mathrm{sec}})$ and $x_d(t_f) = (225.6\,\mathrm{mA}, 15.65\,\mathrm{V}, 69.6\,\mathrm{mA}, 350\,\frac{\mathrm{rad}}{\mathrm{sec}})$ in the x-coordinates. The transition time has been set to $t_f = 400\,$msec. The internal dynamics

$$\dot{\eta} = q(\eta, \xi_d, y_{c,d}^{(3)}) \tag{43}$$

can be found to be unstable. To obtain a bounded solution for η_d the internal dynamics (43) can be integrated backwards in time (see e.g. [14]). As the eigenvalues of the linearized internal dynamics (43) are very large, preactuation is neglectable. The resulting feedforward controller is thus given by

$$u_d = u_{io}(x, v)\big|_{x=\phi^{-1}(\xi_d,\eta_d),\, v=y_{c,d}^{(3)}} \tag{44}$$

where $u_d \in [0, 1]$ is fulfilled (see Figure 3).

Due to the results in this section neither a flatness based controller (as has been implemented e.g. in [15] for a laboratory setup of a dc motor connected to a buck type power converter) nor the input-output linearizing controller (40) can be used to stabilize the tracking of the reference trajectory $y_{c,d}$ for the angular velocity. Therefore in the next section the linearized system of (37) will be investigated to design a linear dynamic output feedback in order to stabilize the linearized tracking error dynamics.

B. Linearized tracking error dynamics

The linearized tracking error system of system (37) is given by (3) where

$$A(t) = \begin{bmatrix} 0 & -\frac{u_d(t)}{L} & 0 & 0 \\ \frac{u_d(t)}{C} & -\frac{1}{RC} & -\frac{1}{C} & \\ 0 & \frac{1}{L_m} & -\frac{R_m}{L_m} & -\frac{K_m}{L_m} \\ 0 & 0 & \frac{K_m}{J} & 0 \end{bmatrix} \quad (45)$$

and

$$b(t) = \begin{bmatrix} -\frac{1}{L}x_{2,d}(t) & \frac{1}{C}x_{1,d}(t) & 0 & 0 \end{bmatrix}^T \quad (46)$$

As pointed out in Section IV all required time derivatives for the analysis of the linear time varying dynamics (3) can be calculated in terms of u and its derivatives. For the given situation the input u_d is given by the feedforward controller (44). Consequently, u_d is given as a function of the reference trajectory $y_{c,d}$ for the output and of the corresponding solution η_d of the internal dynamics. In a similar manner as in Section IV the time deriviatives of u_d can thus be generated iteratively as

$$\begin{aligned} u_d^{(i+1)} &= u_{io}^{(i+1)}(y_{c,d}, \dot{y}_{c,d} \ldots, y_{c,d}^{(i+3)}, \eta_d) \\ &= \frac{\partial u_{io}^{(i)}}{\partial \eta_d} q(\eta_d, y_{c,d}, \dot{y}_{c,d} \ldots, y_{c,d}^{(3)}) + \sum_{j=0}^{\mu-1} \frac{\partial u_{io}^{(i)}}{\partial y_{c,d}^{(j)}} y_{c,d}^{(j+1)}, \\ & i = 0, 1, \ldots \end{aligned} \quad (47)$$

in view of (44) for sufficiently large μ. It has to be emphasized that although the reference trajectory is only known from numeric computation and not symbolically, all necessary time derivatives for the analysis of the time varying tracking error dynamics can be computed using (36) and (47). All states are continuous functions of time. As a consequence, all time derivatives are continuous functions of time, when f and g as well as the reference trajectory for y_c are sufficiently smooth.

For the given trajectory the controllability matrix Q_c as defined in (9) is nonsingular during the whole trajectory and thus as discussed in Section II the linearized system (45)–(46) is flat and can be stabilized by flat feedback as pointed out in Section III-A. For the stabilization it is assumed that the angular velocity and the capacitor voltage can be measured, i.e. the measured output y for the stabilization of the trajectory is given by

$$y = \begin{bmatrix} x_4 & x_2 \end{bmatrix}^T \quad (48)$$

Consequently, the output matrix for the linearized dynamics can be obtained according to (7)

$$C(t) = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (49)$$

The observability matrix (see [5])

$$Q_o(t) = \begin{bmatrix} c_1^T \\ \dot{c}_1^T + c_1^T A \\ c_2^T \\ \dot{c}_2^T + c_2^T A \end{bmatrix} \quad (50)$$

can be found to be nonsingular over the whole trajectory. Thus, the linear time varying system (45)–(46) with output matrix (49) is uniformly controllable and observable over the whole trajectory.

