LEAST SQUARES LINEARIZATION OF A NONLINEAR PERMANENT MAGNET SYNCHRONOUS MOTOR

F. Belkhouche, U. Qidwai and B. Belkhouche

EECS Depatment, Tulane University, New Orleans, Louisiana, 70118, USA
email: belkhouf@eecs.tulane.edu

ABSTRACT

This paper presents a trajectory-based linearized model for a permanent magnet synchronous motor. The linearized model is derived using an optimization technique, when minimizing a cost function along a particular solution. Numerical experiments show good agreement of the method with the nonlinear system for both autonomous and non-autonomous cases. It turns out that unlike the classical linearization, the linearization method used here is not a first order approximation, which allows to include higher order terms and improve the quality of the approximatiom.

1. INTRODUCTION

Since the 1970's there has been a considerable development in studying the dynamic characteristic of various types of motors. In fact industrial applications has made an extensive use of all types of motors and many non-linear control techniques such as feedback linearization, input/output linearization were applied to motor control.

Since mathematical tools and algorithmic techniques for linear systems are very well established and can be implemented in real-time scenarios, there has been a constant interest in linearizing the non-linear behavior of a system.

An important concept in both control theory and power engineering is the concept of stability. The stability has two different aspects, a local aspect which deals with the equilibria of the model and a global aspect. The most classical techniques used for studying the asymptotic stability are based on Lyapunov functions, and linearization methods. It is well known that the mathematical models of most motors are non-linear and coupled. This results in some difficulty when studying the stability of the model describing the motor. In fact, with the difficulty of the construction of Lyapunov functions, linearization methods seem to be more appropriate for the determination of the local stability.

The most classical linearization method is based on Fréchet derivative at the equilibrium point. Unfortunately, this method fails when the equilibrium is non-hyperbolic. For example it fails to give an answer about the local stability of non-linear vector fields at a bifurcation point.

Permanent-magnet synchronous motors $(PMSMs)$ are studied and analyzed heavily in the scientific community due to the advantages and enormous industrial applications. Non-linear behaviors such as bifurcation and chaos in a non-linear model of $(PMSM)$ are investigated in [1]. In [2] the implementation of a rotor position estimator for a slotless $(PMSM)$ is discussed. In this paper, our objective is to use an optimal trajectory-based least squares linearization method to derive an equivalent linear model for a permanent magnet synchronous motor. In fact it was shown [4] that optimal linearization methods present some advantages over the classical linearization, and can be used in cases where the classical linearization fails (the case of non-hyperbolic equilibria is a good example).

2. MODEL FOR THE MOTOR

Permanent-magnet synchronous motors present high power efficiency, and they are widely used for high performance servo application because they present smooth torque. The model for the $(PMSM)$ is non-linear. In this paper, it is considered that the motor is modeled based on the $d-q$ frames [1] as follows

$$\left\{\begin{array}{l} \frac{di_d}{dt} = -\frac{R_S}{L_d}i_d + \frac{L_q}{L_d}\omega i_q + \frac{U_d}{L_d} \\ \frac{di_q}{dt} = -\frac{R_S}{L_q}i_q - \frac{\psi_r}{L_q}\omega - \frac{L_d}{L_q}\omega i_d + \frac{U_d}{L_q} \\ \frac{d\omega}{dt} = \frac{n\psi_r}{J}i_q + \frac{n}{J}i_q i_d\left[L_d - L_q\right] - \frac{\beta}{J}\omega - \frac{T_L}{J} \end{array}\right. \quad (1)$$

Where $i = [i_d, i_q]^T \in I\!R^2$ is the current vector in the $d-q$ frames ($[\cdot]^T$ denotes the transpose matrix), ω is the motor angular velocity, and $u = [U_d, U_q]^T \in I\!R^2$ is the stator vector voltage in the $d-q$ frames, R_S is the stator winding resistance, $L = [L_d, L_q]^T$ is the stator inductance vector in the $d-q$ frames; T_L is the external load torque. β is the viscous damping coefficient. ψ_r is the permanent magnet flux and n is the number of pole-pairs. For simplicity we put

