Proceedings of the American Control Conference
Anchorage, AK May 8-10, 2002

Set Invariance Analysis and Gain-Scheduling Control for LPV Systems Subject to Actuator Saturation

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Abstract: In this paper, a set invariance analysis and gain scheduling control design approach is proposed for the polytopic linear parameter-varying systems subject to actuator saturation. A set invariance condition is first established. By utilizing this set invariance condition, the design of a time-invariant state feedback law is formulated and solved as an optimization problem with LMI constraints. A gain-scheduling controller is then designed to further improve the closed-loop performance. Numerical examples are presented to demonstrate the effectiveness of the proposed analysis and design method.

Keywords: Set invariance, linear parameter-varying systems, gain-scheduling, actuator saturation.

1 Introduction

In recent years there has been significant interest in the study of linear parameter-varying (LPV) systems, which is motivated by the gain scheduling control design methodology [9, 10, 11]. LPV systems are systems that depend on unknown but measurable time-varying parameters. The measurement of these parameters provides real-time information on the variations of the plant's characteristics. Hence, it is desirable to design controllers that are scheduled based on this information. LPV control theory has proven to be useful to simplify the interpolation and realization problems associated with the conventional gain-scheduling. The analysis and synthesis of LPV systems have been investigated recently in [1, 8, 13, 14] by the linear matrix inequality approach. The approach involves the design of several linear time-invariant (LTI) controllers for a parameterized family of linear time-invariant system models and the interpolation of these controller gains.

Actuator saturation can severely degrade the closed-loop system performance and sometimes even make the otherwise stable closed-loop system unstable by some large perturbation. The analysis and synthesis of control systems with actuator saturation nonlinearities have been receiving increasing attention recently (see, for example, [2, 5, 7] and the references therein). Very often, actuator saturation is dealt with by either designing low gain control laws that, for a given bound on the initial conditions, avoid the saturation limits, or estimating the region of attraction in the presence of actuator saturation. In this paper, we will analyze the stability of LPV systems with actuator saturation. The recent analysis approach proposed in [5, 6] is used to analyze the set invariance and then a gain-scheduled optimal control design is proposed. The resulting closed-loop system not only possesses a large domain of attraction that contains *a priori* given set of initial conditions, but also guarantees a minimal performance index.

2 Problem Statement and Preliminary

We consider the polytopic LPV systems, whose system matrices are affine functions of a parameter vector $p(t)$, subject to actuator saturation,

$$\dot{x}(t) = A(p(t))x(t) + B(p(t))\sigma(u(t)), \quad (1)$$
$$z(t) = C(p(t))x(t) + D(p(t))\sigma(u(t)), \quad (2)$$

where

$$A(p(t)) = \sum_{j=1}^{r} p_j(t)A_j, \quad B(p(t)) = \sum_{j=1}^{r} p_j(t)B_j,$$
$$C(p(t)) = \sum_{j=1}^{r} p_j(t)C_j, \quad D(p(t)) = \sum_{j=1}^{r} p_j(t)D_j,$$

with $x \in \mathbb{R}^n$ denoting the state, $u \in \mathbb{R}^m$ the input, $z \in \mathbb{R}^p$ the control output vector and $p(t) = [\ p_1(t) \ \ p_2(t) \ \ \cdots \ \ p_r(t)\ ]^T \in \mathbb{R}^r$ the time-varying parameter vector. It is assumed that vector $p(t)$ belongs to the unit simplex $\mathcal{P}$, where

$$\mathcal{P} := \left\{ \sum_{j=1}^{r} p_j = 1, \ \ 0 \le p_j \le 1 \right\}. \quad (3)$$

Therefore, when $p_i(t) = 1$ and $p_j(t) = 0$ for $j \in [1, \ r]$, $j \neq i$, LPV model (1-2) reduces to its i-th LTI "local" model, *i.e.*, $(A(p), \ B(p), \ C(p), \ D(p)) = (A_i, \ B_i, \ C_i, \ D_i)$. That is, the LPV system matrices vary inside a corresponding polytope Ω whose vertices consist of r local system matrices

$$\Omega = \mathrm{co}\left\{(A_i, \ B_i, \ C_i, \ D_i), \ \ i \in [1, r]\right\}, \quad (4)$$

where co denotes the convex hull.