C. Design of stabilizing dynamic output feedback

Using the methods introduced in Section III the denominator polynomial $n(D,t)$ of the differential operator representation of the linearized tracking dynamics can be derived, resulting in a time varying polynomial differential operator of degree four

$$n(D,t) = \frac{1}{b_n(t)}(D^4 + a_3(t)D^3 + a_2(t)D^2 + a_1(t)D + a_0(t)) \quad (51)$$

The numerator vector can be shown to be of the form

$$Z(D,t) = \begin{bmatrix} z_{11}(t)D + z_{10}(t) \\ z_{23}(t)D^3 + z_{22}(t)D^2 + z_{21}(t)D + z_{20}(t) \end{bmatrix} \quad (52)$$

For the dynamics of the controlled system the numerator polynomial

$$\tilde{n}(D,t) = \frac{1}{b_n(t)}(D + \lambda_r)^4 \quad (53)$$

is assigned (see (29)). The observability index of the observability matrix Q_o in (50) is $n_o = 2$. Using the solvability conditions in [9] for the Diophantine equation (34) it can be derived that a first order dynamic output feedback is sufficient, i.e. $\Delta(D)$ in (33) is chosen to be of degree one

$$\Delta(D) = D + \lambda_o \quad (54)$$

It follows from the structure of (34) that also the vector $Z_y(D,t)$ has to be chosen to be of degree one

$$Z_y(D,t) = \begin{bmatrix} z_{y_1}(D,t) \\ z_{y_2}(D,t \end{bmatrix} = \begin{bmatrix} z_{y,11}(t)D + z_{y,10}(t) \\ z_{y,21}(t)D + z_{y,20}(t) \end{bmatrix} \quad (55)$$

and $z_u(D,t)$ has to be chosen to be of degree zero

$$z_u(D,t) = z_{u,0}(t) \quad (56)$$

The coefficients of $z_u(D,t)$ and $Z_y(D,t)$ have to be calculated from the Diophantine equation (34). For the given configuration the solution is unique, no additional degrees of freedom occur. The additional stabilizing control input $\Delta\hat{u}$ can be computed according to (31) which in this case is given by

$$\Delta(D)\Delta\hat{u} = z_u(D,t)\Delta u + z_{y_1}(D,t)\Delta y_1 + z_{y_2}(D,t)\Delta y_2 \quad (57)$$

As system (45)–(46) is to be stabilized at the equilibrium $\Delta x = 0$ it can be shown using the results in [16] that the control signal $\Delta\hat{u}$ can equivalently be generated by

$$n_c(D,t)\Delta\hat{u} = z_{y1}(D,t)\Delta y_1 + z_{y2}(D,t)\Delta y_2 \quad (58)$$

where $n_c(D,t) = \Delta(D) - z_u(D,t)$. As the degrees of $n_c(D,t)$ and $z_{y_1}(D,t)$ and $z_{y_2}(D,t)$ respectively are equal the controller state

$$\zeta = \Delta\hat{u} - z_{y,11}\Delta y_1 - z_{y,21}\Delta y_2 \quad (59)$$

is introduced to reduce the degree of $z_{y_1}(D,t)$ and $z_{y_2}(D,t)$. Substituting (59) into (58) yields the relation

$$n_c(D,t)\zeta = \bar{z}_{y_1}(t)\Delta y_1 + \bar{z}_{y_2}(t)\Delta y_2 \quad (60)$$

The time varying left differential operator $n_c(D,t)$ in (60) is then transformed into a right polynomial differential operator using the operator identity $m(t)D = Dm(t) - \dot{m}(t)$. As pointed out in [7] the differential equation (60) with the right polynomial differential operators directly corresponds to a state space realization in observer normal form. In the given case a first order output feedback of the kind

$$\dot{\zeta} = a_\zeta(t)\zeta + b_{\zeta 1}(t)\Delta y_1 + b_{\zeta 2}(t)\Delta y_2$$
$$\Delta\hat{u} = \zeta + z_{y,11}(t)\Delta y_1 + z_{y,21}(t)\Delta y_2 \tag{61}$$

results. It has to be mentioned that a nonlinear tracking observer as proposed in [17] would be of order four. Even a reduced order observer would be of order 2. As outlined in Section III-B the low order of the output feedback (61) results from the fact that only the control input (28) is estimated. The first order time varying filter (61) is easy to initialize as demonstrated in the next section.