$$\begin{array}{l} b_{11} = \frac{R_S}{L_d}; m_1 = \frac{L_q}{L_d}; b_{22} = \frac{R_S}{L_q}; b_{23} = \frac{\psi_r}{L_q} \\ m_2 = \frac{L_d}{L_q}; b_{32} = \frac{n\psi_r}{J}; m_3 = \frac{n}{J}\left[L_d - L_q\right] \\ b_{33} = \frac{\beta}{J}; u_1 = \frac{U_d}{L_d}; u_2 = \frac{U_d}{L_q}; u_3 = \frac{T_L}{J} \end{array} \quad (2)$$

and $x = [i_d, i_q, \omega]^T$, where $x(t) = [x_1(t), x_2(t), x_3(t)]^T$ is the state vector. Equation (1) becomes

$$\left\{\begin{array}{l} \frac{dx_1}{dt} = -b_{11}x_1 + m_1x_2x_3 + u_1 \\ \frac{dx_2}{dt} = -b_{22}x_2 - b_{23}x_3 - m_2x_1x_3 + u_2 \\ \frac{dx_3}{dt} = b_{32}x_2 - b_{33}x_3 + m_3x_1x_2 - u_3 \end{array}\right. \quad (3)$$

The model presents three non-linearities which result from the coupling of the state variables. We put fot the initial conditions $x_0 = [i_d(t_0), i_q(t_0), \omega(t_0)]^T$. The motor's state equation model has the following form

$$\left\{\begin{array}{l} \frac{dx}{dt} = F(x(t), u(t)) \\ x(t_0) = x_0 \end{array}\right. \quad (4)$$

where $F = [f_1, f_2, f_3]^T, u(t) = [u_1, u_2, u_3]^T$.

3. DERIVATION OF THE LINEARIZED MODEL

The objective in this section is to derive a linear system which locally describes the dynamics of the non-linear model of the motor. The linear system has the following form

$$\left\{ \begin{array}{l} \frac{dx}{dt} = Ax(t) + Bu(t) \\ x(t_0) = x_0 \end{array} \right. \tag{5}$$

one advantage of replacing a non-linear system by a linear system is that the solution of the linear system is known in closed form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} BU(\tau)\, d\tau \tag{6}$$

which is rarely the case for non-linear systems, furthermore it is easier to design a controller for the equivalent linear system. Our approach is based on a simple least square method ([3],[4]), where a given functional is minimized along a particular trajectory. Consider the following cost function

$$\int_0^{+\infty} \left\| F(x(t), u(t)) - Ax(t) - Bu(t) \right\|^2 dt \tag{7}$$

The minimization of (7) along the given trajectory allows to find the equivalent linear model, for this particular case, since the state and the control are separable, the non-linear function F can be written as

$$F(x(t), u(t)) = G(x(t)) + Bu(t) \tag{8}$$

where the matrix B is given by

$$B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \tag{9}$$

and since the only unknown in (7) is the matrix A, the functional to be minimized is simplified to

$$\Phi(A) = \int_0^{+\infty} \left\| G(x(t)) - Ax(t) \right\|^2 dt \tag{10}$$

the best approximation is obtained for

$$\frac{\partial \Phi}{\partial A}\delta = 2\int_0^{+\infty} \left\langle G(x(t)) - Ax(t), \delta x(t) \right\rangle dt = 0$$

where the matrix δ is defined as

$$\left\{ \begin{array}{ll} \delta_{ij} = 0 & i \neq j \\ \delta_{ij} = 1 & i = j \end{array} \right. \tag{11}$$

in this case we have

$$\begin{aligned} & \int_0^{+\infty} \left\langle G(x(t)) - Ax(t), \delta x(t) \right\rangle dt \\ = & \int_0^{+\infty} \left[G(x(t)) - Ax(t) \right] [x(t)]^T dt \end{aligned} \tag{12}$$

and the solution for the matrix A is given by

$$A = \left[\int_0^{+\infty} \left[G(x(t)) \right] [x(t)]^T dt \right] [M]^{-1} \tag{13}$$

with

$$M = \left[\int_0^{+\infty} [x(t)]\, [x(t)]^T dt \right]$$

Note that the existence and uniqueness of the solutions for the matrix A depend on M, in fact if M is non-singular, then the matrix A exists and is unique.