The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the standard saturation function of appropriate dimensions defined as follows

$$\sigma(u) = \left[\begin{array}{cccc} \sigma(u_1) & \sigma(u_2) & \cdots & \sigma(u_m) \end{array} \right]^{\mathrm{T}},$$

where $\sigma(u_i) = \mathrm{sign}(u_i)\min\{1, |u_i|\}$. Also note that it is without loss of generality to assume unity saturation level. In this paper, we will study the design of a state feedback law

$$u(t) = Fx(t), \qquad (5)$$

or a time-varying parameter-dependent control law

$$u(t) = \bar{F}(t)x(t) = \sum_{i=1}^{r} p_i(t)F_i x(t), \qquad (6)$$

which asymptotically stabilizes LPV system (1) with actuator saturation. Control law (5) is a constant feedback law, while (6) is a time-varying feedback law, which is the so-called gain-scheduled controller.

In this paper, we will consider the optimal control problem of the LPV plants subject to actuator saturation. That is, we will design a control u, which minimizes the following worst-case performance

$$\min_{u(t)} \max_{(A(p),B(p),C(p),D(p))\in\Omega} \left\{ J = \int_0^\infty z^{\mathrm{T}}(t)z(t) \right\}. \quad (7)$$

Let f_i be the i-th row of the matrix F. We define the symmetric polyhedron

$$\mathcal{L}(F) = \{x \in \mathbb{R}^n : |f_i x| \le 1, \quad i = 1, 2, \cdots, m\}.$$

If the control u does not saturate for all $i = 1, 2, \cdots, m$, that is $x \in \mathcal{L}(F)$, then nonlinear system (1) admits the following linear representation

$$\dot{x}(t) = (A(p(t)) + B(p(t))F)x(t). \qquad (8)$$

Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix. For a positive number ρ, denote

$$\Omega(P, \rho) = \{x \in \mathbb{R}^n : x^{\mathrm{T}}Px \le \rho\}.$$

An ellipsoid $\Omega(P, \rho)$ is inside $\mathcal{L}(F)$ if and only if

$$f_i(P/\rho)^{-1}f_i^{\mathrm{T}} \le 1, \quad i = 1, 2, \cdots m.$$

Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as E_i, $i = 1, 2, \cdots, 2^m$, and denote $E_i^- = I - E_i$. Clearly, E_i^- is also an element of $\mathcal{V}$ if $E_i \in \mathcal{V}$.

Lemma 1 [5] *Let $F, H \in \mathbb{R}^{m\times n}$ be given. For $x \in \mathbb{R}^n$, if $x \in \mathcal{L}(H)$, then*

$$\sigma(Fx) \in \mathrm{co}\left\{E_i Fx + E_i^- Hx : \quad i \in [1, 2^m]\right\},$$

This means that we can rewrite $\sigma(Fx)$ as

$$\sigma(Fx) = \sum_{i=1}^{2^m} \eta_i(E_i F + E_i^- H)x,$$

where $0 \le \eta_i \le 1$, $\sum_{i=1}^{2^m} \eta_i = 1$.

Lemma 2 *Suppose that matrices $M_i \in \mathbb{R}^{m\times n}, i = 1, 2, \cdots, r$, and a positive semi-definite matrix $P \in \mathbb{R}^{m\times m}$ are given. If $\sum_{i=1}^r p_i = 1$ and $0 \le p_i \le 1$, then*

$$\left(\sum_{i=1}^r p_i M_i\right)^{\mathrm{T}} P \left(\sum_{i=1}^r p_i M_i\right) \le \sum_{i=1}^r p_i M_i^{\mathrm{T}} P M_i. \quad (9)$$

3 A Set Invariance Condition

For a given LPV system with actuator saturation and a given control law $u = Fx$, we first need to establish a set invariance condition. For simplicity, we will denote

$$\hat{A}_{i,j} = A_i + B_i(E_j F + E_j^- H),$$
$$\hat{C}_{i,j} = C_i + D_i(E_j F + E_j^- H).$$

