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Retrospective Cost Model Reference Adaptive Control for Nonminimum-Phase Systems

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This paper presents a direct model reference adaptive controller for single-input/single-output discrete-time (and thus sampled-data) systems that are possibly nonminimum phase. The adaptive control algorithm requires knowledge of the nonminimum-phase zeros of the transfer function from the control to the output. This controller uses a retrospective performance, which is a surrogate measure of the actual performance, and a cumulative retrospective cost function, which is minimized by a recursive-least-squares adaptation algorithm. This paper develops the retrospective cost model reference adaptive controller and analyzes its stability.

I. Introduction

THE objective of model reference adaptive control (MRAC) is to control an uncertain system so that it behaves like a given reference model in response to specified reference model commands. MRAC has been studied extensively for both continuous-time [1–8] and discrete-time systems [7–12]. In addition, MRAC has been extended to various classes of nonlinear systems [13]. However, the direct adaptive control results of [1–13], as well as related adaptive control techniques [14,15], are restricted to minimum-phase systems.

For nonminimum-phase systems, [16] shows that periodic control can be used, but this approach entails periods of open-loop operation. In [17], an adaptive controller is presented for systems with known nonminimum-phase zeros; however, this controller has only local convergence and stability properties. Another approach to addressing systems with nonminimum-phase zeros is to remove the nonminimum-phase zeros by relocating sensors and actuators or by using linear combinations of sensor measurements. However, constraints on the number and placement of sensors and actuators can make this approach infeasible. For example, a tail-controlled missile with its inertial measurement unit located behind the center of gravity is known to be nonminimum phase [18], and an aircraft's elevator-to-vertical-acceleration transfer function is often nonminimum phase [19].

Retrospective cost adaptive control (RCAC) is a discrete-time adaptive control technique for discrete-time (and thus sampled-data) systems that are possibly nonminimum phase [20–23]. RCAC uses a retrospective performance, which is the actual performance modified based on the difference between the actual past control inputs and the recomputed past control inputs. The structure of the retrospective performance is reminiscent of the augmented error signal presented in [1] and used in [2,4,10,13]; however, the construction and purpose of the retrospective performance differs from the augmented error signal. More specifically, the retrospective performance is constructed using knowledge of the system's nonminimum-phase zeros, thus accounting for their presence. In contrast, the augmented error signals in [1,2,4,10,13] are used to accommodate reference models that are not strictly positive real but do not accommodate nonminimum-phase zeros in the plant.

RCAC has been demonstrated on multi-input/multi-output nonminimum-phase systems [20,21]. Furthermore, the stability of RCAC for single-input/single-output systems is analyzed in [23] for the model reference adaptive control problem and in [22] for command following and disturbance rejection. A related controller construction is used in [24] for continuous-time minimum-phase systems that have real nonminimum-phase zeros due to sampling.

The adaptive laws in [20–23] are derived by minimizing a retrospective cost, which is a quadratic function of the retrospective performance. In particular, [20] uses an instantaneous retrospective cost, which is a function of the retrospective performance at the current time and is minimized by a gradient-type adaptation algorithm. In contrast, [21] uses a recursive-least-squares (RLS) adaptation algorithm to minimize a cumulative retrospective cost that is a function of the retrospective performance at the current time step, as well as all previous time steps.

The present paper develops a retrospective cost model reference adaptive control (RC-MRAC) algorithm for discrete-time systems that are subject to unknown disturbances and potentially nonminimum phase. The reference model is assumed to satisfy a model-matching condition, where the numerator polynomial of the reference model duplicates the nonminimum-phase zeros of the open-loop system. This condition reflects the fact that the nonminimum-phase zeros of the plant cannot be moved through feedback or pole-zero cancellation. Numerical examples show that the plant's nonminimum-phase zeros need not be known exactly.

The present paper goes beyond prior work on retrospective cost adaptive control [20,21] by analyzing the stability of the closed-loop system for plants that are nonminimum phase. In addition, the present paper extends the control architecture of [20,21] to a more general MRAC architecture with unmatched disturbances. The current paper focuses on the single-input/single-output problem for clarity in the presentation of the assumptions, as well as the main stability results. Also, unlike [20], the current paper considers an RLS adaptation algorithm, as in [21]. Preliminary versions of some results in this paper are given in [22,23].

Section II of this paper describes the adaptive control problem, while Sec. III presents the RC-MRAC algorithm. Section IV presents a nonminimal-state-space realization for use in subsequent sections. Section V proves the existence of an ideal fixed-gain controller, and a closed-loop error system is constructed in Sec. VI. Section VII presents the closed-loop stability analysis. Section VIII provides

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numerical examples, including a multiple-degree-of-freedom mass-spring-dashpot system, as well as the NASA Generic Transport Model (GTM). Finally, conclusions are given in Sec. IX.

II. Problem Formulation

Consider the discrete-time system:

$$y(k) = \sum_{i=1}^{n} -\alpha_i y(k-i) + \sum_{i=d}^{n} \beta_i u(k-i) + \sum_{i=0}^{n} \gamma_i w(k-i) \quad (1)$$

where $k \geq 0$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\beta_d, \ldots, \beta_n \in \mathbb{R}$, $\gamma_0, \ldots, \gamma_n \in \mathbb{R}^{1\times l_w}$, $y(k) \in \mathbb{R}$ is the output, $u(k) \in \mathbb{R}$ is the control, $w(k) \in \mathbb{R}^{l_w}$ is the exogenous disturbance, and the relative degree is $d > 0$. Furthermore, for all $i < 0$, $u(i) = 0$ and $w(i) = 0$, and the initial condition is $x_0 \triangleq [\, y(-1) \quad \cdots \quad y(-n)\,]^T \in \mathbb{R}^n$.

Let $\mathbf{q}$ and $\mathbf{q}^{-1}$ denote the forward-shift and backward-shift operators, respectively. For all $k \geq 0$, Eq. (1) can be expressed as

$$\alpha(\mathbf{q})y(k-n) = \beta(\mathbf{q})u(k-n) + \gamma(\mathbf{q})w(k-n) \quad (2)$$

where $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ are coprime, and

$$\alpha(\mathbf{q}) \triangleq \mathbf{q}^n + \alpha_1\mathbf{q}^{n-1} + \alpha_2\mathbf{q}^{n-2} + \cdots + \alpha_{n-1}\mathbf{q} + \alpha_n,$$

$$\beta(\mathbf{q}) \triangleq \beta_d\mathbf{q}^{n-d} + \beta_{d+1}\mathbf{q}^{n-d-1} + \beta_{d+2}\mathbf{q}^{n-d-2} + \cdots + \beta_{n-1}\mathbf{q} + \beta_n,$$

$$\gamma(\mathbf{q}) = \mathbf{q}^n\gamma_0 + \mathbf{q}^{n-1}\gamma_1 + \mathbf{q}^{n-2}\gamma_2 + \cdots + \mathbf{q}\gamma_{n-1} + \gamma_n$$

Next, consider the reference model

$$\alpha_m(\mathbf{q})y_m(k-n_m) = \beta_m(\mathbf{q})r(k-n_m) \quad (3)$$

where $k \geq 0$, $y_m(k) \in \mathbb{R}$ is the reference model output, $r(k) \in \mathbb{R}$ is the bounded reference model command, $\alpha_m(\mathbf{q})$ is a monic asymptotically stable polynomial with degree $n_m > 0$, $\beta_m(\mathbf{q})$ is a polynomial with degree $n_m - d_m \geq 0$, where $d_m > 0$ is the relative degree of Eq. (3), and $\alpha_m(\mathbf{q})$ and $\beta_m(\mathbf{q})$ are coprime. Furthermore, for all $i < 0$, $r(i) = 0$, and the initial condition of Eq. (3) is $x_{m,0} \triangleq [\, y_m(-1) \quad \cdots \quad y_m(-n_m)\,]^T \in \mathbb{R}^{n_m}$.

Next, define the performance:

$$z(k) \triangleq y(k) - y_m(k)$$

The goal is to develop an adaptive output-feedback controller that generates a control signal $u(k)$ such that $y(k)$ asymptotically follows $y_m(k)$ for all bounded reference model commands $r(k)$ in the presence of the disturbance $w(k)$. The goal is thus to drive the performance $z(k)$ to zero. The following assumptions are made regarding the open-loop system (1):

Assumption 1. d is known.

Assumption 2. β_d is known.

Assumption 3. If $\zeta \in \mathbb{C}$, $|\zeta| \geq 1$, and $\beta(\zeta) = 0$, then ζ and its multiplicity are known.

Assumption 4. There exists a known integer $\bar{n}$ such that $n \leq \bar{n}$.

The parameters $\alpha(\mathbf{q})$, $\beta(\mathbf{q})$, $\gamma(\mathbf{q})$, n, and x_0 are otherwise unknown.

Assumption 2 states that the first nonzero Markov parameter β_d from the control to the output is known. In discrete-time adaptive control for minimum-phase systems, it is common to assume that the sign of β_d is known and an upper bound on the magnitude of β_d is known [9,10,14,15], which are weaker assumptions than Assumption 2.

Assumption 3 implies that the nonminimum-phase zeros from the control to the output (i.e., the roots of $\beta(\mathbf{q})$ that lie on or outside the unit circle) are known. Assumption 3 is weaker than the classical direct adaptive control assumption that there are no nonminium-phase zeros from the control to the output [1–15].

While the analysis presented herein relies on Assumptions 2 and 3, numerical examples demonstrate that RC-MRAC is robust to errors in the model information assumed by Assumptions 2 and 3. More specifically, the numerical examples presented [20,25] suggest that Assumption 2 may be able to be weakened to the assumption that the sign of β_d is known and an upper bound on the magnitude of β_d is known. Furthermore, the current paper presents numerical examples that show that RC-MRAC is robust to errors in the nonminimum-phase zero estimates. Additional numerical examples are presented in [25].

Next, the following assumptions are made regarding the exogenous disturbance $w(k)$:

Assumption 5. The signal $w(k)$ is bounded, and, for all $k \geq 0$, $w(k)$ satisfies

$$\alpha_w(\mathbf{q})w(k) = 0$$

where $\alpha_w(\mathbf{q})$ is a nonzero monic polynomial whose roots are on the unit circle and do not coincide with the roots of $\beta(\mathbf{q})$.

Assumption 6. There exists a known integer $\bar{n}_w$ such that $n_w \triangleq \deg \alpha_w(\mathbf{q}) \leq \bar{n}_w$.

The parameters $\alpha_w(\mathbf{q})$ and n_w are otherwise unknown, and $w(k)$ is not assumed to be measured.

Finally, the following assumptions are made regarding the reference model (3):

Assumption 7. If $\zeta \in \mathbb{C}$, $|\zeta| \geq 1$, and $\beta(\zeta) = 0$, then $\beta_m(\zeta) = 0$ and the multiplicity of ζ with respect to $\beta(\mathbf{q})$ equals the multiplicity of ζ with respect to $\beta_m(\mathbf{q})$.

Assumption 8. $d_m \geq d$.

Assumption 9. $\alpha_m(\mathbf{q})$, $\beta_m(\mathbf{q})$, $y_m(k)$, and $r(k)$ are known.

Assumption 7 implies that the numerator polynomial $\beta_m(\mathbf{q})$ of the reference model duplicates the plant's nonminimum-phase zeros from the control to the output. This assumption arises from the model reference architecture and reflects the fact that the nonminimum-phase zeros of the plant cannot be moved through either feedback or pole-zero cancellation. Since the reference model duplicates the plant's nonminimum-phase zeros, its step response may exhibit initial undershoot or directions reversals depending on the number of positive nonminimum-phase zeros in the reference model. However, the reference model can contain additional zeros, which can be chosen to prevent initial undershoot. Furthermore, if $r(k) \equiv 0$, then the reference model (3) simplifies to $\alpha_m(\mathbf{q})y_m(k) = 0$, which does not explicitly depend on $\beta_m(\mathbf{q})$. In this case, the RC-MRAC controller does not depend on $\beta_m(\mathbf{q})$, and letting $\beta_m(\mathbf{q}) = \beta(\mathbf{q})$ trivially satisfies Assumption 7.

Now, consider the factorization of $\beta(\mathbf{q})$ given by

$$\beta(\mathbf{q}) = \beta_d\beta_u(\mathbf{q})\beta_s(\mathbf{q}) \quad (4)$$

where $\beta_u(\mathbf{q})$ and $\beta_s(\mathbf{q})$ are monic polynomials; if $\zeta \in \mathbb{C}$, $|\zeta| \geq 1$, and $\beta(\zeta) = 0$, then $\beta_u(\zeta) = 0$ and $\beta_s(\zeta) \neq 0$; $n_u \leq n - d$ is the degree of $\beta_u(\mathbf{q})$; and $n_s \triangleq n - n_u - d$ is the degree of $\beta_s(\mathbf{q})$. Thus, Assumption 3 is equivalent to the assumption that $\beta_u(\mathbf{q})$ is known (and thus n_u is also known). Furthermore, Assumption 7 is equivalent to the assumption that $\beta_u(\mathbf{q})$ is a factor of $\beta_m(\mathbf{q})$. Thus, $\beta_m(\mathbf{q})$ has the factorization $\beta_m(\mathbf{q}) = \beta_u(\mathbf{q})\beta_r(\mathbf{q})$, where $\beta_r(\mathbf{q})$ is a known polynomial with degree $n_m - d_m - n_u$.

III. Retrospective Performance and the Retrospective Cost Model Reference Adaptive Controller

This section defines the retrospective performance and presents the retrospective cost model reference adaptive control (RC-MRAC) algorithm. First, define

$$r_f(k) \triangleq \mathbf{q}^{-(n_m-d-n_u)}\beta_r(\mathbf{q})r(k)$$

which can be computed from the known reference model command $r(k)$ and the known polynomial $\beta_r(\mathbf{q})$. Let $n_c \geq n$, and, for all $k \geq n_c$, consider the controller

$$u(k) = \sum_{i=1}^{n_c} L_i(k)y(k-i) + \sum_{i=1}^{n_c} M_i(k)u(k-i) + N_0(k)r_f(k) \quad (5)$$

where, for all $i = 1, \ldots, n_c$, $L_i : \mathbb{N} \to \mathbb{R}$ and $M_i : \mathbb{N} \to \mathbb{R}$, and $N_0 : \mathbb{N} \to \mathbb{R}$ are given by the adaptive laws (13) and (14) presented below. The adaptive controller presented in this section may be implemented

with positive controller order $n_c < n$, but the analysis presented in Secs. IV, V, VI, and VII requires that $n_c \geq n$. For example, we require $n_c \geq n$ to prove the existence of an ideal fixed-gain controller that drives the performance to zero. For all $k \geq n_c$, the controller (5) can be expressed as

$$u(k) = \phi^T(k)\theta(k) \tag{6}$$

where

$$\theta(k) \triangleq [\,L_1(k) \quad \cdots \quad L_{n_c}(k) \quad M_1(k) \quad \cdots \quad M_{n_c}(k) \quad N_0(k)\,]^T$$

and, for all $k \geq n_c$,

$$\phi(k) \triangleq [\,y(k-1) \quad \cdots \quad y(k-n_c) \quad u(k-1) \quad \cdots \quad u(k-n_c) \quad r_f(k)\,]^T \tag{7}$$

The controller (5) cannot be implemented for nonnegative $k < n_c$ because, for nonnegative $k < n_c$, $u(k)$ depends on the initial condition x_0 of Eq. (1), which is not assumed to be known. Therefore, for all nonnegative integers $k < n_c$, let $u(k)$ be given by Eq. (6), where, for all nonnegative integers $k < n_c$, $\phi(k) \in \mathbb{R}^{2n_c+1}$ is chosen arbitrarily. The choice of $\phi(k)$ for $k < n_c$ impacts the transient performance of the closed-loop adaptive system. Numerical simulations suggest that letting $\phi(0) = 0$ and inserting new data at each time step as it becomes available tends to mitigate poor transient behavior.

Next, define $\bar{\alpha}_m(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_m}\alpha_m(\mathbf{q})$, $\bar{\beta}_m(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_m}\beta_m(\mathbf{q})$, and $\bar{\beta}_u(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_u-d}\beta_u(\mathbf{q})$. In addition, for all $k \geq 0$, define the *filtered performance*

$$z_f(k) \triangleq \bar{\alpha}_m(\mathbf{q}^{-1})z(k) \tag{8}$$

which can be interpreted as the output of a finite-impulse-response filter whose input is $z(k)$ and whose zeros replicate the reference model poles. For nonnegative $k < n_m$, $z_f(k)$ depends on $z(-1), \ldots, z(-n_m)$ [i.e., the difference between the initial conditions x_0 of Eq. (1) and the initial conditions $x_{m,0}$ of Eq. (3)], which are not assumed to be known. Therefore, for nonnegative $k < n_m$, $z_f(k)$ is given by (8), where the values used for $z(-1), \ldots, z(-n_m)$ are chosen arbitrarily. Furthermore, $z_f(k)$ is computable from the measurements $y(k)$ and $y_m(k)$, as well as the known asymptotically stable polynomial $\alpha_m(\mathbf{q})$.