4. SOME PROPERTIES OF THE APPROXIMATION

Here, we give briefly the main properties of the linearization method, first observe that the motor's non-linear model is a sum of linear and non-linear terms, and $G(x)$ can be written under the following form

$$G(x(t)) = DG(0)\, x(t) + G_2(x(t)) \tag{14}$$

where $DG(0)$ is the Jacobian matrix at the origin

$$DG(0) = \left[\begin{array}{ccc} -b_{11} & 0 & 0 \\ 0 & -b_{22} & -b_{23} \\ 0 & b_{32} & -b_{33} \end{array} \right] \tag{15}$$

and

$$G_2(x(t)) = \left[\begin{array}{c} m_1 x_2 x_3, -m_2 x_1 x_3, m_3 x_1 x_2 \end{array} \right]^T \tag{16}$$

by applying equation (13) to (14), we get (for simplicity x is used instead of $x(t)$)

$$\begin{aligned} A &= \left[\int_0^{+\infty} \left[DG(0)\, x + G_2(x) \right] [x]^T dt \right] [M]^{-1} \\ &= \left[\int_0^{+\infty} \left[DG(0)\, x \right] [x]^T dt \right] [M]^{-1} \\ &\quad + \left[\int_0^{+\infty} G_2(x)\, [x]^T dt \right] [M]^{-1} \qquad (17) \\ &= DG(0) + A_{G_2} \qquad (18) \end{aligned}$$

where A_{G_2} depends on higher order terms, this shows the approximation as a sum of two terms, one is associated to the linear part and the other is associated to the non-linear part of the non-linear model. As a result of this property, if the classical linearization vanishes at the origin $DG(0) = 0$, the matrix A will exist and will depend on higher order terms.

Another important property of the approximation is its convergence. In fact the convergence of the solution for the matrix A requires the solution $x(t)$ of the initial value problem to be bounded. A strong condition for the boundedness of the solutions is obtained when $DG(x_0)$ has negative spectrum, this condition is satisfied near the origin for the model desribing the $(PMSM)$.

It is also possible to compute the matrix A when the solutions for all the state variables of the initial value problem are unbounded. This can be performed by simply considering the backward evolution of the time in the initial value problem. An associated linear system is computed $\frac{dx}{dt} = A^*x, x(t_0) = x_0$ and the matrix A is obtained form the associated linear system by putting $A = -A^*$.

5. NUMERICAL IMPLEMENTATION OF THE PROCEDURE

For the numerical implementation of the procedure, we generalize the algorithm suggested for autonomous systems $u(t) = 0$ in [4] to the case of the motor where $u \neq 0$. The approach consists of an iterative method for calculating the series A_j, where A_j is the matrix associated to the non-linear motor's model at iteration j. For each iteration the linear model is obtained by minimizing the functional

$$\Phi\left(A_j\right) = \int_0^{+\infty} \left\| G\left(x\left(t\right)\right) - A_j x\left(t\right) \right\|^2 dt \tag{19}$$

and replacing $x(t)$ by the solution of linearized system at the previous step

$$x_{j-1}\left(t\right) = e^{A_{j-1}t}x_0 + \int_0^t e^{A_{j-1}(t-\tau)} BU\left(\tau\right) d\tau \tag{20}$$

we get the matrix A for the step j

$$A_j = \left[\int_0^{+\infty} \left[G\left(x_{j-1}\left(t\right)\right)\right] \left[x_{j-1}\left(t\right)\right]^T dt \right] \left[M\right]^{-1} \tag{21}$$

with

$$M = \left[\int_0^{+\infty} \left[x_{j-1}\left(t\right)\right] \left[x_{j-1}\left(t\right)\right]^T dt \right] \tag{22}$$

This equation describes also the relationship between A_j and A_{j-1}. The initial guess matrix A_0 can be any matrix with negative spectrum. Appropriate choices are $DG(0)$ or $DG\left(x_0\right)$. For the numerical implementation of the method, we consider the following numerical values for the motor's parameters $R_S = 2.5\Omega$, $L = \left[0.105, 0.09\right]^T H$ $\beta = 0.5$, $\psi_r = 0.5v/s$, $n = 2$, $J = 0.005kgm^2$. We provide a comparison between the method described here and the classical linearization quantitatively in terms of the error due to the approximations and qualitatively in terms of the stability type of the equilibrium. Both the autonomous and the non-autonomous cases are considered.