Theorem 3 *For a given system (1) and a given state feedback control matrix F, the ellipsoid $\Omega(P, \gamma)$ is an invariant set of the closed-loop system under linear state feedback control law (5) if there exists a matrix $H \in \mathbb{R}^{m\times n}$ satisfying the following inequalities*

$$\hat{A}_{i,j}^{\mathrm{T}}P + P\hat{A}_{i,j} + \hat{C}_{i,j}^{\mathrm{T}}\hat{C}_{i,j} < 0, i \in [1, r], j \in [1, 2^m] \quad (10)$$

and $\Omega(P, \gamma) \subset \mathcal{L}(H)$. Moreover, for any $x_0 \in \Omega(P, \gamma)$, the performance objective function (7) satisfies

$$J \le x_0^{\mathrm{T}}Px_0 \le \gamma.$$

Proof: Choose a Lyapunov function $V(x) = x^{\mathrm{T}}Px$. Then,

$$\dot{V} = 2x^{\mathrm{T}}P[A(p)x + B(p)\sigma(Fx)].$$

By Lemma 1, we have

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{r}\sum_{j=1}^{2^m} p_i\eta_j x^{\mathrm{T}}\left[\left(A_i + B_i(E_j F + E_j^- H)\right)^{\mathrm{T}} P\right.\\ &\quad \left.+P\left(A_i + B_i(E_j F + E_j^- H)\right)\right] x.\end{aligned}$$

On the other hand, (10) implies that

$$\hat{A}_{i,j}^{\mathrm{T}}P + P\hat{A}_{i,j} < 0, \quad \forall i \in [1, r], \ j \in [1, 2^m].$$

So, we have $\dot{V} < 0$, for $x \in \Omega(P, \gamma)\backslash\{0\}$. Thus, if $x_0^{\mathrm{T}}Px_0 \le \gamma$, then $x^{\mathrm{T}}(t)Px(t) \le \gamma$ for $t \ge 0$, *i.e.*, $\Omega(P, \gamma)$ is a positively invariant set. This also implies that system (1) is asymptotically stable at the origin with $\Omega(P, \gamma)$ contained in the domain of attraction.

To complete the proof, we note that

$$J = \int_0^\infty (z^{\mathrm{T}}z + \dot{V}(x))dt + x_0^{\mathrm{T}}Px_0 = \int_0^\infty \bar{J}(t)dt + x_0^{\mathrm{T}}Px_0,$$

where $\bar{J}(t) = z^{\mathrm{T}}(t)z(t) + \dot{V}(x)$. By Lemma 1, we can rewrite (2) as

$$z(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^m} p_i\eta_j \hat{C}_{i,j}x(t).$$

Hence, by Lemma 2,

$$\bar{J}(t) \le \sum_{i=1}^{r}\sum_{j=1}^{2^m} p_i\eta_j x^{\mathrm{T}}(\hat{A}_{i,j}^{\mathrm{T}}P + P\hat{A}_{i,j} + \hat{C}_{i,j}^{\mathrm{T}}\hat{C}_{i,j})x.$$

It is easy to see that if (10) hold, *i.e.*,

$$\hat{A}_{i,j}^{\mathrm{T}}P + P\hat{A}_{i,j} + \hat{C}_{i,j}^{\mathrm{T}}\hat{C}_{i,j} < 0,$$

then $\bar{J}(t) \le 0$, which implies $J \le x_0^{\mathrm{T}}Px_0 \le \gamma$. ∎

Remark 1 *If we don't consider the optimal performance index (7), Theorem 3 is a set invariance condition of LPV system subject to actuator saturation. For the special case of $r = 1$, Theorem 3 recovers the set invariance condition for an LTI system with actuator saturation [5]. Additionally, Theorem 3 also addressed the quadratic performance problem for linear systems subject to actuator saturation.*

By Theorem 3, we can present the following optimization problem minimizing the upper bound of performance function (7) for a given initial condition set $\mathcal{X}_0$:

$$\min_{P>0,F,H} \gamma, \quad \text{s.t.} \qquad (11)$$

$$\begin{aligned}
&a) \quad \mathcal{X}_0 \subset \Omega(P,\gamma),\\
&b) \quad \text{inequalities (10)}, \quad i \in [1,r],\ j \in [1,2^m],\\
&c) \quad |h_i x| \le 1, \quad \forall x \in \Omega(P,\gamma), \quad i = [1,m],
\end{aligned}$$

where h_i denotes the i-th row of H.