Now, let $\hat{\theta} \in \mathbb{R}^{2n_c+1}$ be an optimization variable used to develop the adaptive controller update equations, and, for all $k \geq 0$, define the *retrospective performance*

$$\begin{aligned}\hat{z}(\hat{\theta}, k) &\triangleq z_f(k) + \beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\hat{\theta} - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(k)\\ &= z_f(k) + \Phi^T(k)\hat{\theta} - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(k)\end{aligned} \tag{9}$$

where the *filtered regressor* is defined by

$$\Phi(k) \triangleq \beta_d\bar{\beta}_u(\mathbf{q}^{-1})\phi(k) \tag{10}$$

where, for all $k < 0$, $\phi(k) = 0$. The retrospective performance (9) can be interpreted as a modification to the filtered performance $z_f(k)$ based on the difference between the actual past control inputs and the recomputed past control inputs assuming that the controller parameter vector $\hat{\theta}$ was used in the past. Next, for all $k \geq 0$, define the *retrospective performance measure*:

$$\begin{aligned}z_r(k) &\triangleq \hat{z}(\theta(k), k)\\ &= z_f(k) + \beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\theta(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\theta(k)]\end{aligned} \tag{11}$$

Note that $\beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\theta(k)$ and $\beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\theta(k)]$, which appear in Eq. (11), are not generally equal because $\mathbf{q}^{-1}[a(k)b(k)]$ is not generally equal to $[\mathbf{q}^{-1}a(k)]b(k)$. However, if $\theta(k)$ is constant, then

$$\beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\theta(k) = \beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\theta(k)]$$

and in this case, Eq. (11) implies that $z_r(k) = z_f(k)$, that is, the retrospective performance measure equals the filtered performance. This provides an intuitive interpretation of the RC-MRAC adaptation law, which is presented in Theorem 1 below. Specifically, the goal of RC-MRAC is to minimize $z_r(k)$ and by extension $z_f(k)$, since $z_r(k)$ can be viewed as a surrogate measure of $z_f(k)$.

To develop the RC-MRAC law, define the cumulative retrospective cost function:

$$J(\hat{\theta}, k) \triangleq \sum_{i=0}^{k}\lambda^{k-i}\hat{z}^2(\hat{\theta}, i) + \lambda^k[\hat{\theta} - \theta(0)]^T R[\hat{\theta} - \theta(0)] \tag{12}$$

where $\lambda \in (0, 1]$ and $R \in \mathbb{R}^{(2n_c+1)\times(2n_c+1)}$ is positive definite. The scalar λ is a forgetting factor, which allows more recent data to be weighted more heavily than past data. The next result along with the controller (5) provides the RC-MRAC algorithm.

Theorem 1. Let $P(0) = R^{-1}$ and $\theta(0) \in \mathbb{R}^{2n_c+1}$. Then, for each $k \geq 0$, the unique global minimizer of the cumulative retrospective cost function (12) is given by

$$\theta(k+1) = \theta(k) - \frac{P(k)\Phi(k)z_r(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)} \tag{13}$$

where

$$P(k+1) = \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^T(k)P(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)}\right] \tag{14}$$

Proof. Let $\bar{P}(0) = R^{-1}$, and, for all $k \geq 0$, define

$$\begin{aligned}\bar{P}(k+1) &\triangleq \left[\lambda^k R + \sum_{i=0}^{k}\lambda^{k-i}\Phi(i)\Phi^T(i)\right]^{-1}\\ &= [\lambda\bar{P}^{-1}(k) + \Phi(k)\Phi^T(k)]^{-1}\end{aligned}$$

Using the matrix inversion lemma ([5], Lemma 2.1) implies that

$$\begin{aligned}\bar{P}(k+1) &= \frac{1}{\lambda}\bar{P}(k) - \frac{1}{\lambda^2}\bar{P}(k)\Phi(k)\\ &\quad\times\left[1 + \frac{1}{\lambda}\Phi^T(k)\bar{P}(k)\Phi(k)\right]^{-1}\Phi^T(k)\bar{P}(k)\\ &= \frac{1}{\lambda}\left[\bar{P}(k) - \frac{\bar{P}(k)\Phi(k)\Phi^T(k)\bar{P}(k)}{\lambda + \Phi^T(k)\bar{P}(k)\Phi(k)}\right]\end{aligned} \tag{15}$$

Now, it follows from Eqs. (14) and (15) that $P(k)$ and $\bar{P}(k)$ satisfy the same difference equation. Since, in addition, $P(0) = \bar{P}(0)$, it follows that $P(k) = \bar{P}(k)$.

Next, it follows from Eq. (9) that

$$J(\hat{\theta}, k) = \hat{\theta}^T\Gamma_1(k)\hat{\theta} + \Gamma_2(k)\hat{\theta} + \Gamma_3(k)$$

where

$$\Gamma_1(k) \triangleq \bar{P}^{-1}(k+1) = P^{-1}(k+1)$$

$$\Gamma_2(k) \triangleq -2\lambda^k\theta^T(0)R + 2\sum_{i=0}^{k}\lambda^{k-i}[z_f(i) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(i)]\Phi^T(i)$$

$$\Gamma_3(k) \triangleq \lambda^k\theta^T(0)R\theta(0) + \sum_{i=0}^{k}\lambda^{k-i}[z_f(i) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(i)]^2$$

The cost function (12) has the unique global minimizer

$$\theta(k+1) \triangleq -\tfrac{1}{2}\Gamma_1^{-1}(k)\Gamma_2^T(k)$$

which implies that

$$
\begin{aligned}
&\theta(k+1)\\
&\quad = P(k+1)\left[\lambda^k R\theta(0) - \sum_{i=0}^{k}\lambda^{k-i}[z_f(i) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(i)]\Phi(i)\right]\\
&\quad = P(k+1)\left[-\frac{\lambda}{2}\Gamma_2^T(k-1) - [z_f(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(k)]\Phi(k)\right]\\
&\quad = P(k+1)[\lambda P^{-1}(k)\theta(k) - [z_f(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})u(k)]\Phi(k)]
\end{aligned}
$$

Adding and subtracting $\Phi(k)\Phi^T(k)\theta(k)$ to the right-hand side and using Eq. (11) yields

$$
\begin{aligned}
\theta(k+1) &= P(k+1)[\lambda P^{-1}(k)\theta(k) + \Phi(k)\Phi^T(k)\theta(k) - \Phi(k)z_r(k)]\\
&= P(k+1)[P^{-1}(k+1)\theta(k) - \Phi(k)z_r(k)]\\
&= \theta(k) - P(k+1)\Phi(k)z_r(k) \qquad (16)
\end{aligned}
$$

Finally, it follows from Eq. (14) that

$$
\begin{aligned}
P(k+1)\Phi(k) &= \frac{1}{\lambda}\left[\frac{\lambda + \Phi^T(k)P(k)\Phi(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)}P(k)\Phi(k)\right.\\
&\quad \left. - \frac{P(k)\Phi(k)\Phi^T(k)P(k)\Phi(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)}\right]\\
&= \frac{P(k)\Phi(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)} \qquad (17)
\end{aligned}
$$

and combining Eq. (16) with Eq. (17) yields Eq. (13). □

Therefore, the RC-MRAC algorithm is given by Eqs. (6), (13), and (14), where $\phi(k)$, $\Phi(k)$, and $z_r(k)$ are given by Eqs. (7), (10), and (11), respectively. The RC-MRAC architecture is shown in Fig. 1. RC-MRAC uses the RLS-based adaptive laws (13) and (14), where $P(k)$ is the RLS covariance matrix. The initial condition $P(0) = R^{-1}$ of the covariance matrix impacts the transient performance and convergence speed of the adaptive controller, and is the primary tuning parameter for the adaptive controller. For example, increasing the singular values of $P(0)$ tends to increase the speed of convergence; however, convergence behavior is affected by other factors, such as the initial condition $\theta(0)$ and the persistency of excitation in $\Phi(k)$.

The remainder of this paper is devoted to analyzing the stability properties of the closed-loop adaptive system and providing numerical examples.

IV. Nonminimal-State-Space Realization

A nonminimal-state-space realization of the time-series model (1) is used to analyze the stability of the closed-loop adaptive system. The state $\phi(k)$ of this nonminimal-state-space realization consists entirely of measured information, specifically, past values of y and u, as well as the current value of r_f. To construct this realization, define

$$
\mathcal{N}_p \triangleq \begin{bmatrix} 0 & \cdots & 0 & 0\\ 1 & & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p}, \qquad E_p \triangleq \begin{bmatrix} 1\\ 0_{(p-1)\times 1} \end{bmatrix} \in \mathbb{R}^p
$$

where p is a positive integer. Next, for all $k \geq n_c$, consider the $(2n_c+1)$th-order nonminimal-state-space realization of Eq. (1) given by

$$
\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}_1\psi_w(k) + \mathcal{D}_r r_f(k+1) \qquad (18)
$$

$$
y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2\psi_w(k) \qquad (19)
$$

where

$$
\mathcal{A} \triangleq \mathcal{A}_{\text{nil}} + E_{2n_c+1}\mathcal{C} \in \mathbb{R}^{(2n_c+1)\times(2n_c+1)} \qquad (20)
$$

$$
\mathcal{A}_{\text{nil}} \triangleq \begin{bmatrix} \mathcal{N}_{n_c} & 0_{n_c\times n_c} & 0_{n_c\times 1}\\ 0_{n_c\times n_c} & \mathcal{N}_{n_c} & 0_{n_c\times 1}\\ 0_{1\times n_c} & 0_{1\times n_c} & 0 \end{bmatrix} \in \mathbb{R}^{(2n_c+1)\times(2n_c+1)} \qquad (21)
$$

$$
\mathcal{B} \triangleq \begin{bmatrix} 0_{n_c\times 1}\\ E_{n_c}\\ 0 \end{bmatrix} \in \mathbb{R}^{(2n_c+1)\times 1} \qquad (22)
$$

$$
\begin{aligned}
\mathcal{C} \triangleq [\, &-\alpha_1 \quad \cdots \quad -\alpha_n \quad 0_{1\times(n_c-n)}\\
&0_{1\times(d-1)} \quad \beta_d \quad \cdots \quad \beta_n \quad 0_{1\times(n_c-n+1)}] \in \mathbb{R}^{1\times(2n_c+1)}
\end{aligned} \qquad (23)
$$

$$
\begin{aligned}
\mathcal{D}_1 &\triangleq E_{2n_C+1}\mathcal{D}_2 \in \mathbb{R}^{(2n_c+1)\times l_w(n+1)}\\
\mathcal{D}_2 &\triangleq [\,\gamma_0 \quad \cdots \quad \gamma_n\,] \in \mathbb{R}^{1\times l_w(n+1)}\\
D_r &\triangleq [\,0_{1\times 2n_c} \quad 1\,]^T \in \mathbb{R}^{(2n_c+1)\times 1}
\end{aligned} \qquad (24)
$$

and $\psi_w(k) \triangleq [\,w^T(k) \quad \cdots \quad w^T(k-n)\,]^T \in \mathbb{R}^{l_w(n+1)}$.

The triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is stabilizable and detectable but is neither controllable nor observable. In particular, $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has n controllable and observable eigenvalues, while $(\mathcal{A}, \mathcal{B})$ has $n_c - n + 1$ uncontrollable eigenvalues at 0, and $(\mathcal{A}, \mathcal{C})$ has $2n_c - n + 1$ unobservable eigenvalues at 0.

V. Ideal Fixed-Gain Controller

This section proves the existence of an ideal fixed-gain controller for the open-loop system (1). This controller, whose structure is illustrated in Fig. 2, is used in the next section to construct an error system for analyzing the closed-loop adaptive system. An ideal fixed-gain controller consists of four parts, specifically, a feedforward controller whose input is r_f; a precompensator that cancels the stable zeros of the open-loop system (i.e., the roots of $\beta_s(\mathbf{q})$); an internal model of the exogenous disturbance dynamics $\alpha_w(\mathbf{q})$; and a feedback controller that stabilizes the closed loop.

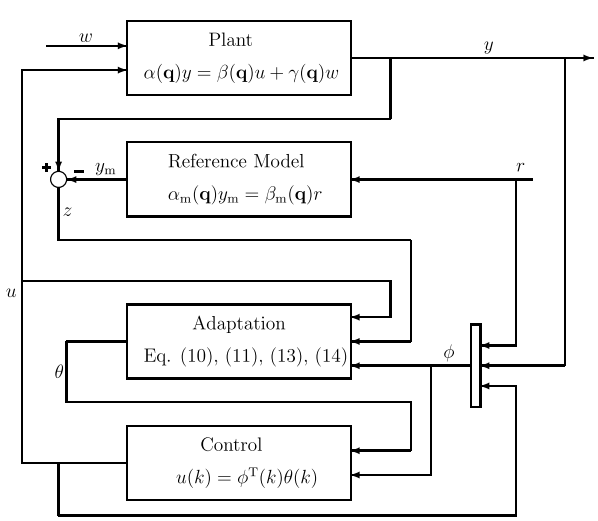

Fig. 1 Schematic diagram of the RC-MRAC architecture given by Eqs. (6), (13), and (14).

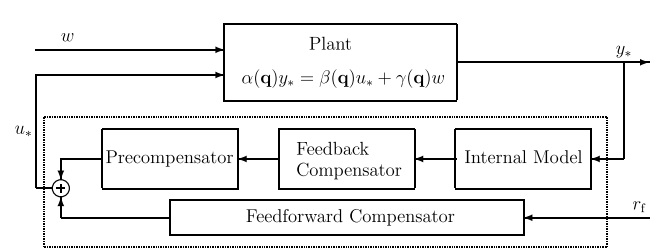

Fig. 2 Schematic diagram of the closed-loop system with the ideal fixed-gain controller.

For more information on internal model control in discrete time, see [26].

For all $k \geq n_c$, consider the system (1) with $u(k) = u_*(k)$, where $u_*(k)$ is the ideal control. More precisely, for all $k \geq n_c$, consider the system

$$y_*(k) = -\sum_{i=1}^{n} \alpha_i y_*(k-i) + \sum_{i=d}^{n} \beta_i u_*(k-i) + \sum_{i=0}^{n} \gamma_i w(k-i) \tag{25}$$

where, for all $k \geq n_c$, $u_*(k)$ is given by the strictly proper ideal fixed-gain controller:

$$u_*(k) = \sum_{i=1}^{n_c} L_{*,i} y_*(k-i) + \sum_{i=1}^{n_c} M_{*,i} u_*(k-i) + N_* r_f(k) \tag{26}$$

where $L_{*,1}, \ldots, L_{*,n_c} \in \mathbb{R}$, $M_{*,1}, \ldots, M_{*,n_c} \in \mathbb{R}$, $N_* \in \mathbb{R}$, and the initial condition at $k = n_c$ for Eqs. (25) and (26) is

$$\phi_{*,0} \triangleq [y_*(n_c-1) \quad \cdots \quad y_*(0) \quad u_*(n_c-1) \quad \cdots \quad u_*(0) \quad r_f(n_c)]^T$$

For all $k \geq n_c$, the ideal control $u_*(k)$ can be written as

$$u_*(k) = \phi_*^T(k)\theta_* \tag{27}$$

where

$$\theta_* \triangleq [L_{*,1} \quad \cdots \quad L_{*,n_c} \quad M_{*,1} \quad \cdots \quad M_{*,n_c} \quad N_*]^T$$

and

$$\phi_*(k) \triangleq [y_*(k-1) \quad \cdots \quad y_*(k-n_c) \quad u_*(k-1) \quad \cdots \quad u_*(k-n_c) \quad r_f(k)]^T$$

Therefore, it follows from Eqs. (18–24) and (27) that, for all $k \geq n_c$, the ideal closed-loop system (25) and (26), has the $(2n_c+1)$th order nonminimal-state-space realization

$$\phi_*(k+1) = \mathcal{A}_*\phi_*(k) + \mathcal{D}_1\psi_w(k) + \mathcal{D}_r r_f(k+1) \tag{28}$$

$$y_*(k) = \mathcal{C}\phi_*(k) + \mathcal{D}_2\psi_w(k) \tag{29}$$

where

$$\mathcal{A}_* \triangleq \mathcal{A} + \mathcal{B}\theta_*^T = \mathcal{A}_{\text{nil}} + \begin{bmatrix} E_{n_c}\mathcal{C} \\ E_{n_c}\theta_*^T \\ 0 \end{bmatrix} \tag{30}$$

and the initial condition is $\phi_*(n_c) = \phi_{*,0}$.

The following result guarantees the existence of an ideal fixed-gain controller of the form in Eq. (26) with certain properties that are needed for the subsequent stability analysis.

Theorem 2. Let n_c satisfy

$$n_c \geq \max(n + 2n_w, n_m - n_u - d) \tag{31}$$

Then there exists an ideal fixed-gain controller (26) of order n_c such that the following statements hold for the ideal closed-loop system consisting of Eqs. (25) and (26), which has the $(2n_c+1)$th-order nonminimal-state-space realization (28–30):

1) For all initial conditions $\phi_{*,0}$ and for all $k \geq k_0 \triangleq 2n_c + n_u + d$,

$$\bar{\alpha}_m(\mathbf{q}^{-1})y_*(k) = \bar{\beta}_m(\mathbf{q}^{-1})r(k) \tag{32}$$

and thus,

$$\bar{\alpha}_m(\mathbf{q}^{-1})y_*(k) = \bar{\alpha}_m(\mathbf{q}^{-1})y_m(k) \tag{33}$$

2) $\mathcal{A}_*$ is asymptotically stable.

3) For all initial conditions $\phi_{*,0}$, $u_*(k)$ is bounded.

4) For all $k \geq k_0$ and all sequences $e(k)$,

$$\beta_d\bar{\beta}_u(\mathbf{q}^{-1})e(k) = \bar{\alpha}_m(\mathbf{q}^{-1})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right] \tag{34}$$

The proof of Theorem 2 is in Appendix A. The lower bound on the controller order, given by Eq. (31), is a sufficient condition to guarantee the existence of an ideal fixed-gain controller. If there is no disturbance (i.e., $n_w = 0$) and the reference model is selected such that its order satisfies $n_m \leq n + n_u + d$, then Eq. (31) is satisfied by a controller order greater than or equal to the order n of the plant.

