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FUZZY-SLIDING STATE-FEEDBACK CONTROL OF NONLINEAR BALL SUSPENSION SYSTEM

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Abstract: This paper presents a fuzzy-sliding control, based on state feedback methods, to control nonlinear magnetic ball suspension system. Simulations results show that the control speed of fuzzy-sliding-mode controllers is more than the classical sliding mode controllers, but the steady state error of later is less that the earlier. This is due to the limited number of membership functions defined for the input variables of the system. Here, a combination of fuzzy-sliding-mode and state-feedback control is presented. Simulation results show that this controller contains the advantages of both the fuzzy-sliding-mode method (fast response) and the state-feedback scheme (low steady-state error). *Copyright © 2002 IFAC*

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1. INTRODUCTION

Although fuzzy logic controller (FLC) was proposed almost three decades ago, and were successfully applied in many applications, but the accuracy of this controller depends on number of membership functions for input variables. That is, large number of membership functions usually guarantee small steady state error. On the other hand, large number of membership functions results in enormous number of fuzzy rules, which is referred to as the curse of dimensionality. This makes the design and application of the fuzzy controllers very cumbersome. In this paper a combined control law will be presented which has two modes: fuzzy-sliding mode and state-feedback mode. This combined control law takes advantage of both the fuzzy sliding and the state feedback. That is, High speed response due to the fuzzy-sliding part and small steady state error because of the state-feedback part. This controller works as follows: when the states are far from the desired states (or from the sliding surface) the fuzzy-sliding law is used, and when the states are near them, the control law uses the state feedback law. The reminder of this paper is organized as follows. Section 2 discusses the characteristics of the nonlinear magnetic ball suspension system (MBSS). In section 3 the classical sliding-mode controller will be designed for MBSS. In section 4 the fuzzy sliding-mode controller, with two different number of membership functions, will be given and the results will be compared with the classical sliding-mode controller. The combined controller (fuzzy-sliding and state feedback) is presented in section 5. Simulation results show the superiority of the proposed method as compared to the fuzzy sliding and the classical sliding-mode controllers.

2. MAGNETIC BALL SUSPENSION SYSTEM

The magnetic ball suspension system (MBSS) is shown Fig.1. The problem considered here is to stabilize the steel ball to the desired position, with different initial conditions, by controlling the magnetic levitation. The dynamic equations of the MBSS are as follows. The magnetic force equation can written as

$$f = \frac{ci^2}{y^2}$$

where i is the winding current, c is a positive constant and y is the vertical distance from the steel ball to the edge of the core, as are shown in Fig. 1. The motion equation is

$$M\frac{d^2y}{dt^2} = Mg - f$$

where M is the mass of the ball and g is the gravitational acceleration (9.81 m/s^2). The electric circuit equation can be written as

$$L\frac{di}{dt} = -Ri + u$$

where R is the resistance of the winding, u is the volatge applied to the winding and L is the inductance of the winding, which is equal to

$$L = \frac{L_0}{y}$$

in which L_0 is a constant Defining the state variables as $x_1 = y, x_2 = \dot{y}, x_3 = i$, the state equations of MBSS can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \frac{cx_3^2}{mx_1^2} \\ \dot{x}_3 &= \frac{-R}{L_0}x_1x_3 + \frac{x_1}{L_0}u \end{aligned} \quad (1)$$

In the simulations m=1 kg, R=1.2 Ω and L_0=10 mH (Hwang, et al., 1993). In the next section, the sliding-mode control law for the MBSS will be derived.

3. CLASSICAL SLIDING-MODE CONTROL

The set of equations in (1) can written in the following vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\, u \quad (2)$$

where

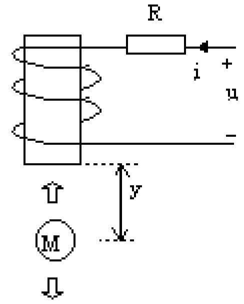

Fig. 1. Schematic diagram of MBSS.

$$\mathbf{f} = \begin{bmatrix} x_2 \\ g - \frac{c}{m}(\frac{x_3}{x_1})^2 \\ \frac{-R}{L}x_1x_3 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ \frac{x_1}{L} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (3)$$