The feasibility of the above optimization problem (11) ensures the existence of a stabilizing state feedback matrix F such that the given initial condition set $\mathcal{X}_0$ is contained in the domain of attraction of the system (1)-(2), and the performance index $J \le \gamma$. On the other hand, for a given constant control matrix F designed for the systems without considering actuator saturation, (11) can also be used to determine if an initial condition set $\mathcal{X}_0$ is contained in the domain attraction of the origin when the system is subject to actuator saturation. In what follows, we will show that the optimization problem (11) can be solved as an LMI optimization problem.

For simplicity, we assume that the initial condition set $\mathcal{X}_0$ is the combination of some given points,

$$\mathcal{X}_0 := \mathrm{co}\left\{x_0^1, x_0^2, \cdots, x_0^l\right\},$$

where $x_0^i \in \mathbb{R}^n, i = 1,2,\cdots,l$, are l given points. Let $Q = (P/\gamma)^{-1}$, $Y = FQ$, $Z = HQ$. Then, Condition $a)$ is equivalent to

$$(x_0^i)^{\mathrm{T}}Px_0^i \le \gamma \Longleftrightarrow \begin{bmatrix} 1 & (x_0^i)^{\mathrm{T}} \\ x_0^i & Q \end{bmatrix} \ge 0.$$

Condition $b)$ is equivalent to

$$\begin{bmatrix} A_iQ + B_i(E_jY + E_j^- Z) + (*)^{\mathrm{T}} & * \\ C_iQ + D_i(E_jY + E_j^- Z) & -\gamma I \end{bmatrix} < 0. \quad (12)$$

Condition $c)$ is equivalent to

$$h_i\left(\frac{P}{\gamma}\right)^{-1} h_i^{\mathrm{T}} \le 1 \Longleftrightarrow \begin{bmatrix} 1 & h_iQ \\ Qh_i^{\mathrm{T}} & Q \end{bmatrix} \ge 0.$$

Also let the i-th row of Z be z_i, *i.e.*, $z_i = h_iQ$. The optimization problem (11) can then be reduced to the following one with LMI constraints,

$$\min_{Q>0,Y,Z} \gamma, \quad \text{s.t.} \qquad (13)$$

$$\begin{aligned}
&a) \quad \begin{bmatrix} 1 & (x_0^i)^{\mathrm{T}} \\ x_0^i & Q \end{bmatrix} \ge 0, \quad i \in [1,l],\\
&b) \quad \text{LMI (12)}, \quad i \in [1,r],\ j \in [1,2^m],\\
&c) \quad \begin{bmatrix} 1 & z_i \\ z_i^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \quad i \in [1,m].
\end{aligned}$$

Theorem 4 *For a given system (1), the state feedback control matrix F that minimizes the upper bound of performance function (7) can be solved by*

$$F = YQ^{-1},$$

where $(Q > 0, Y)$ is the solution of the LMI optimization problem (13).

In the optimization problem (13), the amplitude of control law (5) is not constrained, *i.e.*, there is no control amplitude constraint on the control law. In [6], the authors proved that this controller design method is less conservative than the approaches based on circle criterion and Popov criterion [4]. On the other hand, to avoid the controller gain being too large, we may constrain it to be bounded by $\mu_0 > 1$, *i.e.*, $|f_i x| \le \mu_0$, which is equivalent to the following LMI

$$\begin{bmatrix} \mu_0^2 & y_i \\ y_i^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \quad i \in [1,m],$$

where y_i denotes i-th row of Y.

If we require $Y = Z$, then we recover the design algorithm which constrains the optimal control law to be unsaturated [3]. The unsaturated control algorithm can be described as:

$$\min_{Q>0,Y} \gamma, \quad \text{s.t.} \qquad (14)$$

$$\begin{aligned}
&a) \quad \begin{bmatrix} 1 & (x_0^i)^{\mathrm{T}} \\ x_0^i & Q \end{bmatrix} \ge 0, \quad i \in [1,l],\\
&b) \quad \begin{bmatrix} (A_iQ + B_iY) + (*)^{\mathrm{T}} & * \\ C_iQ + D_iY & -\gamma I \end{bmatrix} < 0, \ i \in [1,r],\\
&c) \quad \begin{bmatrix} 1 & y_i \\ y_i^{T} & Q \end{bmatrix} \ge 0, \quad i \in [1,m].
\end{aligned}$$

Note that constraints (14) imply that $\Omega(Q^{-1},1) \subset \mathcal{L}(F)$ and hence the control $u = Fx$ will never reach saturation limits. In (13), we permit the control to be saturated and hence our algorithm will result in a larger domain of attraction. It is known that low-gain controllers that avoid saturation will often result in low levels of performance, especially for the cases where the disturbance is intermediate or small amplitude.