Property 4 of Theorem 2 is a time-domain property that has the $\mathbf{z}$-domain interpretation

$$\mathcal{C}(\mathbf{z}I - \mathcal{A}_*)^{-1}\mathcal{B} = \frac{\beta_d\beta_u(\mathbf{z})\mathbf{z}^{n_m-n_u-d}}{\alpha_m(\mathbf{z})} \tag{35}$$

which implies that the nonminimum-phase zeros of the closed-loop transfer function (35) are exactly the nonminimum-phase zeros of the open-loop system, that is, the roots of $\beta_u(\mathbf{q})$. Furthermore, Eq. (35) is the closed-loop transfer function from a control input perturbation e (that is, the amount that the actual control signal differs from the control signal generated by the ideal controller) to the performance z. In the subsequent sections of this paper, Eq. (34) is used to relate $z_f(k)$ and $z_r(k)$ to the controller-parameter-estimation error $\theta(k) - \theta_*$.

VI. Error System

Now, an error system is constructed using the ideal fixed-gain controller (which is not implemented) and the adaptive controller presented in Sec. III. Since n and n_w are unknown, the lower bound for the controller order n_c given by Eq. (31) is unknown. Thus, for the remainder of this paper, let n_c satisfy the lower bound

$$n_c \geq \max(\bar{n} + 2\bar{n}_w, n_m - n_u - d) \tag{36}$$

where Assumptions 1, 3, 4, 6, and 9 imply that the lower bound on n_c given by Eq. (36) is known. Furthermore, since, by Assumptions 4 and 6, $n \leq \bar{n}$ and $n_w \leq \bar{n}_w$, it follows that Eq. (36) implies Eq. (31).

Next, let $\theta_* \in \mathbb{R}^{2n_c+1}$ denote the ideal fixed-gain controller given by Theorem 2, and, for all $k \geq n_c$, let $\phi_*(k)$ denote the state of the ideal closed-loop system (28) and (29), where the initial condition is $\phi_{*,0} = \phi_*(n_c) = \phi(n_c)$. Furthermore, define $k_0 \triangleq 2n_c + n_u + d$. For all $k \geq n_c$, the closed-loop system consisting of Eqs. (6), (18), and (19) becomes

$$\phi(k+1) = \mathcal{A}_*\phi(k) + \mathcal{B}\phi^T(k)\tilde{\theta}(k) + \mathcal{D}_1\psi_w(k) + \mathcal{D}_r r_f(k+1) \tag{37}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2\psi_w(k) \tag{38}$$

where $\tilde{\theta}(k) \triangleq \theta(k) - \theta_*$ and $\mathcal{A}_*$ is given by Eq. (30)

Now, construct an error system by combining the ideal closed-loop system (28) and (29) with the adaptive closed-loop system (37) and (38). For all $k \geq n_c$, define the error state

$$\tilde{\phi}(k) \triangleq \phi(k) - \phi_*(k)$$

and subtract Eqs. (28) and (29) from Eqs. (37) and (38) to obtain, for all $k \geq n_c$,

$$\tilde{\phi}(k+1) = \mathcal{A}_*\tilde{\phi}(k) + \mathcal{B}\phi^T(k)\tilde{\theta}(k) \tag{39}$$

$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k) \tag{40}$$

where

$$\tilde{y}(k) \triangleq y(k) - y_*(k)$$

The following result relates $z_f(k)$ to $\tilde{\theta}(k)$.

Lemma 1. Consider the open-loop system (1) with the feedback (6). Then, for all initial conditions x_0, all sequences $\theta(k)$, and all $k \geq k_0$,

$$z_f(k) = \beta_d \bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\tilde{\theta}(k)] \tag{41}$$

Proof. For all $k \geq n_c$, the error system (39) and (40) has the solution

$$\tilde{y}(k) = \mathcal{C}\mathcal{A}_*^{k-n_c}\tilde{\phi}(n_c) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^T(k-i)\tilde{\theta}(k-i)$$

Since $\phi_*(n_c) = \phi(n_c)$ it follows that $\tilde{\phi}(n_c) = 0$, and thus, for all $k \geq n_c$,

$$\tilde{y}(k) = \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^T(k-i)\tilde{\theta}(k-i)$$

which implies that, for all $k \geq n_c + n_m$,

$$\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \bar{\alpha}_m(\mathbf{q}^{-1})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^T(k-i)\tilde{\theta}(k-i)\right]$$

Next, it follows from property 4 of Theorem 2 with $e(k) = \phi^T(k)\tilde{\theta}(k)$ that, for all $k \geq k_0$,

$$\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \beta_d \bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\tilde{\theta}(k)]$$

Finally, note that

$$\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \bar{\alpha}_m(\mathbf{q}^{-1})y(k) - \bar{\alpha}_m(\mathbf{q}^{-1})y_*(k)$$

and it follows from statement 1 of Theorem 2 that $\bar{\alpha}_m(\mathbf{q}^{-1})y_*(k) = \bar{\alpha}_m(\mathbf{q}^{-1})y_m(k)$. Therefore, for all $k \geq k_0$, $z_f(k) = \bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k)$, thus verifying Eq. (41). □

Lemma 1 relates $z_f(k)$ to $\tilde{\theta}(k)$. Although Eq. (41) is not a linear regression in $\tilde{\theta}(k)$, the following result expresses the retrospective performance measure $z_r(k)$ as a linear regression in $\tilde{\theta}(k)$.

Lemma 2. Consider the open-loop system (1) with the feedback (6). Then, for all initial conditions x_0, all sequences $\theta(k)$, and all $k \geq k_0$,

$$z_r(k) = \Phi^T(k)\tilde{\theta}(k) \tag{42}$$

Proof. Adding and subtracting $\beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\theta_*$ to the right-hand side of Eq. (11) yields, for all $k \geq 0$,

$$z_r(k) = z_f(k) - \beta_d \bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\tilde{\theta}(k)] + \beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\tilde{\theta}(k)$$

Next, it follows from Lemma 1 that, for all $k \geq k_0$, $z_f(k) - \beta_d \bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\tilde{\theta}(k)] = 0$, which implies that, for all $k \geq k_0$,

$$z_r(k) = \beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^T\tilde{\theta}(k) = \Phi^T(k)\tilde{\theta}(k)$$

thus verifying Eq. (42). □

VII. Stability Analysis

This section analyzes the stability of the RC-MRAC algorithm (6), (13), and (14), as well as the stability of the closed-loop system. The following lemma provides the stability properties of RC-MRAC. The proof is in Appendix B.

Lemma 3. Consider the open-loop system (1) satisfying Assumptions 1–9, and the cumulative retrospective cost model reference adaptive controller (6), (13), and (14), where n_c satisfies Eq. (36). Furthermore, define

$$\eta(k) \triangleq \frac{1}{1 + \Phi^T(k)P(0)\Phi(k)} \tag{43}$$

Then, for all initial conditions x_0 and $\theta(0)$, the following properties hold:

1) $\theta(k)$ is bounded.

2) $\lim_{k\to\infty} \sum_{j=0}^{k} \eta(j)z_r^2(j)$ exists.

3) For all positive integers N,

$$\lim_{k\to\infty} \sum_{j=N}^{k} \|\theta(j) - \theta(j-N)\|^2$$

exists.

4) If $\lambda = 1$, then $P(k)$ is bounded.

Notice that property 4 of Lemma 3 applies only if the forgetting factor $\lambda = 1$. If $\lambda < 1$ and the regressor $\Phi(k)$ is not sufficiently rich, then $P(k)$ can grow without bound ([5], pp. 473–480; [10], pp. 224–228). In practice, this effect can be mitigated by periodically resetting the covariance matrix $P(k)$ or by adopting the techniques discussed in [5], pp. 473–480, and [10], pp. 224–228.

Next, let $\xi_1, \ldots, \xi_{n_u} \in \mathbb{C}$ denote the n_u roots of $\beta_u(\mathbf{z})$, and define

$$M(\mathbf{z}, k) \triangleq \mathbf{z}^{n_c} - M_1(k)\mathbf{z}^{n_c-1} - \cdots - M_{n_c-1}(k)\mathbf{z} - M_{n_c}(k)$$

which can be interpreted as the denominator polynomial of the controller (6) at each time k. Before presenting the main result of the paper, the following additional assumption is made:

Assumption 10. There exist $\epsilon > 0$ and $k_1 > 0$ such that, for all $k \geq k_1$ and for all $i = 1, \ldots, n_u$, $|M(\xi_i, k)| \geq \epsilon$.

Assumption 10 asymptotically bounds the instantaneous controller poles (i.e., the roots of $M(\mathbf{z}, k)$) away from the nonminimum-phase zeros of Eq. (1). Thus, Assumption 10 implies that unstable pole-zero cancellation between the plant zeros and the controller poles does not occur asymptotically in time.

The following theorem is the main result of the paper. The proof is in Appendix C.

Theorem 3. Consider the open-loop system (1) satisfying Assumptions 1–10, and the cumulative retrospective cost model reference adaptive controller (6), (13), and (14), where n_c satisfies Eq. (36). Then, for all initial conditions x_0 and $\theta(0)$, $\theta(k)$ is bounded, $u(k)$ is bounded, and $\lim_{k\to\infty} z(k) = 0$.

Theorem 3 invokes the assumption that there exist $\epsilon > 0$ and $k_1 > 0$ such that, for all $k \geq k_1$ and for all $i = 1, \ldots, n_u$, $|M(\xi_i, k)| \geq \epsilon$. This assumption cannot be verified a priori. However, the assumption $|M(\xi_i, k)| \geq \epsilon$ for some arbitrarily small $\epsilon > 0$ can be verified at each time step since $M(\xi_i, k)$ can be computed from known values (i.e., the roots of $\beta_u(\mathbf{z})$ and the controller parameter $\theta(k)$). In fact, if, for some arbitrarily small $\epsilon > 0$, the condition $|M(\xi_i, k)| \geq \epsilon$ is violated at a particular time step, then the controller parameter $\theta(k)$ can be perturbed to ensure $|M(\xi_i, k)| \geq \epsilon$. For example, $\theta(k)$ can be orthogonally projected a distance ϵ away from the hyperplane in θ space defined by the equation $M(\xi_i, k) = 0$; however, determining the direction and analyzing the stability properties of this projection is an open problem. Techniques developed to prevent pole-zero cancellation for indirect adaptive control [27] may have application to this problem. Nevertheless, numerical examples suggest that asymptotic unstable pole-zero cancellation does not occur [20,21,25].

VIII. Numerical Examples

This section presents numerical examples to demonstrate RC-MRAC. In all simulations, the adaptive controller is initialized to zero (i.e., $\theta(0) = 0$) and $\lambda = 1$. For all examples, the objective is to minimize the performance $z = y - y_m$. Unless otherwise stated, the examples rely on the plant-parameter information assumed by 1–4. No additional knowledge of the plant parameters is assumed, and no known uncertainty sets are used.

Example 1. Lyapunov-stable, nonminimum-phase system without disturbance. Consider the Lyapunov-stable-but-not-asymptotically-stable, nonminimum-phase system

$$(\mathbf{q} - 0.7)^3(\mathbf{q}^2 + 1)y(k) = 0.25(\mathbf{q} - 1.3) \times (\mathbf{q} - 1 - \jmath)(\mathbf{q} - 1 + \jmath)u(k)$$

where $y(0) = -1$. For this example, it follows that $n = 5$, $n_u = 3$, $d = 2$, $\beta_d = 0.25$, and $\beta_u(\mathbf{q}) = (\mathbf{q} - 1.3)(\mathbf{q} - 1 - \jmath)(\mathbf{q} - 1 + \jmath)$. Next, consider the reference model (3), where

$$\alpha_m(\mathbf{q}) = (\mathbf{q} - 0.5)^5, \qquad \beta_m(\mathbf{q}) = \frac{\alpha_m(1)}{\beta_u(1)}\beta_u(\mathbf{q})$$

Note that the leading coefficient of $\beta_m(\mathbf{q})$ is chosen such that the reference model has unity gain at $\mathbf{z} = 1$. Finally, let $r(k)$ be a sequence of doublets with a period of 100 samples and an amplitude of 10.

A controller order $n_c \geq 5$ is required to satisfy Eq. (36). Let $n_c = 10$. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $P(0) = I_{2n_c+1}$. The closed-loop system is simulated for 500 time steps, and Fig. 3 shows the time history of y, y_m, z, and u. The closed-loop adaptive system experiences transient responses for approximately half of a period of the reference model doublet. Then RC-MRAC drives the performance $z = y - y_m$ to zero, and thus y follows y_m.

Next, the controller order n_c is increased to explore the sensitivity of the closed-loop performance to the value of n_c. For $n_c = 10, 20, \ldots, 100$, the closed-loop system is simulated, where all

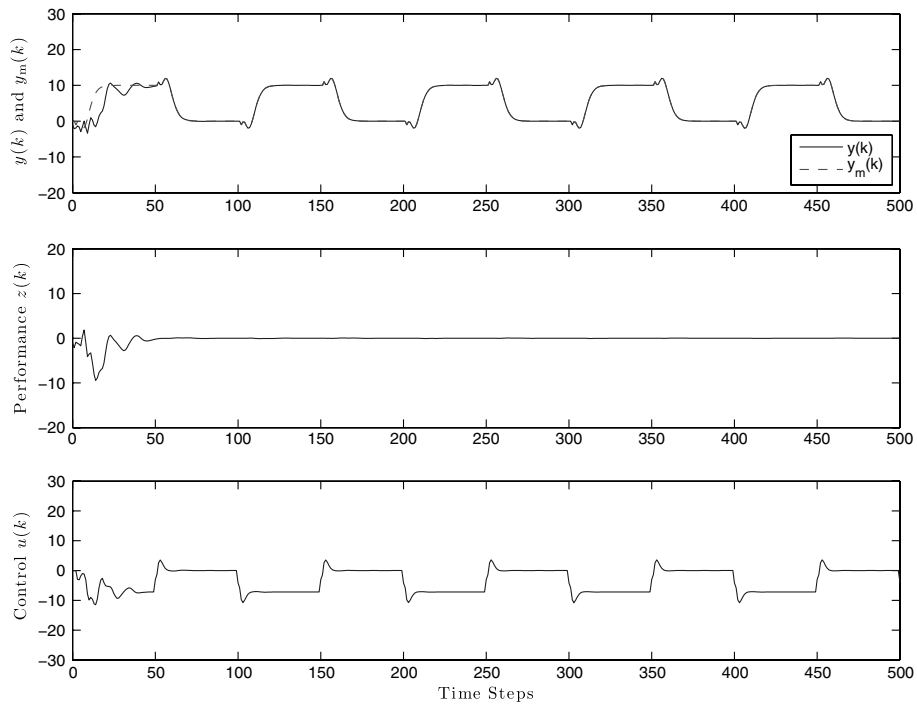

Fig. 3 Lyapunov-stable, nonminimum-phase plant without disturbance. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 10$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The adaptive controller drives z to zero.

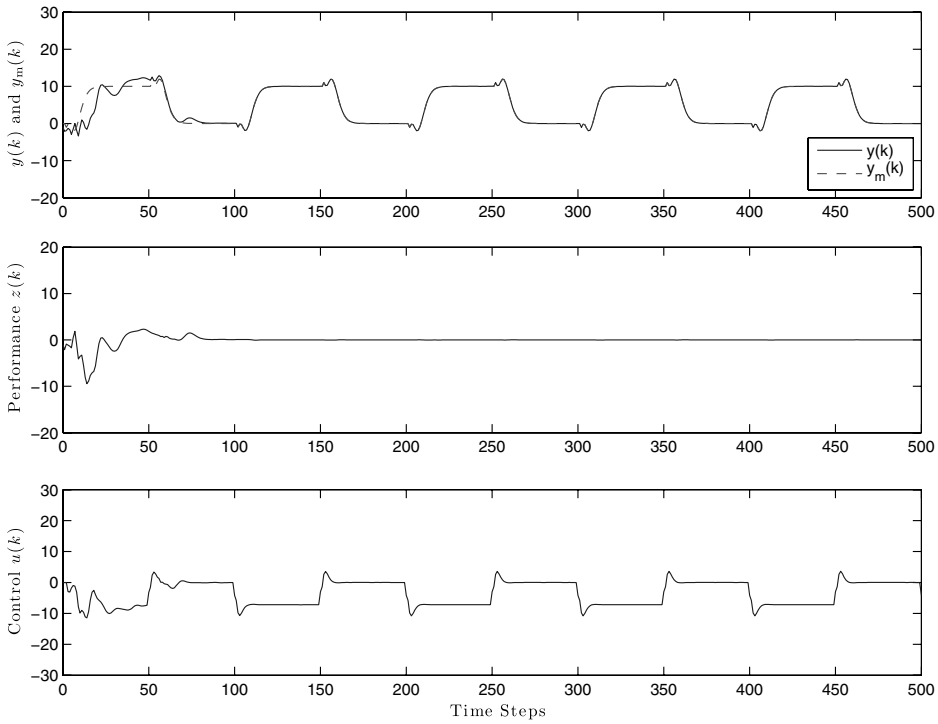

Fig. 4 Lyapunov-stable, nonminimum-phase plant without disturbance. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 40$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The closed-loop performance is comparable to that shown in Fig. 3.

parameters other than n_c are the same as above. The closed-loop performance in this example is insensitive to the choice of n_c provided that $n_c \geq 5$, which is required to satisfy Eq. (36). For this example, the worst performance is obtained by letting $n_c = 40$. Figure 4 shows the time history of y, y_m, z, and u with $n_c = 40$. Over the interval of approximately $k = 30$ to $k = 80$, the closed-loop performance shown in Fig. 4 is slightly worse than the closed-loop performance shown in Fig. 3; however, the closed-loop performances are comparable over the rest of the time history.