5.1. Autonomous case

Here we consider the non-linear model of the $(PMSM)$ without external inputs, for the numerical values described previously, the non-linear autonomous system is the following

$$\left\{ \begin{array}{l} \frac{di_d}{dt} = -23.8095 i_d + 1.1667 \omega i_q \\ \frac{di_q}{dt} = -27.7778 i_q - 2.3810 \omega - 0.8571 \omega i_d \\ \frac{d\omega}{dt} = 100 i_q + 6 i_q i_d - 100 \omega \end{array} \right. \tag{23}$$

with $\left[i_d(t_0), i_q(t_0), \omega(t_0)\right]^T = \left[2, 3, 0.4\right]^T$ the initial state. We suggest to linearize system (23) in the neighborhood of the origin by the classical and the least square linearization methods, recall that the classical linearization is given by

$$\frac{dx}{dt} = DG\left(0\right) x\left(t\right) \tag{24}$$

where $DG\left(0\right)$ is the same as in (15), and the classical linearized system is the following

$$\left\{ \begin{array}{l} \frac{di_d}{dt} = -23.8095 i_d \\ \frac{di_q}{dt} = -27.7778 i_q - 2.3810 \omega \\ \frac{d\omega}{dt} = 100 i_q - 100 \omega \end{array} \right. \tag{25}$$

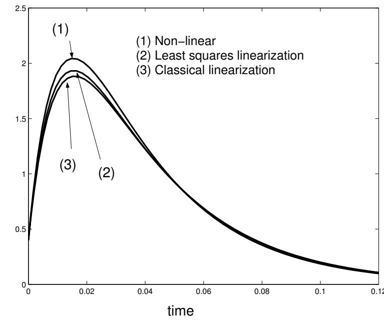

Figure 1: Time evolution for the angular velocity

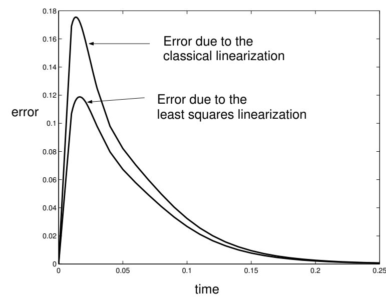

Figure 2: Absolute error due to the approximation

The linearization in the least square sense gives the following system

$$\left\{ \begin{array}{l} \frac{di_d}{dt} = 23.9396 i_d + 0.5669 i_q - 0.1019 \omega \\ \frac{di_q}{dt} = 0.1284 i_d - 28.0808 i_q - 2.4648 \omega \\ \frac{d\omega}{dt} = -0.6921 i_d + 113.4843 i_q - 107.6720 \omega \\ \left[i_d(t_0), i_q(t_0), \omega(t_0)\right]^T = \left[2, 3, 0.4\right]^T \end{array} \right. \tag{26}$$

Both matrices A and $DG\left(0\right)$ have eigenvalues with negative real parts, which implies the asymptotic stability of the origin, furthermore we observe that the elements of $DG\left(0\right)$ that are equal to zero (namely $b_{12}, b_{13}, b_{21}, b_{31},$) are replaced by small terms in the matrix A (namely $a_{12} = 0.5669, a_{13} = -0.1019, a_{21} = 0.1284, a_{31} = -0.6921$). These terms are due to the presence of the non-linearities. Unlike the classical linearization, the least square linearization is not a first order approximation, this can be seen also from equation (18).

The solution for $\omega\left(t\right)$ is plotted in figure 1, and the absolute error $\left\|x_{linear}\left(t\right) - x_{non-linear}\left(t\right)\right\|$ is depicted in figure 2, from which we observe the following remarks

1. The error goes to zero uniformly when the solution goes to its steady state.

2. The error due the linearization by system (26) is smaller than the error due to the classical linearization (25) during the transient time.