4 Gain-Scheduled Control Law Design

The approach to gain-scheduling involves the design of several LTI controllers for a parameterized family of time-invariant system models and the interpolation of these controller gains. If the time-varying parameter vector $p(t)$ can be measured or estimated on-line, then we may design a gain-scheduled control law (6). It is reasonable to expect that this kind of control laws can result in a larger domain of attraction and better performance. Note that F in (5) is a constant matrix, while $\tilde{F}$ in (6) is a time-varying matrix function of $p(t)$ although matrices F_j's are constant.

With control law (6), the closed-loop system (1-2) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} p_i A_i x(t) + \sum_{i=1}^{r} p_i B_i \sigma\left(\tilde{F}x(t)\right), \\ z(t) &= \sum_{i=1}^{r} p_i C_i x(t) + \sum_{i=1}^{r} p_i D_i \sigma\left(\tilde{F}x(t)\right). \end{aligned}$$

By Lemma 1, we have that for any matrix $\tilde{H}$ of the same dimensions of $\tilde{F}$ such that $x \in \mathcal{L}(\tilde{H})$,

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} p_i \sum_{s=1}^{2^m} \eta_s \left[A_i + B_i\left(E_s\tilde{F} + E_s^- \tilde{H}\right)\right] x, \\ z &= \sum_{s=1}^{2^m} \eta_s \sum_{i=1}^{r} p_i \left[C_i + D_i\left(E_s\tilde{F} + E_s^- \tilde{H}\right)\right] x, \end{aligned}$$

where $0 \le \eta_s(t) \le 1$, $\sum_{s=1}^{2^m} \eta_s(t) = 1$, for all $s \in [1, 2^m]$. If we let $\tilde{H} = \sum_{j=1}^{r} p_j H_j$, then

$$\dot{x}(t) = \sum_{s=1}^{2^m} \eta_s(t) \sum_{i=1}^{r} p_i \sum_{j=1}^{r} p_j \tilde{A}_{s,i,j} x(t), \tag{15}$$

$$z(t) = \sum_{s=1}^{2^m} \eta_s(t) \sum_{i=1}^{r} p_i \sum_{j=1}^{r} p_j \tilde{C}_{s,i,j} x(t), \tag{16}$$

where

$$\begin{aligned} \tilde{A}_{s,i,j} &:= A_i + B_i(E_s F_j + E_s^- H_j), \\ \tilde{C}_{s,i,j} &:= C_i + D_i(E_s F_j + E_s^- H_j). \end{aligned}$$

Remark 2 *It is easy to find that the closed-loop system described by (15)-(16) can be further simplified if the subsystems (A_i, B_i, C_i, D_i) possess common input matrices B and D, namely $B_i = B$, $D_i = D$ for all i. In this case, the closed-loop system (1)-(2) can be simplified as*

$$\begin{aligned} \dot{x} &= \sum_{s=1}^{2^m} \eta_s \sum_{i=1}^{r} p_i \left(A_i + B(E_s F_i + E_s^- H_i)\right) x, \\ z &= \sum_{s=1}^{2^m} \eta_s \sum_{i=1}^{r} p_i \left(C_i + D(E_s F_i + E_s^- H_i)\right) x. \end{aligned}$$

Theorem 5 *Suppose that system (1)-(2) and the local state feedback control matrices F_j, $j = 1, 2, \cdots, r$, are given. The ellipsoid $\Omega(P,\gamma)$ is an invariant set of the closed-loop system under the gain-scheduled state feedback law (6) if there exist matrices $H_j \in \mathbb{R}^{m\times n}$, $j = 1, 2, \cdots, r$, satisfying*

$$\tilde{A}_{s,i,j}^{\mathrm{T}} P + P\tilde{A}_{s,i,j} + \tilde{C}_{s,i,j}^{\mathrm{T}} \tilde{C}_{s,i,j} < 0, \tag{17}$$

$$i, j \in [1, r], \; s \in [1, 2^m], \tag{18}$$

and $\Omega(P,\gamma) \subset \bigcap_{i=1}^{r} \mathcal{L}(H_i)$. Moreover, for any $x_0 \in \Omega(P,\gamma)$, the performance objective function (7) satisfies