Example 2. Lyapunov-stable, nonminimum-phase system with disturbance. Reconsider the Lyapunov-stable, nonminimum-phase system from Example 1 with an unknown external disturbance. More specifically, consider

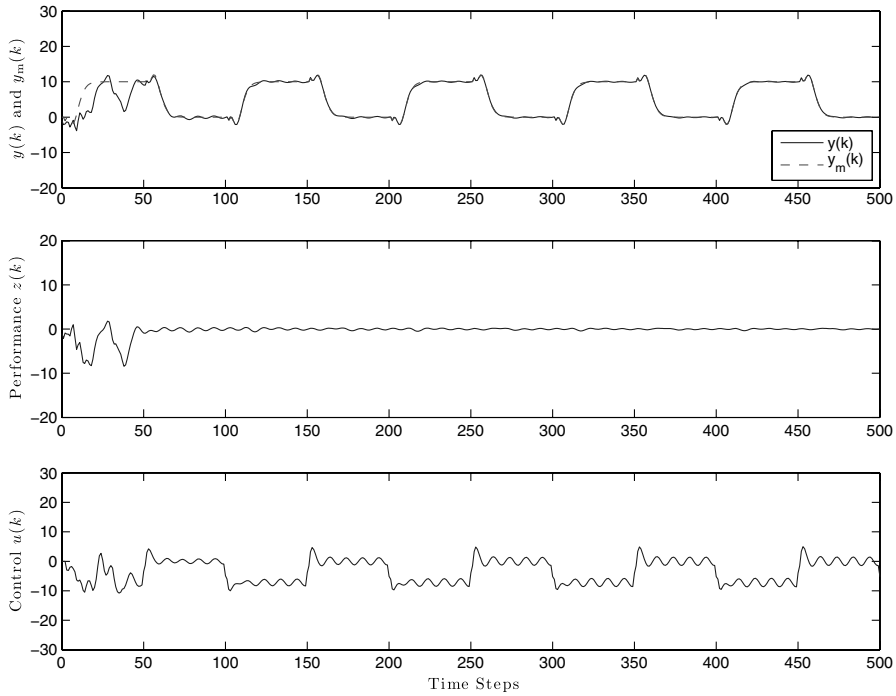

Fig. 5 Lyapunov-stable, nonminimum-phase plant with disturbance. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 10$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The adaptive controller drives z to zero. Thus, y follows y_m while rejecting w.

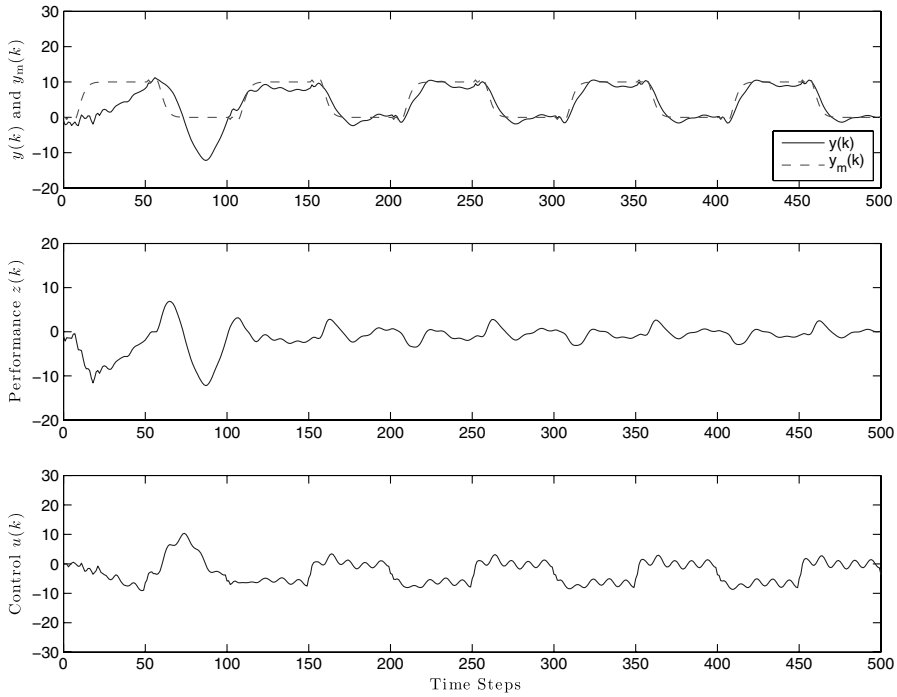

Fig. 6 Lyapunov-stable, nonminimum-phase plant with disturbance and 10% error in the estimates of the nonminimum-phase zeros. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 10$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The adaptive controller yields over 70% improvement in the performance z relative to the open-loop performance.

$$(\mathbf{q}-0.7)^3(\mathbf{q}^2+1)y(k) = 0.25(\mathbf{q}-1.3)(\mathbf{q}-1-\jmath)$$
$$\times(\mathbf{q}-1+\jmath)u(k) + (\mathbf{q}-2)(\mathbf{q}-0.9)w(k)$$

where the external disturbance is $w(k) = 0.3\sin(0.2\pi k)$. Notice that the disturbance-to-performance transfer function is not matched with the control-to-performance transfer function. Thus, the disturbance must be rejected through the system dynamics. Furthermore, note that no information about the disturbance is available to the adaptive controller, that is, the amplitude, frequency, and phase of the disturbance are unknown.

The controller order is $n_c = 10$, which satisfies Eq. (36). All other parameters remain the same as in Example 1. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $P(0) = I_{2n_c+1}$. The closed-loop system is simulated for 500 time steps, and Fig. 5 shows the time history of y, y_m, z, and u. RC-MRAC drives the performance z to zero, and thus y follows y_m while rejecting the unknown exogenous disturbance w.

Example 3. Lyapunov-stable, nonminimum-phase system with disturbance and uncertain nonminimum-phase zeros. Reconsider the Lyapunov-stable, nonminimum-phase system with disturbance from Example 2, but let the estimates of the nonminimum-phase zeros used by the controller have 10% error. Specifically, let the estimate of $\beta_u(\mathbf{q})$, which is used by the reference model as well as the adaptive law, be given by $(\mathbf{q}-1.43)(\mathbf{q}-1.1-\jmath 1.1)(\mathbf{q}-1.1+\jmath 1.1)$. All other parameters remain the same as in Example 2. The closed-loop system is simulated for 500 time steps, and Fig. 6 shows the time history of y, y_m, z, and u. Figure 6 shows that there is some performance degradation relative to Example 2 because the closed-loop system is unable to match the reference model as required by Assumption 7. However, the performance z is bounded and is reduced by over 70% relative to the open-loop performance. In this example, the error in the nonminimum-phase zero estimates can be increased to approximately 18% without causing the closed-loop performance to become unbounded.

Example 4. Stabilization of a plant that is not strongly stabilizable. Consider the unstable, nonminimum-phase system

$$\mathbf{q}(\mathbf{q}-0.1)(\mathbf{q}-1.2)y(k) = -2(\mathbf{q}-1.1)u(k) \tag{44}$$

where $y(0) = 2$. The reference command and disturbance are identically zero; thus, $z(k) = y(k)$ and the control objective is output stabilization. Note that Eq. (44) is not strongly stabilizable; that is, an unstable linear controller is required to stabilize Eq. (44) [28]. For this problem, $n = 3$, $n_u = 1$, $d = 2$, $\beta_d = -2$, and $\beta_u(\mathbf{q}) = (\mathbf{q}-1.1)$. Let $n_c = 3$, which satisfies (36). The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $\alpha_m(\mathbf{q}) = (\mathbf{q}-0.1)^6$ and $P(0) = I_{2n_c+1}$. Figure 7 shows the time

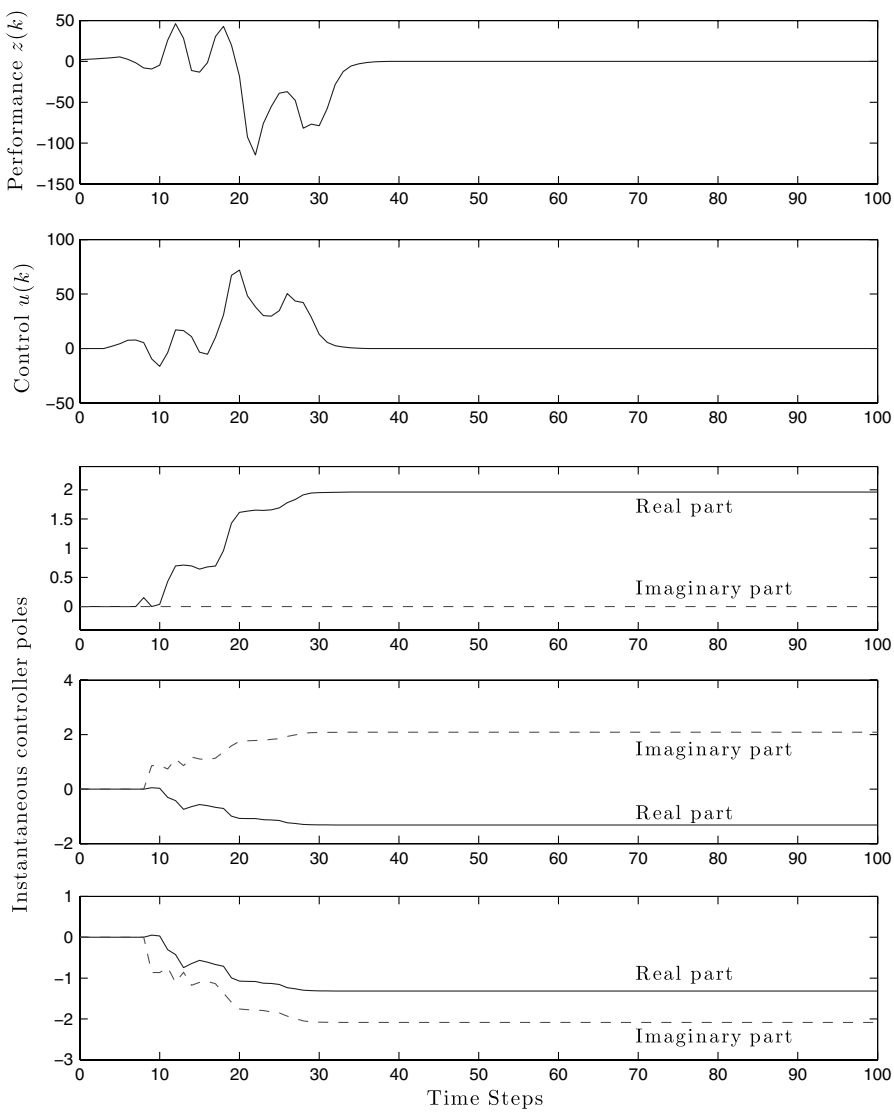

Fig. 7 Stabilization of a plant that is not strongly stabilizable. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 3$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The adaptive controller forces z asymptotically to zero, thus stabilizing the plant, which is not strongly stabilizable.

history of z, u and the three instantaneous controller poles. The closed-loop system is simulated for 100 time steps, z tends to zero, and the controller poles converge. For each $k > 50$, the instantaneous adaptive controller has an unstable positive pole at approximately 1.96. Recall that an unstable pole is required to stabilize Eq. (44); in particular, it follows from root locus arguments that a positive pole larger than 1.2 is required to stabilize Eq. (44).

Next, assume that the nonminimum-phase zero that is located at 1.1 is uncertain. The RC-MRAC adaptive controller stabilizes the output of Eq. (44) for all estimates of the nonminimum-phase zero in the interval [1.04, 1.199]. Notice that the upper bound on this interval is constrained by the location of the unstable pole at 1.2. Figures 8 and 9 show the time history of z, u and the three instantaneous controller poles for the cases where the estimate of the nonminimum-phase zero is 1.04 and 1.199, respectively.

Example 5. Sampled-data, three-mass structure. Consider the serially connected, three-mass structure shown in Fig. 10, which is given by

$$M\ddot{q} + C\dot{q} + Kq = \mu[u \quad 0 \quad 0]^T + \mu[0 \quad w \quad 0]^T \tag{45}$$

where $M \triangleq \mathrm{diag}(m_1, m_2, m_3)$, $q \triangleq [q_1 \quad q_2 \quad q_2]^T$,

$$C \triangleq \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

$$K \triangleq \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

u is the control, w is the exogenous disturbance, and the input gain is $\mu = 10^2$. For this example, the masses are $m_1 = 0.1$ kg, $m_2 = 0.2$ kg and $m_3 = 0.1$ kg; the damping coefficients are $c_1 = 5$ kg/s, $c_2 = 3$ kg/s, and $c_3 = 4$ kg/s; and the spring constants are $k_1 = 11$ kg/s^2, $k_2 = 12$ kg/s^2, and $k_3 = 5$ kg/s^2.

The control objective is to force the position of m_3 to follow the output y_m of a reference model. The continuous-time system (45) is sampled at 20 Hz with input provided by a zero-order hold. Thus, the sample time is $T_s = 0.05$ s. Although the continuous-time system (45) from u to y is minimum-phase [29], the sampled-data system has a nonminimum-phase sampling zero located at approximately -3.4. Thus, let $\beta_u(\mathbf{q}) = \mathbf{q} + 3.4$. In addition, $d = 1$, and $\beta_d = 2/45$.

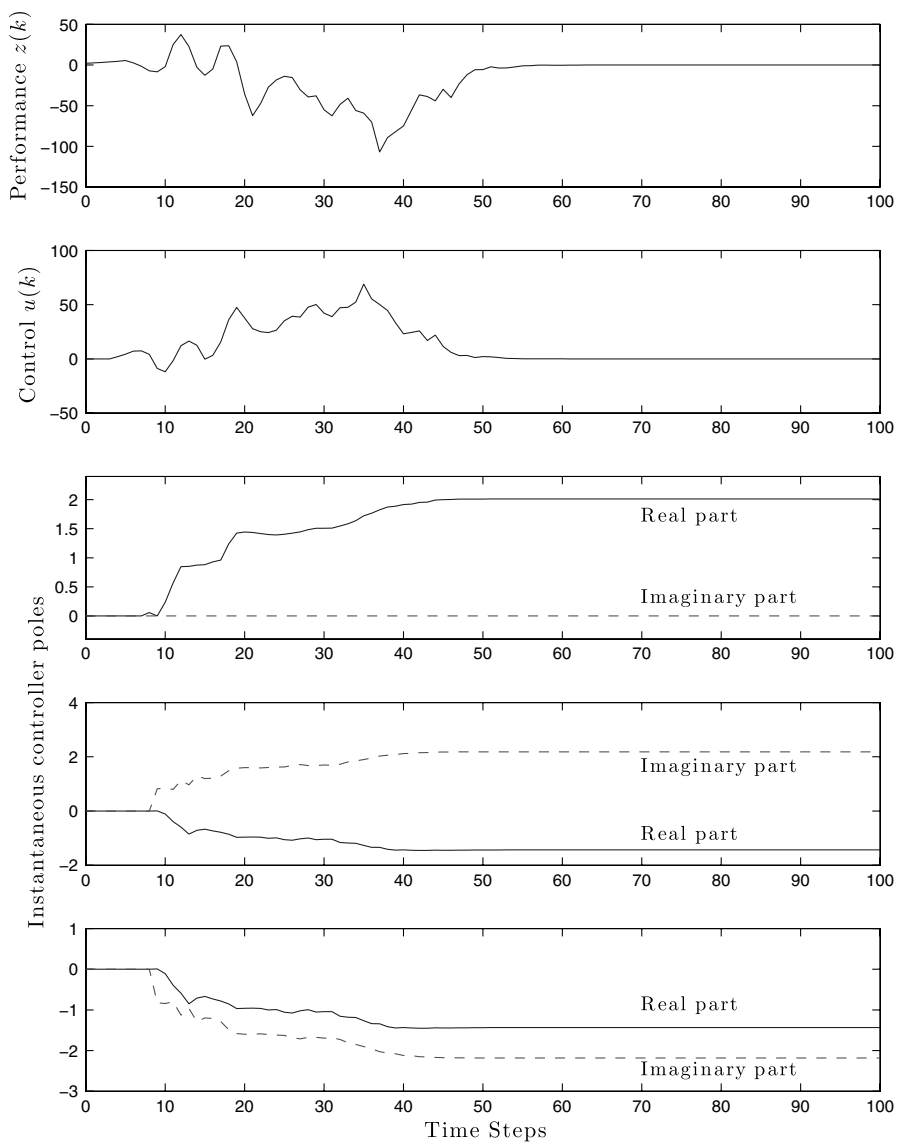

Fig. 8 Stabilization of a plant that is not strongly stabilizable with error in the estimate of the nonminimum-phase zero. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 3$, $\lambda = 1$, $P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The plant's nonminimum-phase zero is located at 1.1, and RC-MRAC uses an estimate of the nonminimum-phase zero given by 1.04. The adaptive controller stabilizes the plant, which is not strongly stabilizable.