D. Simulation results

Figure 1 shows the simulation results when the initial condition for the converter with motor setup is not the equilibrium point $x_d(0)$ (see Section V-A) but the equilibrium point $x(0) = (118.7\,\mathrm{mA}, 9.4\,\mathrm{V}, 69.6\,\mathrm{mA}, 205\,\frac{\mathrm{rad}}{\mathrm{sec}})$. The controller parameters were chosen to be $\lambda_r = 250$ and $\lambda_o = 600$. To get a better impression of the transient behaviour in Figure 2 the state space trajectory for the first 70 ms is plotted. The linear time varying filter (61) has been initialized by the construction of a static feedback controller which results when assuming that the output matrix (7) of the linearized tracking error system is the four dimensional unit matrix and choosing for the controller parameter again $\lambda_r = 250$. This leads to a feedback law of the kind

$$\Delta u_{stat} = \tilde{z}_1(t)\Delta x_1 + \tilde{z}_2(t)\Delta x_2 + \tilde{z}_3(t)\Delta x_3 + \tilde{z}_4(t)\Delta x_4 \tag{62}$$

As Δx_1 and Δx_3 are not available for measurement $\zeta(0)$ is chosen such that

$$\Delta\hat{u}(0) = \tilde{z}_2(0)\Delta x_2(0) + \tilde{z}_4(0)\Delta x_4(0) \tag{63}$$

holds, i.e. $\Delta x_1(0) = \Delta x_3(0) = 0$ is assumed. For the given system this initialization provided very good initial values. To demonstrate the performance of the estimation for the control input an additional initial error $\Delta\zeta$ was introduced such that for the initial relative error of the control input $\frac{\Delta\hat{u}(0) - \Delta u_{stat}(0)}{\Delta u_{stat}(0)} \approx 0.25$ was provided for the simulation results in Figures 1 and 2. On the left hand side of Figure 3 the normalized input signal error $\frac{\Delta\hat{u} - \Delta u_{stat}}{\Delta u_{stat,max}}$ during the first 20 ms is plotted, where $\Delta u_{stat}(t)$ has been obtained by inserting the resulting trajectories as seen in Figure 1 into (62). For comparison the error is plotted when the controller (61) is applied to the time varying linear error system (45)–(46), (49). For the linear time varying system the transient behaviour exactly matches the desired time invariant first order dynamics, assigned in (33). Due to the nonlinearity of system (37) the input error does not perfectly match the desired behaviour. Small deviations can be observed for the time intervals when the system is far away from the reference trajectory. For comparison, the feedback law (62) has been implemented using a nonlinear tracking observer as proposed in ([17]). In the nominal case this output feedback yielded comparable results as the output feedback (61). However, the nonlinear tracking observer seemed to be more sensitive against parameter uncertainties as e.g. an increased friction torque τ. The inferior robustness properties of the nonlinear tracking observer might stem from the fact, that a fourth order order nonlinear observer has to be stabilized in contrast to the first order system (61). However, due to the limitited space, this cannot be disscussed in more detail. The robustness property of the controller (61) can furhter be improved by including an additional integral error feedback. For the construction of the controller it is realized that the flat output (10) of the boost converter is almost coincident with the angular velocity. Thus, the integral of the flat output can be very well approximated by the integral over the tracking error for the angular velocity. In view of the controller normal form (13) when introducing the additional state $z_0 = \int z_1 dt$ the new flat output of the extended system is given by $y_f = z_0$. As a consequence, the numerator polynomial of the differential operator representation for the extended system is given by

$$n_e(D,t) = n(D,t)D \tag{64}$$

The controller state z_0 can be used for the feedback, so the numerator vector is extended to

$$Z_e(D,t) = \begin{bmatrix} Z^T(D,t)D & 1 \end{bmatrix}^T \tag{65}$$

For the extended system (64)–(65) again a first order output feedback can be used. Specifying $\Delta(D)$ as in (54) the following dynamic output feedback results

$$\dot{z}_0 = \Delta y_1$$
$$\dot{\kappa} = a_\kappa(t)\kappa + b_{\kappa 1}(t)\Delta y_1 + b_{\kappa 2}(t)\Delta y_2 + b_{\kappa 3}(t)\Delta z_0$$
$$\Delta\hat{u} = \kappa + z_{y11}(t)\Delta y_1 + z_{y21}(t)\Delta y_2 \tag{66}$$