$$J \le x_0^{\mathrm{T}} P x_0 \le \gamma.$$

Corollary 6 *For the special case of $B_i = B$ and $D_i = D$ for all i, the ellipsoid $\Omega(P,\gamma)$ is an invariant set of the closed-loop system under the gain-scheduled state feedback control law (6), if there exist r matrices $H_i \in \mathbb{R}^{m\times n}$, satisfying*

$$\tilde{A}_{s,i,i}^{\mathrm{T}} P + P\tilde{A}_{s,i,i} + \tilde{C}_{s,i,i}^{\mathrm{T}} \tilde{C}_{s,i,i} < 0, \; \forall i, \; \forall s, \tag{19}$$

and $\Omega(P,\gamma) \subset \bigcap_{i=1}^{r} \mathcal{L}(H_i)$. Moreover, for any $x_0 \in \Omega(P,\gamma)$, the performance objective function (7) satisfies $J \le x_0^{\mathrm{T}} P x_0 \le \gamma$.

In what follows, we present a less conservative set invariance condition. Let

$$\begin{aligned} \bar{p}_l &:= \begin{cases} p_i^2 & l = i^2 \\ 2p_i p_j & l = i\cdot j \end{cases}, \text{ for } i < j = [1, r], \\ \bar{A}_{s,l} &:= \begin{cases} \tilde{A}_{s,i,i} & l = i^2 \\ (\tilde{A}_{s,i,j} + \tilde{A}_{s,j,i})/2 & l = i\cdot j \end{cases}, \text{ for } i < j, \\ \bar{C}_{s,l} &:= \begin{cases} \tilde{C}_{s,i,i} & l = i^2 \\ (\tilde{C}_{s,i,j} + \tilde{C}_{s,j,i})/2 & l = i\cdot j \end{cases}, \text{ for } i < j. \end{aligned}$$

Then, $0 \le \bar{p}_l \le 1$, $\sum_{l=1}^{r(r+1)/2} \bar{p}_l = 1$. Thus, system (15)-(16) can be rewritten as

$$\begin{aligned} \dot{x} &= \sum_{s=1}^{2^m} \eta_s \sum_{l=1}^{r(r+1)/2} \bar{p}_l \bar{A}_{s,l} x, \\ z &= \sum_{s=1}^{2^m} \eta_s \sum_{l=1}^{r(r+1)/2} \bar{p}_l \bar{C}_{s,l} x. \end{aligned}$$

Hence,

$$\dot{V}(x) = x^{\mathrm{T}} \left[\sum_{s=1}^{2^m} \eta_s \sum_{l=1}^{r(r+1)/2} \bar{p}_l \left(\bar{A}_{s,l}^{\mathrm{T}} P + P\bar{A}_{s,l}\right)\right] x,$$

$$\begin{aligned} \bar{J}(t) = x^{\mathrm{T}} &\left[\sum_{s=1}^{2^m} \eta_s \sum_{l=1}^{r(r+1)/2} \bar{p}_l \left(\bar{A}_{s,l}^{\mathrm{T}} P + P\bar{A}_{s,l}\right)\right] x \\ &+ x^{\mathrm{T}} \Big(\sum_{s=1}^{2^m} \sum_{l=1}^{r(r+1)/2} \eta_s \bar{p}_l \bar{C}_{s,l}\Big)^{\mathrm{T}} \Big(\sum_{s=1}^{2^m} \sum_{l=1}^{r(r+1)/2} \eta_s \bar{p}_l \bar{C}_{s,l}\Big) x. \end{aligned}$$