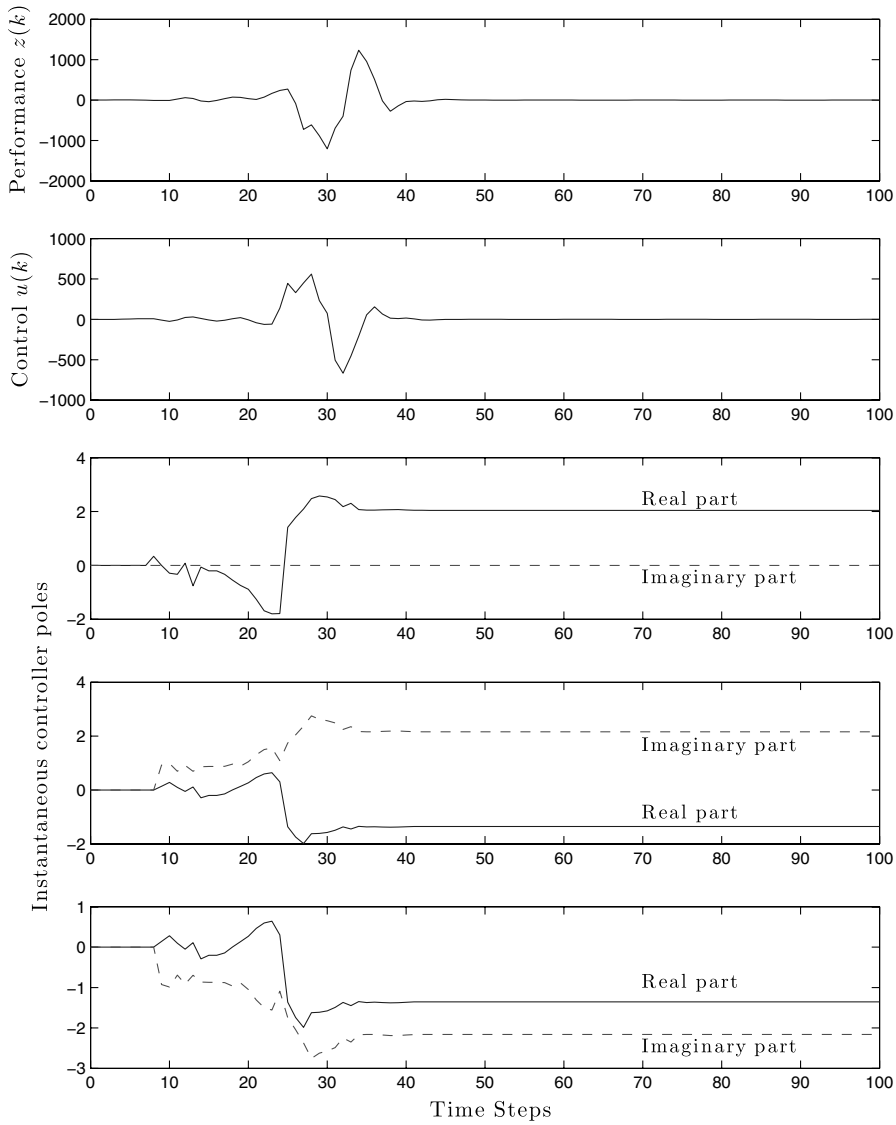

Fig. 9 Stabilization of a plant that is not strongly stabilizable with error in the estimate of the nonminimum-phase zero. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 3, \lambda = 1, P(0) = I_{2n_c+1}$, and $\theta(0) = 0$. The plant's nonminimum-phase zero is located at 1.1, and RC-MRAC uses an estimate of the nonminimum-phase zero given by 1.199. The adaptive controller stabilizes the plant, which is not strongly stabilizable.

Next, consider the reference model (3), where $\alpha_m(\mathbf{q}) = (\mathbf{q} - 0.3)^2$, and

$$\beta_m(\mathbf{q}) = \frac{\alpha_m(1)}{\beta_u(1)}\beta_u(\mathbf{q})$$

Furthermore, for $t = kT_s \leq 8$ s, let the reference model command $r(k)$ be a sampled sequence of 1 s doublets with amplitude 0.5 m, and, for $t = kT_s > 8$ s, let the reference model command $r(k)$ be a sampled sinusoid with frequency 2 Hz and amplitude 1 m. Finally, the unknown disturbance is a sampled sinusoid with frequency 3.5 Hz, amplitude 0.25 m, and a constant bias of 0.1 m. More specifically, $w(k) = 0.25\sin(7\pi T_s k) + 0.1$.

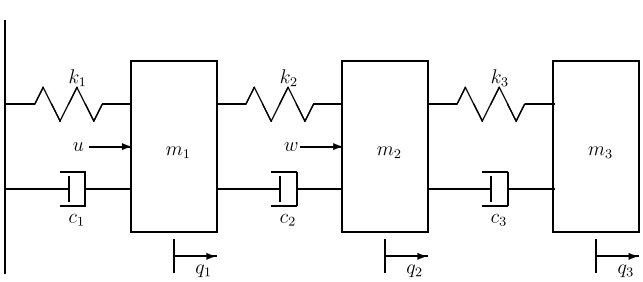

Fig. 10 A serially connected three-mass structure subjected to disturbance w and control u.

The open-loop system is given the initial conditions $q(0) = [0.1 \quad 0.2 \quad 0.1]^T$ m and $\dot{q}(0) = [0 \quad 0 \quad 0]^T$ m/s. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 16$ [which satisfies Eq. (36)] and $P(0) = 10^2 I_{2n_c+1}$. Figure 11 shows the time history of y, y_m, z, and u. The closed-loop adaptive system experiences transient responses for approximately two periods of the reference model doublet. The adaptive controller subsequently drives the performance z to zero, that is, y follows y_m and rejects w. Furthermore, at 8 s, the reference model input r is changed to the 2 Hz sinusoid, and the output y continues to follow y_m with minimal transient behavior.

Example 6. NASA's GTM. This example demonstrates RC-MRAC controlling NASA's GTM [30,31] linearized about a nominal flight condition with the following parameters:

1) Flight-path angle is 0 deg and angle of attack is 3 deg.

2) Body x-axis, y-axis, and z-axis velocities are 161.66, 0, and 7.12 ft/s, respectively.

3) Angular velocities in roll, pitch, and yaw are 0, 0, and 0 deg/s, respectively.

4) Latitude, longitude, and altitude are 0 deg, 0 deg, and 800 ft, respectively.

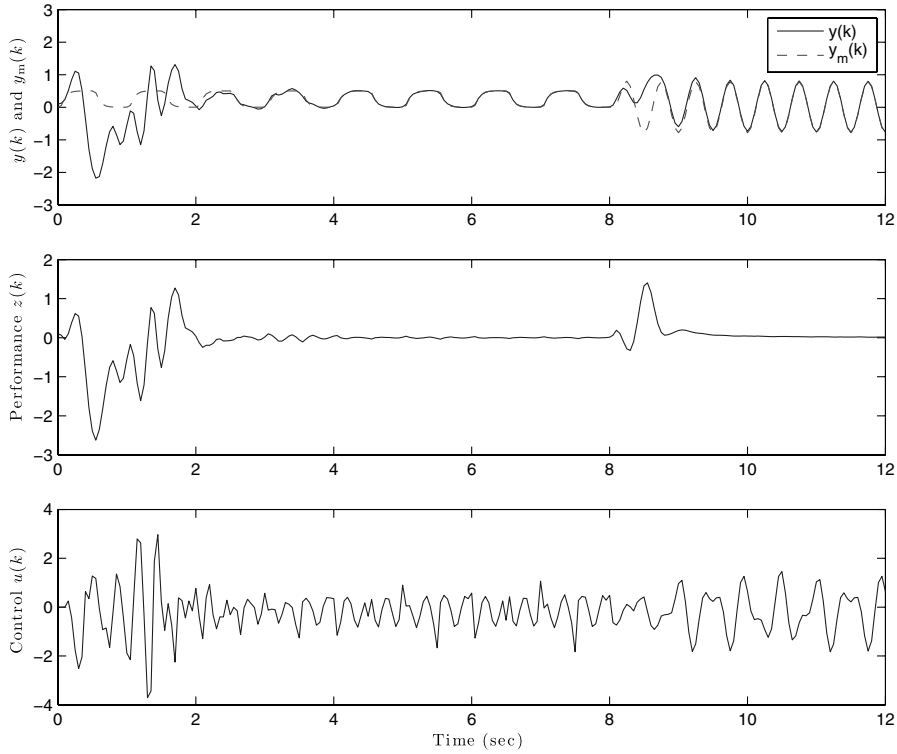

Fig. 11 Sampled-data, three-mass structure. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 16$, $\lambda = 1$, $P(0) = 10^2 I_{2n_c+1}$, and $\theta(0) = 0$. The adaptive controller forces z asymptotically to zero; thus, the position of m_3 follows y_m while rejecting w. Note that y continues to follow the command y_m after 8 s when r is changes to a 2 Hz sinusoid.

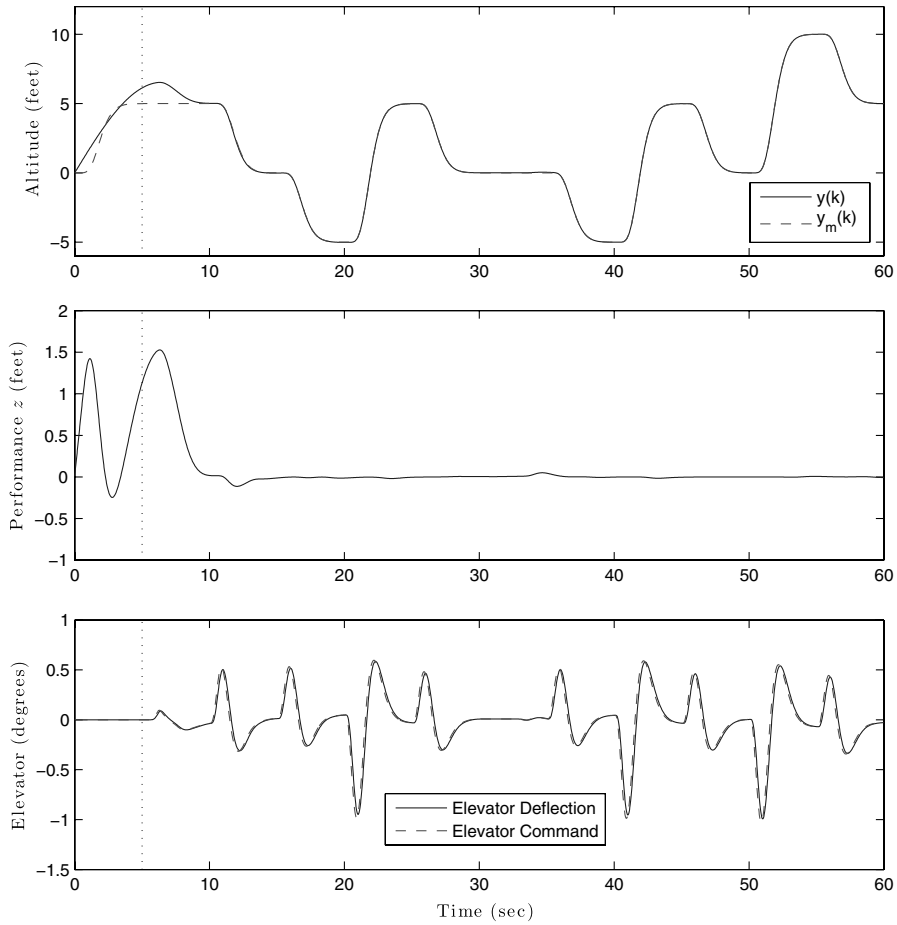

Fig. 12 NASA's GTM. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 20$, $\lambda = 1$, and $P(0) = 10^{20} I_{2n_c+1}$. The adaptive controller force the aircraft's altitude y to follow the reference model y_m.

5) Roll, pitch, and yaw angles are 0.07, 3, and 90 deg, respectively.

6) Elevator, aileron, and rudder angles are 2.7, 0, and 0 deg, respectively.

RC-MRAC is implemented to control GTM's behavior from elevator to altitude. Thus, $u(k)$ is the elevator command from its nominal value, and $y(k)$ is the altitude deviation from its nominal value. The control objective is to force the altitude $y(k)$ to follow the output of a reference model $y_m(k)$. The elevator dynamics are assumed to be first order with a time constant $\tau = 1/10$ ([32], p. 59). More specifically, the actual elevator deflection $u_e(t)$ is the output of the elevator dynamics

$$\tau \dot{u}_e(t) + u_e(t) = u_{\mathrm{ZOH}}(t)$$

where $u_e(0) = 0$ and $u_{\mathrm{ZOH}}(t)$ is the zero-order hold of $u(k)$, which is generated by RC-MRAC. The linearized elevator-to-altitude transfer function for the continuous-time GTM model has a real nonminimum-phase zero. With 50 Hz sampling, the sampled-data system has a nonminimum-phase zero located at approximately 1.706, as well as a nonminimum-phase sampling zero located at approximately -3.29. Thus, let $\beta_u(\mathbf{q}) = (\mathbf{q} - 1.706)(\mathbf{q} + 3.29)$. In addition, $d = 1$, and $\beta_d = 10^{-5}$.

Next, consider the reference model (3), where $\alpha_m(\mathbf{q}) = (\mathbf{q} - 0.92)^8$ and

$$\beta_m(\mathbf{q}) = \frac{\alpha_m(1)}{\beta_u(1)}\beta_u(\mathbf{q})$$

The reference model is chosen such that its gain is unity at $\mathbf{z} = 1$ and its step response settles in approximately 4 s without overshoot. This reference model results in a smooth output $y_m(k)$ for the reference model command $r(k)$, which consists of a sequence of 5 ft and 10 ft step commands.

GTM is given nonzero initial conditions relative to the nominal flight condition described above. The RC-MRAC algorithm (6), (13), and (14) is implemented in feedback with $n_c = 20$ [which satisfies Eq. (36)], $\lambda = 1$, and $P(0) = 10^{20} I_{2n_c+1}$. Note that the singular values of $P(0)$ are 10^{20}, which allows the RC-MRAC controller to adapt quickly. Figure 12 shows the time history of the altitude y, the reference model altitude y_m, the performance $z = y - y_m$, the elevator command u, and the actual elevator deflection u_e. GTM is allowed to run in open loop for 5 s in order to demonstrate the uncontrolled response; note that the altitude drifts upward due to the nonzero initial condition and the rigid-body altitude mode (e.g., an initial altitude velocity causes the uncontrolled aircraft to climb in altitude without bound). After 5 s, the adaptive controller is turned on, and the altitude follows the reference model after a transient period of approximately 3 to 4 s.

IX. Conclusions

The retrospective cost model reference adaptive control (RC-MRAC) algorithm for single-input/single-output discrete-time (including sampled-data) systems was shown to be effective for plants that are possibly nonminimum phase and possibly subjected to disturbances with unknown spectra. The stability analysis presented in this paper relies on knowledge of the first nonzero Markov parameter and the nonminimum-phase zeros of the plant. Numerical examples demonstrated that RC-MRAC is robust to errors in the nonminimum-phase zero estimates; however, quantification of this robustness remains an open problem. Thus, the examples demonstrated that RC-MRAC can provide both command following and disturbance rejection with limited modeling information, which need not be precisely known.

Appendix A: Proof of Theorem 2

Proof. In this proof, the ideal fixed-gain controller (26), which is depicted in Fig. 2, is constructed and shown to satisfy statements 1–4 of Theorem 2.

Since $r_f(k) = \mathbf{q}^{-(n_m-d-n_u)}\beta_r(\mathbf{q})r(k)$, multiplying Eq. (26) by $\mathbf{q}^{n_c}$ yields, for all $k \geq 0$,

$$M_*(\mathbf{q})u_*(k) = L_*(\mathbf{q})y_*(k) + N_*\beta_r(\mathbf{q})\mathbf{q}^{n_1}r(k) \tag{A1}$$

where

$$M_*(\mathbf{q}) = \mathbf{q}^{n_c} - M_{*,1}\mathbf{q}^{n_c-1} - \cdots - M_{*,n_c-1}\mathbf{q} - M_{*,n_c}$$
$$L_*(\mathbf{q}) = L_{*,1}\mathbf{q}^{n_c-1} + \cdots + L_{*,n_c-1}\mathbf{q} + L_{*,n_c}$$
$$n_1 \triangleq n_c + n_u + d - n_m$$

Note that it follows from Eq. (31) that $n_1 \geq 0$. Thus, it suffices to show that there exists $L_*(\mathbf{q})$, $M_*(\mathbf{q})$, and N_* such that statements 1–4 are satisfied.

Define $n_f \triangleq n_c - n_s - n_w$, and it follows from Eq. (31) that

$$n_f = n_c - n_s - n_w \geq \max(n_u + d + n_w, n_m - n - n_w)$$

Next, let

$$M_*(\mathbf{q}) = M_f(\mathbf{q})\alpha_w(\mathbf{q})\beta_s(\mathbf{q}) \tag{A2}$$

where $M_f(\mathbf{q})$ is a monic polynomial with degree n_f. Now, it suffices to show that there exists $L_*(\mathbf{q})$, $M_f(\mathbf{q})$, and N_*, such that statements 1–4 are satisfied.