To demonstrate the performance of this controller it is assumed that the constant friction torque τ is by 30 % larger than assumed in Section V-A. Figure 4 shows the resulting trajectory when starting at $x(0) = (117.4\,\mathrm{mA}, 9.31\,\mathrm{V}, 69.6\,\mathrm{mA}, 203\,\frac{\mathrm{rad}}{\mathrm{sec}})$. The controller parameters have again been chosen to be $\lambda_r = 250$ and $\lambda_o = 600$. It can be verified that the controller increases the converter current by about 20 % to compensate for the additional friction. The angular velocity very well tracks the specified reference trajectory. When an input-output

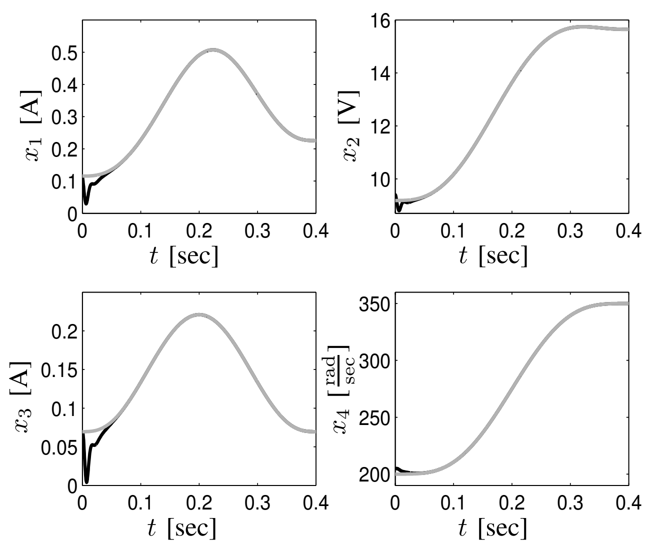

Fig. 1. state space reference trajectory (grey) for the boost converter with motor and actual trajectory for the system with controller (black)

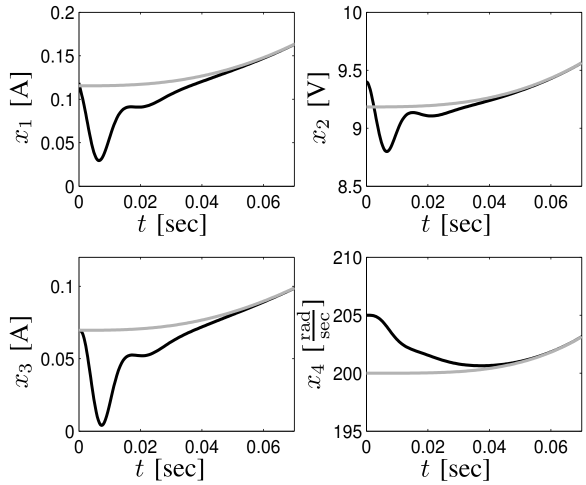

Fig. 2. state space reference trajectory (grey) for the boost converter with motor and actual trajectory for the system with controller (black)

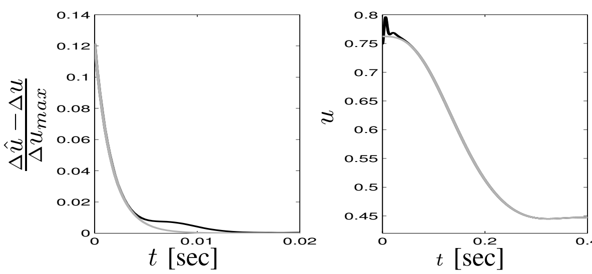

Fig. 3. left hand side: input signal error for the linear time varying system (grey) and for the nonlinear system (black), right hand side reference feedforward signal (grey) and control input of the stabilized system (black)

linearizing controller with respect to the converter current would have been designed, the converter current would have been forced to track the reference value for the current. For the final reference current of $x_{2,d}(t_f) = 225.6\,\mathrm{mA}$ an equilibrium angular velocity of about $300\,\frac{\mathrm{rad}}{\mathrm{sec}}$ would result. An input-output linearizing controller with respect to the converter current could thus not compensate for such a model uncertainty. Additionally, the internal dynamics with respect to the converter current exhibit an almost non hyperbolic component for the investigated setup. An alternative way to construct a controller with integral error feedback using the presented approach would have been to extend the nonlinear system (37) with an additional state which represents the integral of the angular velocity and to design then a controller as outlined in the preceding Sections for the extended nonlinear system.

VI. CONCLUSIONS

In this contribution the approach in [2] to stabilize the linearized tracking error dynamics for nonlinear flat systems using a differential operator representation has been generalized to non-flat systems. A tracking controller was designed that stabilizes the trajectory of a dc motor / converter setup. This has been shown to be a challenging task as the system is not flat and the internal dynamics with respect to the angular velocity are unstable. It was furtherhmore shown that the controller design using the Diophantine equation leads to a low order stabilizing output feedback and that an integral error feedback can be included in the framework.