Theorem 7 *Suppose that the system (1)-(2) and the local state feedback control matrices F_j, $j \in [1,r]$, are given. The ellipsoid $\Omega(P,\gamma)$ is an invariant set of the closed-loop system under the gain-scheduled control law (6), if there exist matrices $H_j \in \mathbb{R}^{m\times n}$, satisfying*

$$\tilde{A}^{\mathrm{T}}_{s,i,i}P + P\tilde{A}_{s,i,i} + \tilde{C}^{\mathrm{T}}_{s,i,i}\tilde{C}_{s,i,i} < 0, \qquad (20)$$

$$(\tilde{A}_{s,i,j} + \tilde{A}_{s,j,i})^{\mathrm{T}}P + P(\tilde{A}_{s,i,j} + \tilde{A}_{s,j,i}) + \frac{1}{2}(\tilde{C}_{s,i,j} + \tilde{C}_{s,j,i})^{\mathrm{T}}(\tilde{C}_{s,i,j} + \tilde{C}_{s,j,i}) < 0, \qquad (21)$$

for $i \in [1,r]$, $j < i$, $s \in [1,2^m]$, and $\Omega(P,\gamma) \subset \bigcap_{i=1}^{r} \mathcal{L}(H_i)$. Moreover, for any $x_0 \in \Omega(P,\gamma)$, the performance objective function (7) satisfies $J \le \gamma$.

Remark 3 *In Comparison with Theorem 5, the number of matrix inequalities in Theorem 7 is reduced by $r(r-1)\cdot 2^{m-1}$. In Comparison with Corollary 6, another $r(r-1)\cdot 2^{m-1}$ matrix inequalities can be removed for the special case of $B_i = B$ and $D_i = D, \forall i$.*

Let $Q = (P/\gamma)^{-1}$, $Y_j = F_jQ$, $Z_j = H_jQ$. Denote the i-th row of the matrix Z_j as z_i^j. Then (20) and (21) are equivalent to the following LMIs

$$\begin{bmatrix} A_iQ + B_i(E_sY_i + E_s^-Z_i) + (*)^{\mathrm{T}} & * \\ C_iQ + D_i(E_sY_i + E_s^-Z_i) & -\gamma I \end{bmatrix} < 0; \qquad (22)$$

$$\begin{bmatrix} A_iQ + B_i(E_sY_j + E_s^-Z_j) + (*)^{\mathrm{T}} + A_jQ + B_j(E_sY_i + E_s^-Z_i) + (*)^{\mathrm{T}} & * \\ C_iQ + D_i(E_sY_j + E_s^-Z_j) + C_jQ + D_j(E_sY_i + E_s^-Z_i) & -2\gamma I \end{bmatrix} < 0, \qquad (23)$$

respectively. Then, we have the following theorem.

Theorem 8 *Suppose that system (1-2) and local control matrices F_j are given. Then gain-scheduled state feedback control law (6) minimizing the upper bound of performance function (7) can be solved by*

$$F_j = Y_jQ^{-1}, \ \forall j \in [1,r],$$

where $(Q > 0, Y_j)$ is a solution of the following LMI optimization problem

$$\min_{Q>0,Y_j,Z_j} \gamma, \quad \text{s.t.} \qquad (24)$$

a) $\begin{bmatrix} 1 & (x_0^i)^{\mathrm{T}} \\ x_0^i & Q \end{bmatrix} \ge 0, \quad \forall i \in [1,l],$

b) LMI (22), (23), $\quad \forall i \in [1,r], j < i, s \in [1,2^m]$,

c) $\begin{bmatrix} 1 & z_i^j \\ (z_i^j)^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \quad \forall i \in [1,m],\ j \in [1,r].$

5 Numerical Examples

Example 1. First, we consider a simple LPV system with the following system matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.1 \end{bmatrix}, \ C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0.1 & 1 \end{bmatrix}, \ B_1 = B_2 = D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The system output is $y = x_1$. The input is subject to saturation $u_{\max} = 1$. We are interested in designing an optimal controller such that the initial condition $x_0 = \begin{bmatrix} 4 & -1 \end{bmatrix}^{\mathrm{T}}$ is contained in the domain of attraction of the origin. The unsaturated control algorithm (14), the constant control algorithm (13) and the gain-scheduling control algorithm (24) are used to design the control law respectively, and the minimum γ obtained are 120.4165, 120.3905 and 32.6565 respectively.

Figure 1 shows the outputs and inputs of the system by the different controllers with

$$p_1 = 0.5 - 0.5\sin(0.1\pi t - 0.5\pi), \quad p_2 = 1 - p_1.$$

The dotted curves correspond to the controller computed by algorithm (14), the dotted dash curves correspond to algorithm (13) and the solid curves correspond to algorithm (24). It is obvious that the gain-scheduled controller has the shortest rising time and the smallest performance cost while the unsaturated controller has the longest rising time with the largest cost.