To show statement 1, consider the closed-loop system consisting of Eqs. (25) and (A1). First, it follows from Eqs. (4) and (25) that, for all $k \geq n_c$,

$$\alpha(\mathbf{q})y_*(k) = \beta_d\beta_u(\mathbf{q})\beta_s(\mathbf{q})u_*(k) + \gamma(\mathbf{q})w(k) \tag{A3}$$

Next, multiplying Eq. (A3) by $M_f(\mathbf{q})\alpha_w(\mathbf{q})$ and using Eq. (A2) yields

$$M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q})y_*(k) = \beta_d\beta_u(\mathbf{q})M_*(\mathbf{q})u_*(k) + M_f(\mathbf{q})\alpha_w(\mathbf{q})\gamma(\mathbf{q})w(k)$$

Using Eq. (A1) yields, for all $k \geq n_c$,

$$[M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q})]y_*(k) = \beta_d N_*\beta_u(\mathbf{q})\beta_r(\mathbf{q})\mathbf{q}^{n_1}r(k) + M_f(\mathbf{q})\alpha_w(\mathbf{q})\gamma(\mathbf{q})w(k) \tag{A4}$$

Since $\alpha_w(\mathbf{q})$ is a scalar polynomial, it follows from Assumption 5 that

$$M_f(\mathbf{q})\alpha_w(\mathbf{q})\gamma(\mathbf{q})w(k) = M_f(\mathbf{q})\gamma(\mathbf{q})\alpha_w(\mathbf{q})w(k) = 0$$

Therefore, for all $k \geq n_c$, Eq. (A4) becomes

$$[M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q})]y_*(k) = \beta_d N_*\beta_u(\mathbf{q})\beta_r(\mathbf{q})\mathbf{q}^{n_1}r(k) \tag{A5}$$

Next, let $N_* = 1/\beta_d$, and since $\beta_m(\mathbf{q}) = \beta_u(\mathbf{q})\beta_r(\mathbf{q})$, it follows from Eq. (A5) that, for all $k \geq n_c$,

$$[M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q})]y_*(k) = \beta_m(\mathbf{q})\mathbf{q}^{n_1}r(k) \tag{A6}$$

Next, show that there exist polynomials $L_*(\mathbf{q})$ and $M_f(\mathbf{q})$ such that

$$M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q}) = \alpha_m(\mathbf{q})\mathbf{q}^{n_1}$$

First, note that

$$\deg M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) = n_f + n_w + n = n_c + n_u + d = \deg \alpha_m(\mathbf{q})\mathbf{q}^{n_1}$$

Next, the degree of $M_f(\mathbf{q})$ is n_f and the degree of $L_*(\mathbf{q})$ is at most $n_c - 1$. Since, in addition, $X(\mathbf{q}) \triangleq \alpha_w(\mathbf{q})\alpha(\mathbf{q})$ and $Y(\mathbf{q}) \triangleq -\beta_d\beta_u(\mathbf{q})$ are coprime, it follows from the Diophantine equation (see [8], Theorem A.2.5) that the roots of $M_f(\mathbf{q})X(\mathbf{q}) + Y(\mathbf{q})L_*(\mathbf{q})$ can be assigned arbitrarily by choice of $L_*(\mathbf{q})$ and $M_f(\mathbf{q})$. Therefore, there exist polynomials $L_*(\mathbf{q})$ and $M_f(\mathbf{q})$ such that

$$M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q}) = \alpha_m(\mathbf{q})\mathbf{q}^{n_1}$$

Thus, for all $k \geq n_c$, Eq. (A6) becomes

$$\alpha_m(\mathbf{q})\mathbf{q}^{n_1} y_*(k) = \beta_m(\mathbf{q})\mathbf{q}^{n_1} r(k)$$

which implies that, for all $k \geq n_c + n_1$,

$$\alpha_m(\mathbf{q}) y_*(k) = \beta_m(\mathbf{q}) r(k) \tag{A7}$$

Thus, for all $k \geq k_0 \triangleq n_c + n_1 + n_m = 2n_c + n_u + d$,

$$\bar{\alpha}_m(\mathbf{q}^{-1}) y_*(k) = \bar{\beta}_m(\mathbf{q}^{-1}) r(k)$$

thus, confirming statement 1.

To show statement 2, note that, for all $k \geq n_c$, the closed-loop system (28) and (29) is a $(2n_c + 1)$th-order nonminimal-state-space realization of the closed-loop system (25) and (26), which has the closed-loop characteristic polynomial

$$\begin{aligned} &M_*(\mathbf{q})\alpha(\mathbf{q}) - \beta(\mathbf{q})L_*(\mathbf{q}) \\ &\quad = \beta_s(\mathbf{q})[M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q})] \\ &\quad = \beta_s(\mathbf{q})\alpha_m(\mathbf{q})\mathbf{q}^{n_1} \end{aligned}$$

Thus, the spectrum of $\mathcal{A}_*$ consists of the $n_c + n$ roots of $\beta_s(\mathbf{q})\alpha_m(\mathbf{q})\mathbf{q}^{n_1}$ along with $n_c - n + 1$ eigenvalues located at 0, which are exactly the uncontrollable eigenvalues of the open-loop dynamics: that is, the uncontrollable eigenvalues of $(\mathcal{A}, \mathcal{B})$. Therefore, since $\alpha_m(\mathbf{q})$ and $\beta_s(\mathbf{q})$ are asymptotically stable, it follows that $\mathcal{A}_*$ is asymptotically stable. Thus, verifying statement 2.

To show statement 3, it follows from Assumption 5 that $w(k)$ is bounded. Since, in addition, $\mathcal{A}_*$ is asymptotically stable, it follows from Eq. (28) that $\phi_*(k)$ is the state of an asymptotically stable linear system with the bounded inputs $\psi_w(k)$ and $r_f(k)$. Thus, $\phi_*(k)$ is bounded. Finally, since $u_*(k)$ is a component of $\phi_*(k+1)$, it follows that $u_*(k)$ is bounded.

To show statement 4, consider the $(2n_c + 1)$th-order nonminimal-state-space realization (28) and (29), which, for all $k \geq n_c$, has the solution

$$\begin{aligned} y_*(k) &= \mathcal{C}\mathcal{A}_*^{k-n_c}\phi_*(n_c) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_r r_f(k-i+1) \\ &\quad + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1\psi_w(k-i) + \mathcal{D}_2\psi_w(k) \end{aligned}$$

which implies that

$$\begin{aligned} \alpha_m(\mathbf{q}) y_*(k) &= \alpha_m(\mathbf{q})[\mathcal{C}\mathcal{A}_*^{k-n_c}\phi_*(n_c)] \\ &\quad + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_r r_f(k-i+1)\right] \\ &\quad + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1\psi_w(k-i) + \mathcal{D}_2\psi_w(k)\right] \end{aligned} \tag{A8}$$

Comparing Eqs. (A7) and (A8) yields, for all $k \geq n_c + n_1$,

$$\begin{aligned} \beta_m(\mathbf{q}) r(k) &= \alpha_m(\mathbf{q})[\mathcal{C}\mathcal{A}_*^{k-n_c}\phi_*(n_c)] \\ &\quad + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_r r_f(k-i+1)\right] \\ &\quad + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1\psi_w(k-i) + \mathcal{D}_2\psi_w(k)\right] \end{aligned} \tag{A9}$$

Next, for all $k \geq n_c$, consider the system (1), where $u(k)$ consists of two components: one that is generated from the ideal controller and one that is an arbitrary sequence $e(k)$. More precisely, for all $k \geq n_c$, consider the system

$$y_e(k) = -\sum_{i=1}^{n} \alpha_i y_e(k-i) + \sum_{i=d}^{n} \beta_i u_e(k-i) + \sum_{i=0}^{n} \gamma_i w(k-i) \tag{A10}$$

where, for all $k \geq n_c$, $u_e(k)$ is given by

$$u_e(k) = \sum_{i=1}^{n_c} L_{*,i} y_e(k-i) + \sum_{i=1}^{n_c} M_{*,i} u_e(k-i) + N_* r_f(k) + e(k) \tag{A11}$$

where $L_{*,1}, \ldots, L_{*,n_c}, M_{*,1}, \ldots, M_{*,n_c}, N_*$ are the ideal controller parameters, and the initial condition at $k = n_c$ for Eqs. (A10) and (A11) is

$$\phi_{e,0} = [\, y_e(n_c - 1) \quad \cdots \quad y_e(0) \quad u_e(n_c - 1) \quad \cdots \quad u_e(0) \quad r_f(n_c)\,]^T$$

Furthermore, let Eqs. (A10) and (A11) have the same initial condition as the ideal closed-loop system (25) and (26), that is, let $\phi_{e,0} = \phi_{*,0}$. For all $k \geq n_c$, Eq. (A10) implies

$$\alpha(\mathbf{q}) y_e(k) = \beta(\mathbf{q}) u_e(k) + \gamma(\mathbf{q}) w(k) \tag{A12}$$

and Eq. (A11) implies

$$M_*(\mathbf{q}) u_e(k) = L_*(\mathbf{q}) y_e(k) + N_*\mathbf{q}^{n_c} r_f(k) + \mathbf{q}^{n_c} e(k)$$

or equivalently,

$$\begin{aligned} M_f(\mathbf{q})\alpha_w(\mathbf{q})\beta_s(\mathbf{q}) u_e(k) &= L_*(\mathbf{q}) y_e(k) \\ &\quad + N_*\beta_r(\mathbf{q})\mathbf{q}^{n_1} r(k) + \mathbf{q}^{n_c} e(k) \end{aligned} \tag{A13}$$

Next, closing the feedback loop between Eqs. (A12) and (A13) yields, for all $k \geq n_c$,

$$\begin{aligned} &[M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q})] y_e(k) \\ &\quad = \beta_d N_*\beta_u(\mathbf{q})\beta_r(\mathbf{q})\mathbf{q}^{n_1} r(k) + M_f(\mathbf{q})\alpha_w(\mathbf{q})\gamma(\mathbf{q}) w(k) \\ &\qquad + \beta_d\beta_u(\mathbf{q})\mathbf{q}^{n_c} e(k) \end{aligned}$$

Since Eq. (A11) is constructed with the ideal controller parameters, it follows from earlier in this proof that

$$\begin{gathered} M_f(\mathbf{q})\alpha_w(\mathbf{q})\gamma(\mathbf{q}) w(k) = 0 \\ M_f(\mathbf{q})\alpha_w(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_u(\mathbf{q})L_*(\mathbf{q}) = \alpha_m(\mathbf{q})\mathbf{q}^{n_1} \\ N_* = 1/\beta_d \end{gathered}$$

Therefore, for all $k \geq n_c$,

$$\alpha_m(\mathbf{q})\mathbf{q}^{n_1} y_e(k) = \beta_m(\mathbf{q})\mathbf{q}^{n_1} r(k) + \beta_d\beta_u(\mathbf{q})\mathbf{q}^{n_c} e(k)$$

which implies that, for all $k \geq n_c + n_1$,

$$\alpha_m(\mathbf{q}) y_e(k) = \beta_m(\mathbf{q}) r(k) + \beta_d\beta_u(\mathbf{q})\mathbf{q}^{n_m - n_u - d} e(k) \tag{A14}$$

Next, for all $k \geq n_c$, consider the $(2n_c + 1)$th-order nonminimal-state-space realization (18–24) with the feedback (A11), which has the closed-loop representation

$$\begin{aligned} \phi_e(k+1) &= \mathcal{A}_*\phi_e(k) + \mathcal{B}e(k) + \mathcal{D}_1\psi_w(k) + \mathcal{D}_r r_f(k+1), \\ y_e(k) &= \mathcal{C}\phi_e(k) + \mathcal{D}_1\psi_w(k) \end{aligned}$$

where the $\phi_e(k)$ has the same form as $\phi(k)$ with $y_e(k)$ and $u_e(k)$ replacing $y(k)$ and $u(k)$, respectively. For all $k \geq n_c$, this system has the solution

$$\begin{aligned} y_e(k) &= \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_r r_f(k-i+1) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1\psi_w(k-i) \\ &\quad + \mathcal{D}_2\psi_w(k) + \mathcal{C}\mathcal{A}_*^{k-n_c}\phi_e(n_c) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) \end{aligned}$$

Multiplying both sides by $\alpha_m(\mathbf{q})$ yields, for all $k \geq n_c$,

$$\alpha_m(\mathbf{q})y_e(k) = \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_r r_f(k-i+1) + \sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1\psi_w(k-i) + \mathcal{D}_2\psi_w(k)\right] + \alpha_m(\mathbf{q})[\mathcal{C}\mathcal{A}_*^{k-n_c}\phi_e(n_c)] + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right] \tag{A15}$$

Since $\phi_e(n_c) = \phi_{e,0} = \phi_{*,0} = \phi_*(n_c)$, it follows from Eqs. (A9) and (A15) that, for all $k \geq n_c + n_1$,

$$\alpha_m(\mathbf{q})y_e(k) = \beta_m(\mathbf{q})r(k) + \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right] \tag{A16}$$

Finally, comparing Eqs. (A14) and (A16) yields, for all $k \geq n_c + n_1$,

$$\beta_d\beta_u(\mathbf{q})\mathbf{q}^{n_m-n_u-d}e(k) = \alpha_m(\mathbf{q})\left[\sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right]$$

and multiplying both sides by $\mathbf{q}^{-n_m}$, yields, for all $k \geq k_0$,

$$\beta_d\bar{\beta}_u(\mathbf{q}^{-1})e(k) = \bar{\alpha}_m(\mathbf{q}^{-1})\left[\sum_{i=1}^{k-n_c}\mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right]$$

thus verifying statement 4. □

Appendix B: Proof of Lemma 3

Proof. Subtracting θ_* from both sides of Eq. (13) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{P(k)\Phi(k)z_r(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)} \tag{B1}$$

Next, note from Eq. (14) that

$$P(k+1)\Phi(k) = \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^T(k)P(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)}\right]\Phi(k) = \frac{P(k)\Phi(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)} \tag{B2}$$

and thus,

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - P(k+1)\Phi(k)z_r(k). \tag{B3}$$

Define

$$V_P(P(k),k) \triangleq \lambda^{-k}P^{-1}(k),$$
$$\Delta V_P(k) \triangleq V_P(P(k+1),k+1) - V_P(P(k),k)$$

and note the RLS identity [5,7,10]

$$P^{-1}(k+1) = \lambda P^{-1}(k) + \Phi(k)\Phi^T(k) \tag{B4}$$

Evaluating $\Delta V_P(k)$ along the trajectories of Eq. (B4) yields

$$\Delta V_P(k) = \frac{1}{\lambda^{k+1}}[\lambda P^{-1}(k) + \Phi(k)\Phi^T(k)] - \frac{1}{\lambda^k}P^{-1}(k) = \lambda^{-k-1}\Phi(k)\Phi^T(k) \tag{B5}$$

Since $P^{-1}(0)$ is positive definite and ΔV_P is positive semidefinite, it follows that, for all $k \geq 0$, $V_P(P(k),k)$ is positive definite and $V_P(P(k),k) \geq V_P(P(k-1),k-1)$. Therefore, for all $k \geq 0$, $0 < V_P(P(0),0) \leq V_P(P(k),k)$, which implies that

$$0 < \lambda^k P(k) \leq P(0) \tag{B6}$$

If $\lambda = 1$, then Eq. (B6) implies that $P(k)$ is bounded, which verifies statement 4.

Next, define the positive-definite Lyapunov-like function,

$$V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k) \triangleq \tilde{\theta}^T(k)V_P(P(k),k)\tilde{\theta}(k)$$

and define the Lyapunov-like difference

$$\Delta V_{\tilde{\theta}}(k) \triangleq V_{\tilde{\theta}}(\tilde{\theta}(k+1),P(k+1),k+1) - V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k) \tag{B7}$$

Evaluating $\Delta V_{\tilde{\theta}}(k)$ along the trajectories of the estimator-error system (B3) and using Eq. (B5) yields

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= \tilde{\theta}^T(k)\Delta V_P(k)\tilde{\theta}(k) - 2\lambda^{-k-1}z_r(k)\Phi^T(k)\tilde{\theta}(k) + \lambda^{-k-1}z_r^2(k)\Phi^T(k)P(k+1)\Phi(k)\\ &= \lambda^{-k-1}[\tilde{\theta}^T(k)\Phi(k)\Phi^T(k)\tilde{\theta}(k) - 2z_r(k)\Phi^T(k)\tilde{\theta}(k) + z_r^2(k)\Phi^T(k)P(k+1)\Phi(k)]\end{aligned}$$

Next, it follows from Lemma 2 and Eq. (B2) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= -\lambda^{-k-1}z_r^2(k)(1 - \Phi^T(k)P(k+1)\Phi(k))\\ &= -\lambda^{-k-1}z_r^2(k)\left(1 - \frac{\Phi^T(k)P(k)\Phi(k)}{\lambda + \Phi^T(k)P(k)\Phi(k)}\right)\\ &= -\lambda^{-k-1}z_r^2(k)\frac{\lambda}{\lambda + \Phi^T(k)P(k)\Phi(k)}\\ &= -\bar{\eta}(k)z_r^2(k)\end{aligned} \tag{B8}$$

where

$$\bar{\eta}(k) \triangleq \frac{1}{\lambda^{k+1} + \lambda^k\Phi^T(k)P(k)\Phi(k)} \tag{B9}$$

Since $V_{\tilde{\theta}}$ is a positive-definite radially unbounded function of $\tilde{\theta}(k)$ and, for $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is nonpositive, it follows that $\tilde{\theta}(k)$ is bounded and thus $\theta(k)$ is bounded. Thus, verifying statement 1.

To show statement 2, first show that

$$\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$$

exists. Since $V_{\tilde{\theta}}$ is positive definite, and, for all $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is nonpositive, it follows from Eq. (B7) that

$$\begin{aligned}0 &\leq -\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)\\ &= V_{\tilde{\theta}}(\tilde{\theta}(k_0),P(k_0),k_0) - \lim_{k\to\infty}V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k)\\ &\leq V_{\tilde{\theta}}(\tilde{\theta}(k_0),P(k_0),k_0)\end{aligned}$$

where the upper and lower bounds imply that both limits exist. Since

$$\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$$

exists, Eq. (B8) implies that

$$\lim_{k\to\infty}\sum_{j=k_0}^{k}\bar{\eta}(j)z_r^2(j)$$

exists, and thus

$$\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}(j)z_r^2(j)$$

exists. Since, for all $k \geq 0$, $\lambda^{k+1} \leq 1$ and $\lambda^k P(k) \leq P(0)$, it follows from Eqs. (43) and (B9) that, for all $k \geq 0$, $\eta(k) \leq \bar{\eta}(k)$, which implies that

$$\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_r^2(j) \le \lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}(j)z_r^2(j)$$

Thus,

$$\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_r^2(j)$$

exists, which verifies statement 2.