By using a suitable transformation as

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= \left[\frac{\partial z_1}{\partial \mathbf{x}}\right]^T \mathbf{f} \\ z_3 &= \left[\frac{\partial z_2}{\partial \mathbf{x}}\right]^T \mathbf{f} \end{aligned} \quad (4)$$

the set of equations (1) becomes one dimensional

$$z_1^{(3)} = p(z_1, z_1^{(1)}, z_1^{(2)}) + q(z_1, z_1^{(1)}, z_1^{(2)})u \quad (5)$$

where

$$\begin{aligned} p &= -\left(z_1^{(2)} + g\right)\left(\frac{2z_1^{(1)}}{z_1} + \frac{2R}{L}z_1\right) \\ q &= \frac{2}{L}\sqrt{\frac{c}{m}\left(z_1^{(2)} + g\right)} \end{aligned} \quad (6)$$

Now p in Eq. (6) can be written as

$$\begin{aligned} p(z_1) &= p(z_1) + \Delta p(z_1) \\ |\Delta p(z_1)| &\le P(z_1) \end{aligned} \quad (7)$$

where $p(z_1)$ is the known part, and $\Delta p(z_1)$ is the disturbing part of $p(z_1)$, and where the control gain q (possibly time varying or state dependent) is unknown but its bounds (themselves possibly time-varying or state-dependent) are assumed to be known

$$0 < q_{\min} \le q \le q_{\max} \quad (8)$$

Since the control input enters multiplicatively in the dynamic equation of the system, it is natural to choose the estimate of gain g as the geometric mean of the above bounds

$$q = (q_{\min} q_{\max})^{\frac{1}{2}} \quad (9)$$

Therefore, bounds (8) can be written in the following form:

$$\beta^{-1} \le \frac{q}{q} \le \beta$$

where

$$\beta = \left(\frac{q_{max}}{q_{min}} \right)^{\frac{1}{2}}$$

It can be shown (Slotine, 1991) that the control law

$$u = q^{-1}[-p + z_{1d}^{(3)} - 2\lambda e^{(2)} - \lambda^2 e^{(1)} - K\,\mathrm{sgn}(s)] \qquad (10)$$

under condition

$$K \ge qq^{-1}P + \eta qq^{-1} + \left|qq^{-1} - 1\right| \cdot \left|p - z_{1d}^{(3)} + 2\lambda e^{(2)} + \lambda^2 e^{(1)}\right| \qquad (11)$$

where

$$e = z_1 - z_{1d}$$
$$s = e^{(2)} + \lambda e^{(1)} + \lambda^2 e \qquad (12)$$

satisfies the sliding condition

$$\frac{1}{2}\frac{d}{dt}s^2 \le -\eta|s| \qquad \eta > 0 \qquad (13)$$

The above control law is used to control the MBSS. The simulation results show that the steady state error is zero but the speed of response is not satisfactory. In next section the fuzzy sliding mode control will be studied .

4. FUZZY SLIDING-MODE CONTROL

Consider a single input nth order nonlinear system with the form of Eq. (2), where $\mathbf{x} = [x_1\ x_2\ ...\ x_n]^T$ is the state vector, and u is the control input, which is determined by a Fuzzy Logic Controller (FLC). The ith fuzzy IF -THEN rule, in the fuzzy rule base, has following form:

Rule i: IF x_1 is X_{i1} AND x_2 is X_{i2} AND... AND x_n is X_{in} THEN $u = u_i(\mathbf{x})$

where X_{ij} $(j = 1, \dots, n)$ is a fuzzy set defined on the jth input applied to the FLC. Also, $u_i(\mathbf{x})$ is the control output of the ith rule, which can be single valued or a function of the state variable **x**. A degree of membership $\mu_i \in [0,1]$ is obtain for each rule i . It is assumed that for any **x** in the universe of discourse X , there exists at least one μ_i among all rules that is not equal to zero. By applying the weighted sum defuzzification method, the overall output of the FLC is given by (Wang, 1997)

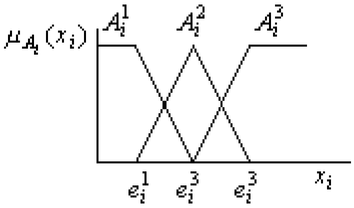

Fig. 2. Membership functions for the fuzzy sets of the ith input variable applied to the FLC.

$$u(\mathbf{x}) = \frac{\sum_{i=1}^{k} \mu_i(\mathbf{x})\ u_i(\mathbf{x})}{\sum_{i=1}^{k} \mu_i(\mathbf{x})} \qquad (14)$$