5.2. Non-autonomous case

In this section, a non-autonomous system with constant input voltage and constant external torque is considered. The following initial conditions $\left[i_d(t_0), i_q(t_0), \omega(t_0)\right]^T = \left[0.2, 0.3, 0.4\right]^T$ and external control $u = \left[50, 50, 20\right]^T$ are used. Our aim is to compute

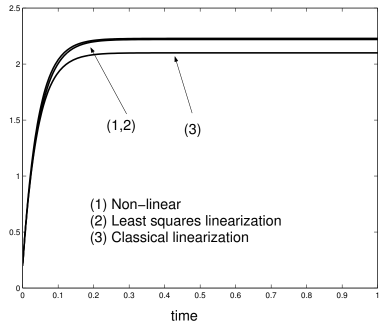

Figure 3: Time evolution of the current in frame d

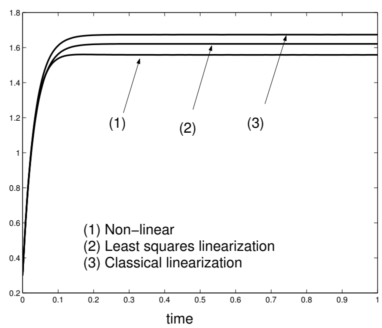

Figure 4: Time evolution of the current in frame q

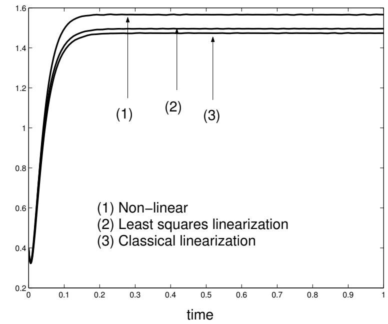

Figure 5: Time evolution of the angular velocity

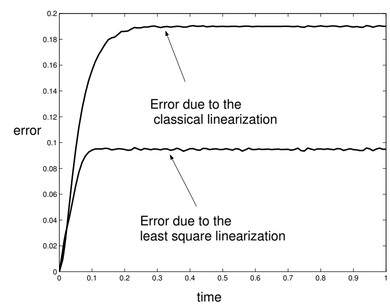

Figure 6: Error due to the approximations

a linear approximation for the non-linear model of the $(PMSM)$ in the presence of the external control.
For the initial state given above, the matrix A is the following

$$A = \begin{bmatrix} -23.8268 & 5.1395 & -3.4960 \\ 0.0150 & -30.1122 & -0.8091 \\ -0.0732 & 112.4109 & -108.3716 \end{bmatrix}$$

Figures 3, 4 and 5 show the solution for $i_d(t)$, $i_q(t)$ and $\omega(t)$. Note the presence of a steady state error for both linearizations, as shown in table 1, this error is smaller for the least square linearization for the three state variables. Figure 6 shows the absolute error due to the approximations.

	Classical method	Least square
i_d	0.1196	0.0092
i_q	0.1152	0.0629
ω	0.0925	0.0704

table 1.
Note that the least square approximation can be used in the critical case when the classical linearization presents a zero eigenvalue. The classical linearized system is not equivalent to the non-linear model of the motor, and it can be proved numerically that even in this case, the linearization in the least square sense presents good agreement with the non-linear system qualitatively.

6. CONCLUSION

A linearized model for the non-linear system modeling a permanent-magnet synchronous motor $(PMSM)$ has been presented. The procedure is easily implimentable numerically and shows good agreement with the non-linear model. Furthermore a comparison with the classical linearization in terms of the error shows good performance of the approximation. In fact for the autonomous case, the error goes uniformly to zero when time goes to infinity. For the non-autonomous case, there exist a steady state error which is smaller than the steady state error for the classical linearization. The main difference between the classical and the least square methods resides in the order of the approximation, since the least square linearization is not a first order approximation and higher order terms are included.

7. REFERENCES

[1] Z. Li, J.B. Park, Y. H. Joo, B. Zhang and G. Chen, "Bifurcation and Chaos in a Permanent-Magnet Synchronous Motor". IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, Vol. 49, No. 3, pp. 383-387, March 2002.

[2] T.D. Batzel and K.Y. Lee, "Slotelless Permanent Magnet Synchronous Motor Operation without a High Resolution Rotor Angle Sensor". IEEE Transaction on Energy Conversion, Vol. 15, No. 5, pp.366-371 December 2000.

[3] A. Jordan, M. Benmouna, A.Bensnane and A. Borucki, "Optimal Linearization Method Applied to Resolution of State Equation", RAIRO APII 21, pp. 175-185, 1987.

[4] T. Benouaz and O. Arino, "Least Square Approximation of a Non-linear Ordinary Differntial Equation", Computers Math. Applic. Vol. 31, No. 8, pp. 69-84, 1996.