To show statement 3, first show that

$$\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2$$

exists. It follows from Eqs. (B1) and (B6) that

$$\begin{aligned}\sum_{j=0}^{\infty}\|\theta(j+1)-\theta(j)\|^2 &= \sum_{j=0}^{\infty}\left\|\frac{P(j)\Phi(j)z_r(j)}{\lambda+\Phi^T(j)P(j)\Phi(j)}\right\|^2\\ &= \sum_{j=0}^{\infty}z_r^2(j)\frac{\Phi^T(j)P^2(j)\Phi(j)}{[\lambda+\Phi^T(j)P(j)\Phi(j)]^2}\\ &= \sum_{j=0}^{\infty}\bar{\eta}(j)z_r^2(j)\left(\frac{\lambda^j\Phi^T(j)P^2(j)\Phi(j)}{\lambda+\Phi^T(j)P(j)\Phi(j)}\right)\\ &\le \sum_{j=0}^{\infty}\bar{\eta}(j)z_r^2(j)\|\lambda^jP(j)\|_F\left(\frac{\Phi^T(j)P(j)\Phi(j)}{\lambda+\Phi^T(j)P(j)\Phi(j)}\right)\\ &\le \|P(0)\|_F\sum_{j=0}^{\infty}\bar{\eta}(j)z_r^2(j)\left(\frac{\Phi^T(j)P(j)\Phi(j)}{\lambda+\Phi^T(j)P(j)\Phi(j)}\right)\end{aligned}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Next, note that, for all $k\ge 0$,

$$\frac{\Phi^T(k)P(k)\Phi(k)}{\lambda+\Phi^T(k)P(k)\Phi(k)} \le 1$$

which implies that

$$\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2 \le \|P(0)\|_F\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}(j)z_r^2(j) \quad \text{(B10)}$$

Since

$$\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}(j)z_r^2(j)$$

exists, it follows from Eq. (B10) that

$$\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2$$

exists. Next, let N be a positive integer and note that

$$\begin{aligned}\sum_{j=N}^{\infty}\|\theta(j)-\theta(j-N)\|^2 &= \sum_{j=N}^{\infty}\|\theta(j)-\theta(j-1)+\theta(j-1)\\ &\quad -\theta(j-2)+\cdots+\theta(j-N+1)-\theta(j-N)\|^2\\ &\le \sum_{j=N}^{\infty}(\|\theta(j)-\theta(j-1)\|+\|\theta(j-1)-\theta(j-2)\|\\ &\quad +\cdots+\|\theta(j-N+1)-\theta(j-N)\|)^2\\ &\le 2^{N-1}\sum_{j=N}^{\infty}(\|\theta(j)-\theta(j-1)\|^2+\|\theta(j-1)-\theta(j-2)\|^2\\ &\quad +\cdots+\|\theta(j-N+1)-\theta(j-N)\|^2) \quad \text{(B11)}\end{aligned}$$

Since all of the limits on the right-hand side of Eq. (B11) exist, it follows that

$$\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j)-\theta(j-N)\|^2$$

exists. This verifies statement 3. □

Appendix C: Proof of Theorem 3

Proof. It follows from statement 1 of Lemma 3 that $\theta(k)$ is bounded. To prove the remaining properties, for all $k\ge k_0$, define the ideal filtered regressor

$$\Phi_*(k) \triangleq \beta_d\bar{\beta}_u(\mathbf{q}^{-1})\phi_*(k) \quad \text{(C1)}$$

and the filtered regressor error

$$\tilde{\Phi}(k) \triangleq \Phi(k)-\Phi_*(k) = \beta_d\bar{\beta}_u(\mathbf{q}^{-1})\tilde{\phi}(k) \quad \text{(C2)}$$

Next, apply the operator $\beta_d\bar{\beta}_u(\mathbf{q}^{-1})$ to Eq. (39) and use Lemma 1 to obtain the filtered error system

$$\begin{aligned}\tilde{\Phi}(k+1) &= \mathcal{A}_*\tilde{\Phi}(k)+\mathcal{B}\beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^T(k)\tilde{\theta}(k)]\\ &= \mathcal{A}_*\tilde{\Phi}(k)+\mathcal{B}z_f(k) \quad \text{(C3)}\end{aligned}$$

which is defined for all $k\ge k_0$.

Next, define the quadratic function

$$J(\tilde{\Phi}(k)) \triangleq \tilde{\Phi}^T(k)\mathcal{P}\tilde{\Phi}(k) \quad \text{(C4)}$$

where $\mathcal{P}>0$ satisfies the discrete-time Lyapunov equation $\mathcal{P}=\mathcal{A}_*^T\mathcal{P}\mathcal{A}_*+\mathcal{Q}+\alpha I$, where $\mathcal{Q}>0$ and $\alpha>0$. Note that $\mathcal{P}$ exists since $\mathcal{A}_*$ is asymptotically stable. Defining

$$\Delta J(k) \triangleq J(\tilde{\Phi}(k+1))-J(\tilde{\Phi}(k)) \quad \text{(C5)}$$

it follows from Eq. (C3) that, for all $k\ge k_0$,

$$\begin{aligned}\Delta J(k) &= -\tilde{\Phi}^T(k)(\mathcal{Q}+\alpha I)\tilde{\Phi}(k)+\tilde{\Phi}^T(k)\mathcal{A}_*^T\mathcal{P}\mathcal{B}z_f(k)\\ &\quad + z_f(k)\mathcal{B}^T\mathcal{P}\tilde{\mathcal{A}}_*\tilde{\Phi}(k)+z_f^2(k)\mathcal{B}^T\mathcal{P}\mathcal{B}\\ &\le -\tilde{\Phi}^T(k)(\mathcal{Q}+\alpha I)\tilde{\Phi}(k)\\ &\quad + z_f^2(k)\mathcal{B}^T\mathcal{P}\mathcal{B}+\alpha\tilde{\Phi}^T(k)\tilde{\Phi}(k)\\ &\quad + \frac{1}{\alpha}z_f^2(k)\mathcal{B}^T\mathcal{P}\mathcal{A}_*\mathcal{A}_*^T\mathcal{P}\mathcal{B}\\ &= -\tilde{\Phi}^T(k)\mathcal{Q}\tilde{\Phi}(k)+\sigma_1 z_f^2 \quad \text{(C6)}\end{aligned}$$

where $\sigma_1 \triangleq \mathcal{B}^T\mathcal{P}\mathcal{B}+\frac{1}{\alpha}\mathcal{B}^T\mathcal{P}\mathcal{A}_*\mathcal{A}_*^T\mathcal{P}\mathcal{B}$.

Now, consider the positive-definite, radially unbounded Lyapunov-like function:

$$V(\tilde{\Phi}(k)) \triangleq \ln(1+a_1J(\tilde{\Phi}(k)))$$

where $a_1>0$ is specified below. The Lyapunov-like difference is thus given by

$$\Delta V(k) \triangleq V(\tilde{\Phi}(k+1))-V(\tilde{\Phi}(k))$$

For all $k\ge k_0$, evaluating $\Delta V(k)$ along the trajectories of Eq. (C3) yields

$$\begin{aligned}\Delta V(k) &= \ln[1+a_1J(\tilde{\Phi}(k))+a_1\Delta J(k)]-\ln[1+a_1J(\tilde{\Phi}(k))]\\ &= \ln\left(1+\frac{a_1\Delta J(k)}{1+a_1J(\tilde{\Phi}(k))}\right) \quad \text{(C7)}\end{aligned}$$

Since, for all $x>0$, $\ln x\le x-1$, and using Eq. (C6) yields

$$\begin{aligned}\Delta V(k) &\le a_1 \frac{\Delta J(k)}{1 + a_1 J(\tilde{\Phi}(k))} \\ &\le -a_1 \frac{\tilde{\Phi}^T(k)\mathcal{Q}\tilde{\Phi}(k)}{1 + a_1\tilde{\Phi}^T(k)\mathcal{P}\tilde{\Phi}(k)} + a_1\sigma_1 \frac{z_f^2(k)}{1 + a_1\tilde{\Phi}^T(k)\mathcal{P}\tilde{\Phi}(k)} \\ &\le -W(\tilde{\Phi}(k)) + a_1\sigma_1\ell^2(k) \end{aligned} \tag{C8}$$

where

$$W(\tilde{\Phi}(k)) \triangleq a_1 \frac{\tilde{\Phi}^T(k)\mathcal{Q}\tilde{\Phi}(k)}{1 + a_1\tilde{\Phi}^T(k)\mathcal{P}\tilde{\Phi}(k)} \tag{C9}$$

$$\ell(k) \triangleq \frac{z_f(k)}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} \tag{C10}$$

Now, show that

$$\lim_{k\to\infty} \sum_{j=0}^{k} \ell^2(j)$$

exists. First, write $\beta_u(\mathbf{q})$ as

$$\beta_u(\mathbf{q}) = \beta_{u,0}\mathbf{q}^{n_u} + \beta_{u,1}\mathbf{q}^{n_u-1} + \cdots + \beta_{u,n_u-1}\mathbf{q} + \beta_{u,n_u}$$

where $\beta_{u,0} = 1$ and $\beta_{u,1}, \ldots, \beta_{u,n_u} \in \mathbb{R}$. It follows from Eq. (11) that, for all $k \ge k_0$,

$$z_f(k) = z_r(k) - \beta_d \sum_{i=d}^{n_u+d} \beta_{u,i-d}\phi^T(k-i)[\theta(k) - \theta(k-i)] \tag{C11}$$

Using Eqs. (C10) and (C11) yields, for all $k \ge k_0$,

$$|\ell(k)| \le \frac{|z_r(k)| + |\beta_d|(\sum_{i=d}^{n_u+d} |\beta_{u,i-d}|\|\phi(k-i)\|\|\theta(k) - \theta(k-i)\|)}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}}$$

It follows from Lemma 3 that $\theta(k)$ is bounded and $\lim_{k\to\infty} \|\theta(k) - \theta(k-1)\| = 0$. Therefore, Lemma 4 in Appendix D implies that there exist $k_2 \ge k_0 > 0$, $c_1 > 0$, and $c_2 > 0$, such that, for all $k \ge k_2$ and all $i = d, \ldots, n_u + d$, $\|\phi(k-i)\| \le c_1 + c_2\|\Phi(k)\|$. In addition, note that $\|\Phi(k)\| = \|\tilde{\Phi}(k) + \Phi_*(k)\| \le \|\tilde{\Phi}(k)\| + \|\Phi_*(k)\| \le \|\tilde{\Phi}(k)\| + \Phi_{*,\max}$, where $\Phi_{*,\max} \triangleq \sup_{k\ge 0}\|\Phi_*(k)\|$ exists because Φ_* is bounded. Therefore, for all $k \ge k_2$, $\|\phi(k-i)\| \le c_1 + c_2\Phi_{*,\max} + c_2\|\tilde{\Phi}(k)\|$, which implies that

$$\begin{aligned}|\ell(k)| &\le \frac{|z_r(k)| + |\beta_d|(c_1 + c_2\Phi_{*,\max} + c_2\|\tilde{\Phi}(k)\|)(\sum_{i=d}^{n_u+d} |\beta_{u,i-d}|\|\theta(k) - \theta(k-i)\|)}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} \\ &\le \frac{|z_r(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} + \frac{c_3 + c_4\|\tilde{\Phi}(k)\|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}}\left(\sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\|\right)\end{aligned}$$

where $c_3 \triangleq (c_1 + c_2\Phi_{*,\max})|\beta_d|(\max_{0\le i\le n_u}|\beta_{u,i}|) > 0$ and $c_4 \triangleq c_2|\beta_d|(\max_{0\le i\le n_u}|\beta_{u,i}|) > 0$. Next, note that $\frac{1}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} \le 1$ and $\frac{\|\tilde{\Phi}(k)\|}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} \le \max(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})})$, which implies that

$$|\ell(k)| \le \frac{|z_r(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} + c_5 \sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\| \tag{C12}$$

where $c_5 \triangleq c_3 + c_4\max(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})}) > 0$.

Next, show that $a_1 > 0$ can be chosen such that the first term of Eq. (C12) is less than a constant times $\sqrt{\eta(k)}|z_r(k)|$, which is square summable according to statement 2 of Lemma 3. It follows from Eq. (43) that

$$\begin{aligned}\frac{1}{\eta(k)} &= 1 + \Phi^T(k)P(0)\Phi(k) \\ &\le 1 + \lambda_{\max}(P(0))[\tilde{\Phi}(k) + \Phi_*(k)]^T[\tilde{\Phi}(k) + \Phi_*(k)] \\ &\le 1 + \lambda_{\max}(P(0))[2\tilde{\Phi}^T(k)\tilde{\Phi}(k) + 2\Phi_*^T(k)\Phi_*(k)] \\ &\le 1 + 2\lambda_{\max}(P(0))\Phi_{*,\max}^2 + 2\lambda_{\max}(P(0))\tilde{\Phi}^T(k)\tilde{\Phi}(k) \\ &= c_6[1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)]\end{aligned}$$

where $a_1 \triangleq \frac{2\lambda_{\max}(P(0))}{\lambda_{\min}(\mathcal{P})[1+2\lambda_{\max}(P(0))\Phi_{*,\max}^2]} > 0$ and $c_6 \triangleq 1 + 2\lambda_{\max}(P(0))\Phi_{*,\max}^2 > 0$. Therefore,

$$\frac{1}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^T(k)\tilde{\Phi}(k)}} \le \sqrt{c_6}\sqrt{\eta(k)}$$

which together with Eq. (C12) implies that, for all $k \ge k_2$,

$$|\ell(k)| \le \sqrt{c_6}\sqrt{\eta(k)}|z_r(k)| + c_5 \sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\|$$

Therefore, for all $k \ge k_2$,

$$\begin{aligned}\ell^2(k) &\le \left[\sqrt{c_6}\sqrt{\eta(k)}|z_r(k)| + c_5 \sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\|\right]^2 \\ &\le 2c_6\eta(k)z_r^2(k) + 2c_5^2\left[\sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\|\right]^2 \\ &\le 2c_6\eta(k)z_r^2(k) + 2^{n_u+1}c_5^2 \sum_{i=d}^{n_u+d} \|\theta(k) - \theta(k-i)\|^2\end{aligned} \tag{C13}$$

It follows from statement 2 of Lemma 3 that

$$\lim_{k\to\infty} \sum_{j=0}^{k} \eta(j)z_r^2(j)$$

exists. Furthermore, it follows from statement 3 of Lemma 3 that, for all $i = d, \ldots, n_u + d$,

$$\lim_{k\to\infty} \sum_{j=0}^{k} \|\theta(j) - \theta(j-i)\|^2$$

exists. Thus, Eq. (C13) implies that

$$\lim_{k\to\infty} \sum_{j=0}^{k} \ell^2(j)$$

exists.

Now, show that $\lim_{k\to\infty} W(\tilde{\Phi}(k)) = 0$. Since W and V are positive definite, it follows from Eq. (C8) that

$$0 \le \lim_{k\to\infty}\sum_{j=0}^{k} W(\tilde{\Phi}(j)) \le \lim_{k\to\infty}\sum_{j=0}^{k} -\Delta V(j) + a_1\sigma_1 \lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j)$$
$$\le V(\tilde{\Phi}(0)) - \lim_{k\to\infty} V(\tilde{\Phi}(k)) + a_1\sigma_1\lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j)$$
$$\le V(\tilde{\Phi}(0)) + a_1\sigma_1\lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j)$$

where the upper and lower bound imply that all limits exist. Thus, $\lim_{k\to\infty} W(\tilde{\Phi}(k)) = 0$. Since $W(\tilde{\Phi}(k))$ is a positive-definite function of $\tilde{\Phi}(k)$, it follows that $\lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0$.

To prove that $u(k)$ is bounded, first note that since $\lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0$ and $\Phi_*(k)$ is bounded, it follows that $\Phi(k)$ is bounded. Next, since $\Phi(k)$ is bounded, it follows from Lemma 4 that $\phi(k)$ is bounded. Furthermore, since $y(k)$ and $u(k)$ are components of $\phi(k+1)$, it follows that $y(k)$ and $u(k)$ are bounded.

To prove that $\lim_{k\to\infty} z(k) = 0$, note that it follows from Eq. (C3) and the fact that $\|\mathcal{B}z_f(k)\| = |z_f(k)|$ that

$$\lim_{k\to\infty}|z_f(k)| \le \lim_{k\to\infty}\|\tilde{\Phi}(k+1)\| + \|\mathcal{A}_*\|_F \lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0$$

Since $\lim_{k\to\infty} z_f(k) = 0$, $z_f(k) = \bar{\alpha}_m(\mathbf{q^{-1}})z(k)$, and $\alpha_m(\mathbf{q}) = \mathbf{q}^{n_m}\bar{\alpha}_m(\mathbf{q}^{-1})$ is an asymptotically stable polynomial, it follows that $\lim_{k\to\infty} z(k) = 0$. □

Appendix D: Lemma used in the Proof of Theorem 3

The following lemma is used in the proof of Theorem 3. This lemma is presented for an arbitrary feedback controller given by Eq. (6) where the controller parameter vector $\theta(k)$ is time varying. More precisely, the following lemma does not depend on the adaptive law used to update $\theta(k)$ provided that such an adaptive law satisfies the assumptions in the lemma.

Lemma 4. Consider the open-loop system (1) satisfying Assumptions 1–9. In addition, consider a feedback controller given by Eq. (6) that satisfies the following assumptions:

Assumption D1. $\theta(k)$ is bounded.

Assumption D2. $\lim_{k\to\infty}\|\theta(k) - \theta(k-1)\| = 0$.

Assumption D3. There exist $\epsilon > 0$ and $k_1 > 0$ such that, for all $k \ge k_1$ and for all $i = 1,\ldots,n_u$, $|M(\xi_i, k)| \ge \epsilon$.