The sliding surface is defined as

$$s = \mathbf{wx} \qquad (15)$$

where **w** is a real and positive vector. A control law can be obtained by considering

$$\dot{s} = \mathbf{w}\dot{\mathbf{x}} = 0$$

Using Eq. 2, the above equation can be written as

$$\mathbf{wf}(\mathbf{x}) + \mathbf{wg}(\mathbf{x})\,u = 0$$

Hence, the sliding input to the plant is

$$u = -(\mathbf{w}\,\mathbf{g}(\mathbf{x}))^{-1}\mathbf{w}\,\mathbf{f}(\mathbf{x})$$

in order to satisfy the sliding condition (13) an extra term must be added to u (because of the uncertainty on dynamics f)

$$u_{\mathrm{fuzzy}} = u - (\mathbf{wg}(\mathbf{x}))^{-1} k_d\ \mathrm{sgn}(s) \qquad (16)$$

where

$$\mathrm{sgn}(s) = \begin{cases} 1 & if \quad s > 0 \\ 0 & if \quad s = 0 \\ -1 & if \quad s < 0 \end{cases}$$

and k_d is a constant. In order to show the effect of the number of membership functions on the performance of the FLC, first, three membership functions, and then five membership functions are defined for every input variable x_i $(i = 1, 2, 3)$. Fig. 2 shows these membership functions for x_i . Then, using Eqs. (14) and (16), the fuzzy-sliding-mode control law can be written as

$$u = \frac{\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{i_3=1}^{3} u_{\mathrm{fuzzy}}(e_1^{i_1}, e_2^{i_2}, e_3^{i_3})\,\mu_{A_1}(x_1)\,\mu_{A_2}(x_2)\,\mu_{A_3}(x_3)}{\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\sum_{i_3=1}^{3} \mu_{A_1}(x_1)\,\mu_{A_2}(x_2)\,\mu_{A_3}(x_3)}$$

The same procedure is repeated for the case of five membership functions for every input variable to the FLC. The simulation results are shown in Fig. 4. As the graphs show, the fuzzy-sliding-mode controller with five membership functions has considerably less steady state error than the three membership functions. Moreover, the fuzzy-sliding-mode controller has a faster response as compared to the classical sliding-mode controller. In order to overcome the shortcomings of the fuzzy-sliding-mode controller, it will be combined with a state-feedback controller, which is explained in the next section.

5. STATE-FEEDBACK CONTROL

If the state equations of MBSS are linearized around the operating point (i.e. the desired position of the ball), then the dynamic equations are

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,u \qquad (17)$$
$$y = \mathbf{c}^T\,\mathbf{x}$$

If the desired output is shown with y_d, the error can be designed as

$$e = y_d - y$$

Then, the goal of the state-feedback control law is to minimize the following performance index, which is in vector-matrix form (McLean, 1990):

$$J = \frac{1}{2}\mathbf{e}^T(\mathrm{T})\,\mathbf{S}\,\mathbf{e}(\mathrm{T}) + \frac{1}{2}\int_0^T \left\{\mathbf{e}^T\mathbf{Q}\,\mathbf{e} + \mathbf{u}^T\mathbf{R}\,\mathbf{u}\right\}dt$$

where **S**, **Q**, and **R** are constant matrices with appropriate dimensions. McLean (1990) has shown that the optimal command control can be obtained as

$$\mathbf{u}_{\mathrm{SFC}} = -\mathbf{R}^{-1}\,\mathbf{B}^T(\mathbf{P}\,\,\mathbf{x}(t) + v(t)) \qquad (18)$$

where **P** is a row vector and is the solution of the following Ricatti equation

$$\dot{\mathbf{P}} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{PFP} + \mathbf{V} = \mathbf{0}$$

in which

$$\mathbf{V} = \mathbf{C}^T\mathbf{Q}\,\mathbf{C} \quad \text{and} \quad \mathbf{F} = \mathbf{BR}^{-1}\mathbf{B}^T$$

and v is obtained from

$$\dot{v} = (\mathbf{PF} - \mathbf{A}^T)v - \mathbf{H}\,y_d$$

with

$$\mathbf{H} = \mathbf{C}^T\mathbf{Q}$$

In the next section, a combination of the fuzzy-sliding-mode controller and the state-feedback controller will be developed.

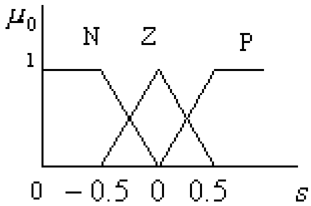

Fig. 3. Membership functions for the fuzzy set s (the sliding surface).