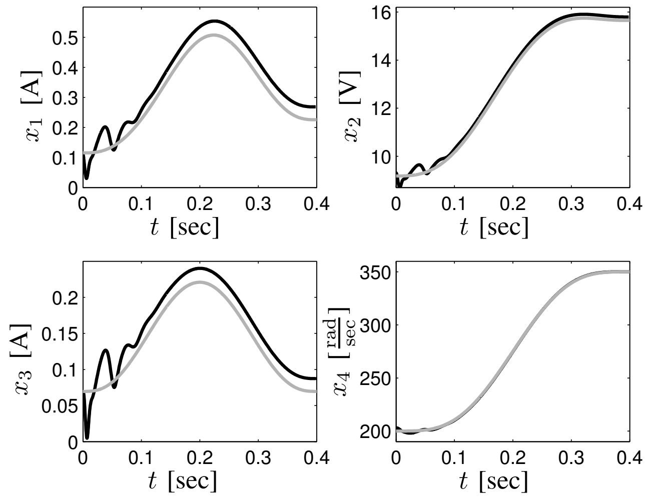

Fig. 4. state space reference trajectory (grey) for the boost converter with motor and actual trajectory for the system with controller with integral error feedback in the case of increased friction torque

The simulation results are encouraging and the controller is going to be implemented on a laboratory setup.

REFERENCES

[1] J. Deutscher, "A linear differential operator approach to flatness based tracking for linear and non-linear systems," *Int. J. Control*, vol. 3, pp. 266–276, 2002.

[2] F. Antritter, B. Müller, and J. Deutscher, "Tracking control for nonlinear flat systems by linear dynamic output feedback," *Proceedings NOLCOS 2004, Stuttgart*, 2004.

[3] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of nonlinear systems: introductory theory and examples," *Int. J. Control*, vol. 61, pp. 1327–1361, 1995.

[4] ——, "A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems," *Trans. Aut. Control*, vol. 44, pp. 922–937, 1999.

[5] L. M. Silverman and H. E. Meadows, "Controllablility and observability in time-variable linear systems," *J. SIAM Control*, vol. 5, pp. 64–73, 1967.

[6] H. Sira-Ramirez and S. K. Agrawal, *Differentially Flat Systems*. New York: Marcel Dekker, 2004.

[7] K. Tsakalis and P. A. Ioannu, *Linear Time-Varying Plants: Control and Adaption*. Englewood Cliffs, New Jersey: Prentice-Hall, 1993.

[8] W. Wolovich, "On the stabilization of controllable systems," *IEEE Trans. Autom. Control*, vol. 13, pp. 569–672, 1968.

[9] S. Limanond and K. Tsakalis, "Adaptive and non-adaptive 'pole-placement' control of multivariable linear time-varying plants," *Int. J. Control*, vol. 74, pp. 507–523, 2001.

[10] D. G. Luenberger, "An introduction to observers," *IEEE Trans. Autom. Control*, vol. 16, pp. 596–602, 1971.

[11] J. Linares-Flores, J. Reger, and H. Sira-Ramirez, "A time-varying linear state feedback tracking controller for a boost-converter driven dc motor," *Proceedings Mechatronics 2006 (submitted)*, 2006.

[12] A. Isidori, *Nonlinear Control Systems*. London: Springer Verlag, 1995.

[13] B. Charlet, J. Lévine, and R. Marino, "Sufficient conditions for dynamic state feedback linearization," *SIAM J. Control and Optimization*, vol. 29, pp. 38–57, 1991.

[14] H. Perez, B. Ogunnaike, and S. Devasia, "Output tracking between operating points for nonlinear processes: Van de vusse example," *IEEE Trans. Contr. Syst. Techn.*, vol. 10, pp. 611–617, 2002.

[15] F. Antritter, P. Maurer, and J. Reger, "Flatness based control of a buck-converter driven dc motor," *Proceedings Mechatronics 2006 (submitted)*, 2006.

[16] P. Hippe and C. Wurmthaler, *Zustandsregelung*. Berlin: Springer-Verlag, 1985.

[17] M. Fliess and J. Rudolph, "Local tracking observers for flat systems," *Proceedings of the Symposium on Control, Optimization and Supervision, CESA '96 IMACS Multiconference, Lille, France*, pp. 213–217, 1996.