If setting $x_0 = 1.3 \times \begin{bmatrix} 4 & -1 \end{bmatrix}^{\mathrm{T}}$, we find the unsaturated control algorithm (14) and the constant control algorithm (13) cannot obtain a feasible solution while the gain-scheduling control algorithm (24) can still work well. This implies that the gain-scheduled controller results in a larger domain of attraction.

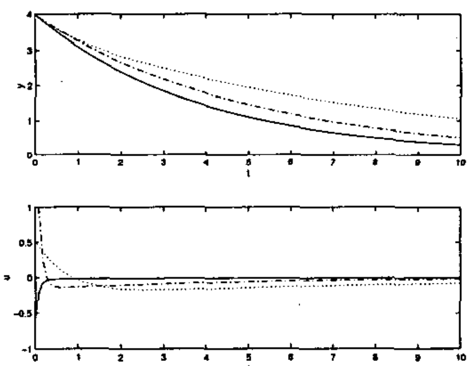

Figure 1: The outputs and inputs of Example 1 with different controllers.

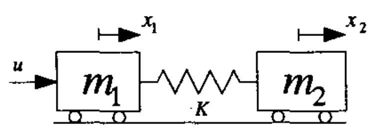

Figure 2: Coupled spring-mass systems.

Example 2. The second example is about the control of a two-mass-spring system shown in Figure 2. The system is given by the following equations [12]

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K}{m_1} & \frac{K}{m_1} & 0 & 0 \\ \frac{K}{m_2} & -\frac{K}{m_2} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u,$$
$$y = x_2.$$

Here, x_1 and x_2 are the positions of the two carts respectively, and x_3 and x_4 are their respective velocities. m_1 and m_2 are the masses of the two bodies and K is the spring constant. For the nominal system $m_1 = m_2 = 1$ with appropriate units. The spring constant is assumed to be uncertain in the range $K_{\min} = 0.5 \le K \le K_{\max} = 2$. It is assumed that exact measurement of the state is available. In the simulation, we set the performance output as

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u),$$

and we assume that the saturation limit is $u_{\max} = 0.5$ and that the time-varying parameter is given as

$$2K = K_{\min}(1 + \sin(0.01\pi t)) + K_{\max}(1 - \sin(0.01\pi t)).$$

We compare the simulation results by the different control algorithms: unsaturated control algorithm (14), the constant control algorithm (13) and the gain-scheduling control algorithm (24). The computed performance bounds γ by the above 3 algorithms are 6.0788, 6.0780 and 5.9481, respectively.

Figure 3 shows the computed outputs and inputs, where the dotted curves are obtained by using the unsaturated control algorithm (14), dotted dash curves by using the constant control algorithm (13) and the solid curves by the gain-scheduling control algorithm (24). We see that the controller computed by the unsaturated control algorithm (14) leads to the largest overshoot and the longest rising time. This is because the input is not able to reach the saturation limit due to the conservatism of algorithm (14). It is also observed that the gain-scheduling control algorithm can result in a faster response than constant control algorithm (13). From simulation results, we can conclude that gain-scheduling control algorithm (24) improve the performance with both output and input reaching the set-point in shorter time.

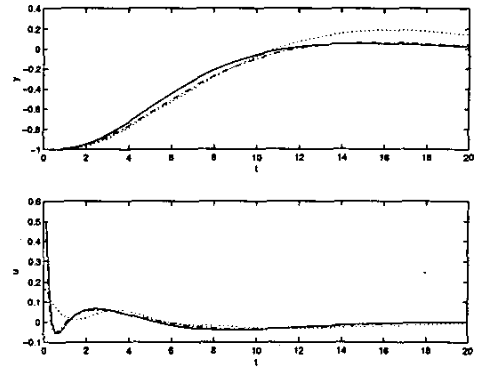

Figure 3: The outputs and inputs of the two-mass-spring system with different controllers.

6 Conclusions

In this paper, we have addressed the set invariance and the gain-scheduling control for LPV systems subject to actuator saturation. The positively invariant set of LPV systems subject to actuator saturation is analyzed by vertex system analysis approach. The optimal control problem for the given initial condition, *i.e.*, steering it to the origin with the minimal performance cost, is solved with the LMI optimization approach. A gain-scheduled controller design method is proposed to reduce the conservativeness. The numerical examples also demonstrate the effectiveness of our design.

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