6. THE COMBINED CONTROL LAW

Now, a combination of the fuzzy-sliding and state-feedback (FSSF) controller will be presented. The fuzzy IF-THEN rules for the FSSF are defined as follows:

Rule 1: IF s is P, THE $u_{\mathrm{FSSF}} = u + (\mathbf{w}\,\,\mathbf{g}(\mathbf{x}))^{-1}k_d$

Rule 2: IF s is Z, THEN $u_{\mathrm{FSSF}} = u_{\mathrm{SFC}}$

Rule 3: IF s is N, THEN $u_{\mathrm{FSSF}} = u - (\mathbf{w}\,\,\mathbf{g}(\mathbf{x}))^{-1}k_d$

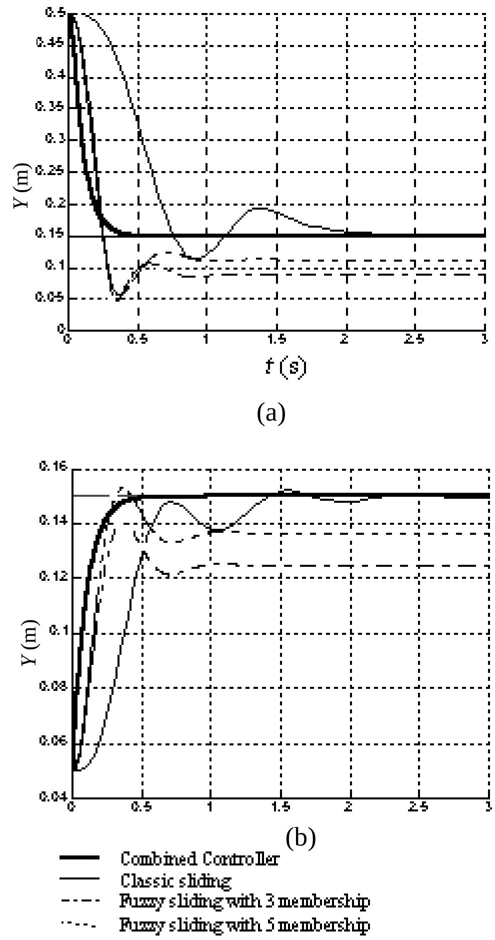

Fig. 4. Simulation results of four controllers applied to the MBSS. (a) Initial position of the ball at 0.5. (b) Initial position of the ball at 0.05. The ball is initially at rest.

where s (i.e. the sliding surface) has been defined as a fuzzy variable in the above fuzzy IF-THEN rules. The membership functions for the fuzzy sets for s ($P \equiv$ $Z \equiv$ $N \equiv$
been shown in Fig. 3. In the first and third rules, the states of the plant are far from the desired states. Hence, the fuzzy-sliding-mode controller is being used, due to its fast response. In the second rule, the states of the plant are near the desired ones. Therefore, in order to have a small steady state error, the state-feedback controller is being called. Fig. 4 shows the simulation results. It is clear that the combined controller has a very fast response. Moreover, the steady state error is negligble.

7. STABILITY ANALYSIS

The idea of stability analysis is to break down the problem of analyzing the stability into analyzing the stability analyzing of the fuzzy subsystems individually. The complexity of the analysis is drastically decreased as it is easier to check whether every fuzzy subsystem can give a negative-definite $\dot{V}$ for a given Lyapunov function V. However, the condition that all fuzzy subsystems have a negative-definite $\dot{V}$ does not directly imply that the whole fuzzy logic control system yields a negative-definite $\dot{V}$ as well. In other words, if the whole system has a negative-definite $\dot{V}$, then the system stability has been proved by Lyapunov stability theorem. The sufficient conditions that make this implication valid are stated in the following theorem (Wang, et al., 2001) and (Wang, et al., 1998).

Theorem 1: Consider a combined fuzzy logic control system as described before. If

1) there exist a positive-definite, continuously differentiable, and radially unbounded scalar function $V = \mathbf{x}^T \mathbf{P} \mathbf{x}$ where $\mathbf{P}$ is an $n \times n$ constant positive-definite matrix, and

2) every fuzzy subsystem gives a negative-definite $\dot{V}$ in the active region of the corresponding fuzzy rule, and

3) the weighted-sum defuzzification method is employed, which for any input, the output u of the FLC lies between u_p and u_q such that $u_p \le u \le u_q$, then according to the Lyapunov theorem, the equilibrium point at the origin is globally asymptotically stable.

Therefore, to guarantee the system stability, we need to find a suitable Lyapunov function V and ensure that every fuzzy subsystem gives a negative-definite $\dot{V}$ in the active region of corresponding fuzzy rule.