Then, for all initial conditions x_0, there exist $k_2 > 0$, $c_1 > 0$, and $c_2 > 0$, such that, for all $k \ge k_2$, and, for all $N = 0,\ldots,n_u$, $\|\phi(k-d-N)\| \le c_1 + c_2\|\Phi(k)\|$.

Proof. For all $k \ge n_c$, consider the $(2n_c+1)$th-order nonminimal-state-space realization of Eq. (6) given by

$$\phi(k+1) = \mathcal{A}_c(k)\phi(k) + \mathcal{B}_c y(k) + \mathcal{D}_r r_f(k+1) \tag{D1}$$

$$u(k) = \theta^T(k)\phi(k) \tag{D2}$$

where

$$\mathcal{A}_c(k) \triangleq \mathcal{A}_{\text{nil}} + \mathcal{B}\theta^T(k), \qquad \mathcal{B}_c \triangleq E_{2n_c+1}$$

Furthermore, note that, for all $k \ge n_c$, $\mathcal{A}_c(k)$ has $n_c + 1$ poles at zero and n_c poles that coincide with the roots of $M(\mathbf{z}, k)$, which implies that, for all $k \ge n_c$, and, for all $i = 1,\ldots,n_u$,

$$\det(\xi_i I_{2n_c+1} - \mathcal{A}_c(k)) = \xi_i^{n_c+1} M(\xi_i, k) \tag{D3}$$

Next, rewrite the closed-loop system (37) and (38) as

$$\phi(k+1) = \tilde{\mathcal{A}}(k)\phi(k) + \mathcal{D}_1\psi_w(k) + \mathcal{D}_r r_f(k+1) \tag{D4}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2\psi_w(k) \tag{D5}$$

where $\tilde{\mathcal{A}}(k) \triangleq \mathcal{A}_* + \mathcal{B}\tilde{\theta}^T(k) = \mathcal{A} + \mathcal{B}\theta^T(k) = \mathcal{A}_{\text{nil}} + \mathcal{B}_c\mathcal{C} + \mathcal{B}\theta^T(k)$, and note that $\tilde{\mathcal{A}}(k) = \mathcal{A}_c(k) + \mathcal{B}_c\mathcal{C}$. Define the closed-loop dynamics matrix of Eqs. (D1), (D2), (D4), and (D5)

$$\tilde{A}_{\text{cl}}(k) \triangleq \begin{bmatrix} \mathcal{A} & \mathcal{B}\theta^T(k) \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c(k) \end{bmatrix}$$

Since, for all $i = 1,\ldots,n_u$, ξ_i is a zero of $(\mathcal{A},\mathcal{B},\mathcal{C})$, it follows that $\mathcal{C}(\xi_i I_{2n_c+1} - \mathcal{A})^{-1}\mathcal{B} = 0$. Since, for all $i = 1,\ldots,n_u$, ξ_i is not an eigenvalue of $\mathcal{A}$, it follows that $\xi_i I_{2n_c+1} - \mathcal{A}$ is nonsingular. Therefore, using Proposition 2.8.3 in [33] implies that

$$\begin{aligned}\det(\xi_i I - \tilde{\mathcal{A}}_{\text{cl}}(k)) &= \det(\xi_i I - \mathcal{A})\det(\xi_i I - \mathcal{A}_c(k) \\ &\quad - \mathcal{B}_c\mathcal{C}(\xi_i I - \mathcal{A})^{-1}\mathcal{B}\theta^T(k)) \\ &= \det(\xi_i I - \mathcal{A})\det(\xi_i I - \mathcal{A}_c(k))\end{aligned} \tag{D6}$$

Next, consider the change of basis

$$\begin{bmatrix} \tilde{\mathcal{A}}(k) & \mathcal{B}\theta^T(k) \\ 0 & \mathcal{A}_{\text{nil}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\tilde{\mathcal{A}}_{\text{cl}}(k)\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

which implies that, for all $k \ge n_c$, and, for all $i = 1,\ldots,n_u$,

$$\begin{aligned}\det(\xi_i I - \tilde{\mathcal{A}}_{\text{cl}}(k)) &= \det(\xi_i I - \tilde{\mathcal{A}}(k))\det(\xi_i I - \mathcal{A}_{\text{nil}}) \\ &= \xi_i^{2n_c+1}\det(\xi_i I - \tilde{\mathcal{A}}(k))\end{aligned} \tag{D7}$$

Combining Eqs. (D3), (D6), and (D7) yields

$$\det(\xi_i I - \tilde{\mathcal{A}}(k)) = \xi_i^{-n_c} M(\xi_i, k)\det(\xi_i I - \mathcal{A})$$

and it follows from Assumption D3 that, for all $k \ge k_1$, and, for all $i = 1,\ldots,n_u$,

$$\begin{aligned}|\det(\xi_i I - \tilde{\mathcal{A}}(k))| &= |\xi_i^{-n_c}||M(\xi_i, k)||\det(\xi_i I - \mathcal{A})| \\ &\ge \epsilon|\xi_i^{-n_c}||\det(\xi_i I - \mathcal{A})|\end{aligned} \tag{D8}$$

Since $\min_{i=1,\ldots,n_u}|\xi_i^{-n_c}| > 0$ and $\min_{i=1,\ldots,n_u}|\det(\xi_i I_{2n_c+1} - \mathcal{A})| > 0$, it follows from Eq. (8) that, for all $k \ge k_1$, and, for all $i = 1,\ldots,n_u$,

$$0 < \epsilon_2 \le |\det(\xi_i I - \tilde{\mathcal{A}}(k))| \tag{D9}$$

where

$$\epsilon_2 \triangleq \epsilon\left(\min_{i=1,\ldots,n_u}|\xi_i^{-n_c}|\right)\left(\min_{i=1,\ldots,n_u}|\det(\xi_i I_{2n_c+1} - \mathcal{A})|\right) > 0$$

Next, write $\beta_u(\mathbf{q})$ as

$$\beta_u(\mathbf{q}) = \beta_{u,0}\mathbf{q}^{n_u} + \beta_{u,1}\mathbf{q}^{n_u-1} + \cdots + \beta_{u,n_u-1}\mathbf{q} + \beta_{u,n_u}$$

where $\beta_{u,0} = 1$ and $\beta_{u,1},\ldots,\beta_{u,n_u} \in \mathbb{R}$. Furthermore, for $i \ge 0$ and $j \ge i$, define

$$\Lambda_{i,j}(k) \triangleq \begin{cases} \tilde{\mathcal{A}}(k-d-i-1)\tilde{\mathcal{A}}(k-d-i-2)\cdots\tilde{\mathcal{A}}(k-d-j), & j > i, \\ I, & j = i \end{cases} \tag{D10}$$

and define

$$\Omega(k) \triangleq \sum_{i=0}^{n_u}\beta_{u,i}\Lambda_{i,n_u}(k) \tag{D11}$$

It follows from Eqs. (D10) and (D11) that

$$\begin{aligned}\Omega(k)\phi(k-d-n_u) &= \beta_{u,0}\tilde{\mathcal{A}}(k-d-1)\cdots\tilde{\mathcal{A}}(k-d-n_u)\phi(k-d-n_u) \\ &\quad + \beta_{u,1}\tilde{\mathcal{A}}(k-d-2)\cdots\tilde{\mathcal{A}}(k-d-n_u)\phi(k-d-n_u) \\ &\quad + \beta_{u,2}\tilde{\mathcal{A}}(k-d-3)\cdots\tilde{\mathcal{A}}(k-d-n_u)\phi(k-d-n_u) \\ &\quad + \cdots + \beta_{u,n_u-1}\tilde{\mathcal{A}}(k-d-n_u)\phi(k-d-n_u) \\ &\quad + \beta_{u,n_u}\phi(k-d-n_u)\end{aligned} \tag{D12}$$

Next, repeatedly substituting Eq. (D4) into Eq. (D12), and using Eqs. (10) and (D10) yields

$$\Omega(k)\phi(k-d-n_u) = \frac{1}{\beta_d}\Phi(k) - \sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}\beta_{u,i}\Lambda_{i,j}(k) \times [\mathcal{D}_1\psi_w(k-d-1-j) + \mathcal{D}_r r_f(k-d-j)] \tag{D13}$$

Now, show that there exist $k_2 \geq 0$ and $\bar{\epsilon} > 0$ such that, for all $k \geq k_2$, $\Omega(k)$ is nonsingular and $\bar{\epsilon} \leq |\det\Omega(k)|$. First, note that, for all $i = 0, \ldots, n_u - 1$,

$$\begin{aligned}\Lambda_{i,n_u}(k) &= \tilde{\mathcal{A}}(k-d-1-i)\tilde{\mathcal{A}}(k-d-2-i)\cdots\tilde{\mathcal{A}}(k-d-n_u)\\ &= \tilde{\mathcal{A}}(k-d-1-i)\cdots\tilde{\mathcal{A}}(k-d-n_u+2)\tilde{\mathcal{A}}^2(k-d-n_u)\\ &\quad + \tilde{\mathcal{A}}(k-d-1-i)\cdots\tilde{\mathcal{A}}(k-d-n_u+2)\\ &\quad \times [\tilde{\mathcal{A}}(k-d-n_u+1) - \tilde{\mathcal{A}}(k-d-n_u)]\tilde{\mathcal{A}}(k-d-n_u)\\ &= \Lambda_{i,n_u-2}(k)\tilde{\mathcal{A}}^2(k-d-n_u)\\ &\quad + \Lambda_{i,n_u-2}(k)\Delta(k-d-n_u+1)\Lambda_{n_u-1,n_u}(k)\end{aligned} \tag{D14}$$

where, for all $k > n_c$,

$$\Delta(k) \triangleq \tilde{\mathcal{A}}(k) - \tilde{\mathcal{A}}(k-1) = \mathcal{B}[\theta(k) - \theta(k-1)]^T \tag{D15}$$

Therefore, repeating the process in Eq. (D14) yields

$$\Lambda_{i,n_u}(k) = \tilde{\mathcal{A}}^{n_u-i}(k-d-n_u) + \sum_{j=1}^{n_u-i-1}\Lambda_{i,n_u-1-j}(k)\Delta(k-d-n_u+j)\Lambda_{n_u-j,n_u}(k) \tag{D16}$$

It follows from Eqs. (D11) and (D16) that $\Omega(k) = \Omega_1(k) + \Omega_2(k)$ where

$$\Omega_1(k) \triangleq \sum_{i=0}^{n_u}\beta_{u,i}\tilde{\mathcal{A}}^{n_u-i}(k-d-n_u) \tag{D17}$$

$$\Omega_2(k) \triangleq \sum_{i=0}^{n_u}\beta_{u,i}\left[\sum_{j=1}^{n_u-i-1}\Lambda_{i,n_u-1-j}(k)\Delta(k-d-n_u+j)\Lambda_{n_u-j,n_u}(k)\right] \tag{D18}$$

It follows from Assumption D2 that $\lim_{k\to\infty}\Delta(k) = 0$. Since, in addition, $\theta(k)$ is bounded according to Assumption D1, it follows that for all $i \geq 0$ and for all $j \geq i$, $\Lambda_{i,j}(k)$ is bounded. Therefore, $\lim_{k\to\infty}\Omega_2(k) = 0$, which implies that there exists $\bar{k}_1 \geq 0$, such that, for all $k \geq \bar{k}_1$,

$$\frac{1}{2}|\det\Omega_1(k)| \leq |\det(\Omega_1(k) + \Omega_2(k))| = |\det\Omega(k)| \tag{D19}$$

Next, note that $\Omega_1(k) = (-1)^{n_u}(\xi_1 I - \tilde{\mathcal{A}}(k-d-n_u))\cdots(\xi_{n_u}I - \tilde{\mathcal{A}}(k-d-n_u))$. Therefore, it follows from Eq. (D9) that, for all $k \geq k_1 + d + n_u$,

$$0 < \epsilon_2^{n_u} \leq |\det\Omega_1(k)| \tag{D20}$$

Combining Eqs. (D19) and (D20) implies that, for all $k \geq k_2 \triangleq \max(\bar{k}_1, k_1 + d + n_u)$,

$$0 < \bar{\epsilon} \leq |\det\Omega(k)| \tag{D21}$$

where $\bar{\epsilon} \triangleq \frac{1}{2}\epsilon_2^{n_u}$, and thus $\Omega(k)$ is nonsingular.

Since, for all $k \geq k_2$, $\Omega(k)$ is nonsingular, it follows from Eq. (D13) that, for all $k \geq k_2$,

$$\begin{aligned}\phi(k-d-n_u) &= \frac{1}{\beta_d}\Omega^{-1}(k)\Phi(k)\\ &\quad - \Omega^{-1}(k)\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}\beta_{u,i}\Lambda_{i,j}(k)\mathcal{D}_r r_f(k-d-j)\\ &\quad - \Omega^{-1}(k)\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}\beta_{u,i}\Lambda_{i,j}(k)\mathcal{D}_1\psi_w(k-d-1-j)\end{aligned} \tag{D22}$$

For all $N = 0, \ldots, n_u$, substituting Eq. (D22) into Eq. (D4) $n_u - N$ times implies

$$\begin{aligned}\phi(k-d-N) &= \frac{1}{\beta_d}\Lambda_{N,n_u}(k)\Omega^{-1}(k)\Phi(k)\\ &\quad - \Lambda_{N,n_u}(k)\Omega^{-1}(k)\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}\beta_{u,i}\Lambda_{i,j}(k)\mathcal{D}_r r_f(k-d-j)\\ &\quad - \Lambda_{N,n_u}(k)\Omega^{-1}(k)\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}\beta_{u,i}\Lambda_{i,j}(k)\mathcal{D}_1\psi_w(k-d-1-j)\\ &\quad + \sum_{i=1}^{n_u-N}\Lambda_{N,n_u-i}(k)[\mathcal{D}_1\psi_w(k-d-n_u+i)\\ &\quad + \mathcal{D}_r r_f(k+1-d-n_u+i)]\end{aligned}$$

which implies

$$\begin{aligned}\|\phi(k-d-N)\| &\leq \frac{1}{|\beta_d|}\frac{\|\Omega^A(k)\|}{|\det\Omega(k)|}\|\Lambda_{N,n_u}(k)\|\|\Phi(k)\|\\ &\quad + \frac{\|\Omega^A(k)\|}{|\det\Omega(k)|}\|\Lambda_{N,n_u}(k)\|\|\mathcal{D}_r\|\\ &\quad \times\left(\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}|\beta_{u,i}|\|\Lambda_{i,j}(k)\||r_f(k-d-j)|\right)\\ &\quad + \frac{\|\Omega^A(k)\|}{|\det\Omega(k)|}\|\Lambda_{N,n_u}(k)\|\|\mathcal{D}_1\|\\ &\quad \times\left(\sum_{i=0}^{n_u-1}\sum_{j=i}^{n_u-1}|\beta_{u,i}|\|\Lambda_{i,j}(k)\|\|\psi_w(k-d-1-j)\|\right)\\ &\quad + \|\mathcal{D}_1\|\left(\sum_{i=1}^{n_u-N}\|\Lambda_{N,n_u-i}(k)\|\|\psi_w(k-d-n_u+i)\|\right)\\ &\quad + \|\mathcal{D}_r\|\left(\sum_{i=1}^{n_u-N}\|\Lambda_{N,n_u-i}(k)\||r_f(k+1-d-n_u+i)|\right)\end{aligned} \tag{D23}$$

where $\|\cdot\|$ is the Euclidean norm for vectors and the corresponding induced norm for matrices, and $\Omega^A(k)$ is the adjugate of $\Omega(k)$.

Since $\theta(k)$ is bounded, it follows that $\Omega(k)$ is bounded, and thus $\Omega^A(k)$ is bounded, which implies that $c_\Omega \triangleq \sup_{k\geq 0}\|\Omega^A(k)\|$ exists. Furthermore, since $\theta(k)$ is bounded, it follows that, for all $i \geq 0$, and, for all $j \geq i$, $\Lambda_{i,j}(k)$ is bounded, and thus $c_\Lambda \triangleq \sup_{i,j=0,\ldots,n_u}(\sup_{k\geq 0}\|\Lambda_{i,j}(k)\|)$ exists. In addition, since $\psi_w(k)$ and $r(k)$ are bounded, it follows that $c_\psi \triangleq \sup_{k\geq 0}\|\psi_w(k)\|$ exists and $c_r \triangleq \sup_{k\geq 0}|r_f(k)|$ exists. Therefore, it follows from Eqs. (D21) and (D23) that

$$\begin{aligned}\|\phi(k-d-N)\| &\leq \frac{c_\Omega c_\Lambda}{\bar{\epsilon}|\beta_d|}\|\Phi(k)\|\\ &\quad + (n_u - N)(\|\mathcal{D}_1\|c_\psi + \|\mathcal{D}_r\|c_r)c_\Lambda\\ &\quad + \frac{c_\Omega(\|\mathcal{D}_1\|c_\psi + \|\mathcal{D}_r\|c_r)c_\Lambda^2}{\bar{\epsilon}}\left(\sum_{i=0}^{n_u-1}(n_u - i)|\beta_{u,i}|\right)\\ &\leq c_1 + c_2\|\Phi(k)\|\end{aligned}$$

where

$$c_1 \triangleq (n_u - N)(\|\mathcal{D}_1\|c_\psi + \|\mathcal{D}_r\|c_r)c_\Lambda + \frac{c_\Omega(\|\mathcal{D}_1\|c_\psi + \|\mathcal{D}_r\|c_r)c_\Lambda^2}{\bar{\epsilon}}\left(\sum_{i=0}^{n_u-1}(n_u - i)|\beta_{u,i}|\right)$$

and

$$c_2 \triangleq \frac{c_\Omega c_\Lambda}{\bar{\epsilon}|\beta_d|}$$

□

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