7.1 SMC Subsystem

Defining a Lyapunov function $V = \frac{1}{2}\mu^2$, where μ is a new sliding plan, defined as

$$\mu = \mathbf{we} = \mathbf{w}(\mathbf{x} - \mathbf{x}_d)$$

results in a new sliding control law as follow:

$$u_{\mathrm{SMC}} = [\mathbf{w} \cdot \mathbf{g}(\mathbf{x})]^{-1} (-\mathbf{w} \cdot \mathbf{f}(\mathbf{x}) - k_d \; sgn(\mu))$$

Therefore, it is straightforward to see that

$$\dot{V} = \mu\dot{\mu} = \mu \, \mathbf{w}\dot{\mathbf{x}} = \mu \big[\mathbf{w}\,\mathbf{f}(\mathbf{x}) + \mathbf{w}\,\mathbf{g}(\mathbf{x}) \big((-\mathbf{w}\,\mathbf{g}(\mathbf{x}))^{-1} \\ (\mathbf{w}\,\mathbf{f}(\mathbf{x}) + k_d \; sgn(\mu)) \big)\big] = -\mu k_d \; sgn(\mu) = -k_d \, |\mu| < 0$$

Hence, according to Lyapunov stability theorem SMC subsystem is stable.

7.2 SFC Subsystem

By linearizing system (2) around the equilibrium point $\mathbf{x}_e = \begin{bmatrix} x_{1e} & 0 & -\sqrt{\frac{mg}{c}}\, x_{1e} \end{bmatrix}$ we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{2g}{x_{1e}} & 0 & \dfrac{2}{x_{1e}}\sqrt{\dfrac{cg}{m}} \\ -\dfrac{R}{L}\sqrt{\dfrac{mg}{c}}\, x_{1e} & 0 & -\dfrac{R}{L} x_{1e} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix}$$

So, the linear system is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ and the proposed control law (18) can bring the state variables near $\mathbf{x}_e$. Now, by choosing $\mathbf{e} = \mathbf{x} - \mathbf{x}_d$ the dynamic equation of error is

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{b}u + \mathbf{A}\mathbf{x}_d \qquad (19)$$

It can be shown that if $\mathbf{A}\mathbf{x}_d = 0$, and if the state variables of the new system (19) converge to zero, then the state variables of system (17) will converge to $\mathbf{x}_d$. Kirk (1970) has shown that the following state feedback control law can drive the states of (19) to zero

$$u_{\mathrm{SFC}} = \left(-\mathbf{R}^{-1}\mathbf{b}^{\mathbf{T}}\mathbf{P}\right) \cdot \mathbf{e} \qquad (20)$$

Hence

$$\dot{V} = \mu\dot{\mu} = \mathbf{e}^T \mathbf{w}^T \mathbf{w}(\mathbf{Ae} - \mathbf{bR}^{-1}\mathbf{b}^T \mathbf{Pe}) = \\ \mathbf{e}^T (\mathbf{w}^T \mathbf{w}(\mathbf{A} - \mathbf{bR}^{-1}\mathbf{b}^T \mathbf{P}))\mathbf{e} = \mathbf{e}^T \boldsymbol{\Lambda} \mathbf{e}$$

where $\boldsymbol{\Lambda}$ is a negative-definite matrix, and for the ball suspension example in this paper, the following numerical entries apply:

$$\boldsymbol{\Lambda} = \begin{pmatrix} -53.9727 & -2.2894 & 16.9079 \\ -10.7945 & -0.4579 & 3.3816 \\ -5.3973 & -0.2289 & 1.6908 \end{pmatrix}$$

So, we have

$$\dot{V} = \mathbf{e}^T \mathbf{\Lambda}\, \mathbf{e} < 0 \qquad (21)$$

Hence SFC subsystem is stable, and according to discussed theorem the stability of proposed fuzzy-sliding and state feedback (FSSF) controller is proved.

8. CONCLUSIONS

A combination of the fuzzy-sliding-mode and the state-feedback control laws were used to control a nonlinear magnetic ball suspension system. This new control law has the advantages of both control methods. That is, the high speed response of fuzzy-sliding-mode and the small steady state error of the state-feedback approach. In this way, the number of the membership functions for the input variables to the fuzzy controller can be reduced significantly. It has been shown with simulations that the proposed fuzzy controller has a quite fast response as well as negligble steady state error. It Has also been shown, anlitically, that the proposed control method (i.e. the FSSF control law) is stable.

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