ISA Transactions 50 (2011) 268–276

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

An analytical method for PID controller tuning with specified gain and phase margins for integral plus time delay processes

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ARTICLE INFO

Article history:
Received 3 May 2010
Received in revised form
3 December 2010
Accepted 3 January 2011
Available online 1 February 2011

Keywords:
IPTD processes
PI/PD/PID controller tuning
Gain margin
Phase margin
Crossover frequencies

ABSTRACT

In this paper, an analytical method is proposed for proportional–integral/proportional–derivative/proportional–integral–derivative (PI/PD/PID) controller tuning with specified gain and phase margins (GPMs) for integral plus time delay (IPTD) processes. Explicit formulas are also obtained for estimating the GPMs resulting from given PI/PD/PID controllers. The proposed method indicates a general form of the PID parameters and unifies a large number of existing rules as PI/PD/PID controller tuning with various GPM specifications. The GPMs realized by existing PID tuning rules are computed and documented as a reference for control engineers to tune the PID controllers.

1. Introduction

Proportional–integral–derivative (PID) control has been widely applied in industry—more than 90% of the applied controllers are PID controllers [1–4]. In the absence of the derivative action, proportional–integral (PI) control is also broadly deployed, since in many cases the derivative action cannot significantly enhance the performance or may not be appropriate for the noisy environment [1–4]. Another special form of PID control without the integral action, proportional–derivative (PD) control is also applied [1–4]. Unlike the previous two cases, however, PD control cannot achieve zero steady-state error subject to load disturbances, which limits its applications [1–4].

Due to the prevailing applications of PI/PD/PID control, research on tuning PI/PD/PID controllers has been of much interest in the past decades [1–6]. As a particular case, tuning PI/PD/PID controllers for integral plus time delay (IPTD) processes has attracted a lot of attention, dating back to 1940s and lasting even today [1,7–18]. Lots of results have been accumulated. There are more than fifty PI/PD/PID tuning rules for IPTD processes according to a survey made by O'Dwyer [1]. The actual number is even much higher [7–9,18,19]. Close observations reveal that many of these rules are sharing a common form. Such observations motivate our exploration of a general solution for the PI/PD/PID controller tuning on an IPTD process in this paper.

Tuning PI/PD/PID controllers based on gain and phase margin (GPM) specifications has been extensively studied in the literature [1,14,20–23]. However, general analytic solutions of the controller parameters are not available, because of the nonlinearity and solvability of such problems. The existing solutions are limited by assuming certain constraints on GPM or by approximations that are valid only for certain regions of process parameters [1,14,20–22]. This paper is devoted to solving the PI/PD/PID parameters for an IPTD process with a specified GPM. Different from the existing results, analytic solutions are obtained for the whole domain of the process parameters. The derived PI/PD/PID tuning formulas unify a large number of existing rules as PI/PD/PID controller tuning with various GPM specifications. As reverse solutions, explicit expressions of GPMs for given PI/PD/PID settings on an IPTD process are also obtained. These GPM formulas estimate GPMs with high accuracy and are applied to estimate the GPM attained by each relevant PI/PD/PID tuning rule collected in [1].

The rest of the paper is organized as follows. In Section 2, analytic expressions of the PI/PD/PID parameters with a specified GPM and the reverse solution of GPM for a given PI/PD/PID setting are derived. During the derivations, numerical evaluations are employed to validate any approximations involved. In Section 3, the derived PI/PD/PID formulas are applied to unify the existing rules as PI/PD/PID controller tuning with different GPM specifications, and the derived GPM formulas are applied to estimate the GPMs attained by the existing rules. Finally, Section 4 concludes the paper.

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doi:10.1016/j.isatra.2011.01.001

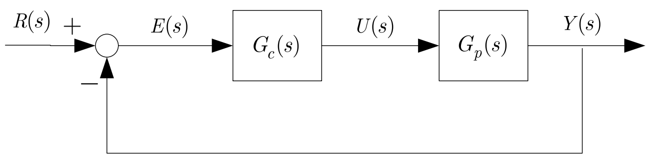

Fig. 1. Control system loop.

2. Derivation of the PI/PD/PID tuning formulas and the GPM formulas

The ideal unity-feedback control system is considered, as shown in Fig. 1, where $G_c(s)$ denotes a PI/PD/PID controller and $G_p(s)$ denotes an IPTD process. Specifically, the transfer functions are

$$G_p(s) = K_p e^{-\tau s}/s, \quad \tau > 0, \tag{1}$$

where K_p is the process gain and τ the time delay, and

$$G_c(s) = \begin{cases} K_c\left(1+\dfrac{1}{sT_i}\right), & \text{PI controller;} \\ K_c(1+T_d s), & \text{PD controller;} \\ K_c\left(1+\dfrac{1}{sT_i}+T_d s\right), & \text{PID controller,} \end{cases} \tag{2}$$

where K_c, T_i and T_d are the proportional, integral and derivative parameters, respectively. With this closed-loop system, explicit PI/PD/PID parameters are solved for achieving a given GPM. While the way of properly specifying a GPM depends on specific design requirements and demands extra studies, it is assumed that a GPM has been specified by the designer throughout the paper. Some brief discussions on this are given in the next section.

Although PI and PD controller tunings are special cases of PID controller tuning, their tuning formulas and corresponding GPM formulas are derived independently, adopting different approximations for accuracy and simplicity.

2.1. PI tuning formula and GPM-PI formula

Suppose the GPM of the closed-loop system is specified as (A_m, ϕ_m), where A_m denotes the gain margin and ϕ_m denotes the phase margin. Given a PI controller in (2), the PI parameters (K_c, T_i) are to be solved. According to the definition of GPM, we have

$$\arg[G(j\omega_p)] = -\pi + \arctan(\omega_p T_i) - \omega_p\tau = -\pi, \tag{3}$$

$$\frac{1}{A_m} = |G(j\omega_p)| = \frac{K_c K_p\sqrt{1+\omega_p^2 T_i^2}}{\omega_p^2 T_i}, \tag{4}$$

$$1 = |G(j\omega_g)| = \frac{K_c K_p\sqrt{1+\omega_g^2 T_i^2}}{\omega_g^2 T_i}, \tag{5}$$

$$\phi_m = \arg[G(j\omega_g)] + \pi = \arctan(\omega_g T_i) - \omega_g\tau, \tag{6}$$

where ω_p and ω_g are the phase and the gain crossover frequencies, respectively. Due to the nonlinearity of the equations, the four variables ω_g, ω_p, K_c and T_i are normally analytically unsolvable, preventing derivation of a general PI tuning formula [1]. By introducing two intermediate variables, however, all these four variables can be solved. Specifically, let $\alpha := \omega_g T_i$ and $\beta := \omega_p T_i$. From (3)–(6), the following solution is obtained:

$$\begin{cases} \omega_g = \dfrac{1}{\tau}(\arctan\alpha - \phi_m), \\ \omega_p = \dfrac{\arctan\beta}{\tau} = \dfrac{\beta}{\alpha}\omega_g, \\ K_c = \dfrac{\alpha\omega_g}{K_p\sqrt{1+\alpha^2}}, \\ T_i = \alpha/\omega_g, \end{cases} \tag{7}$$

where (α, β) is solved from

$$\begin{cases} \phi_m = \arctan\alpha - \dfrac{\alpha}{\beta}\arctan\beta, \\ A_m = \dfrac{\beta^2}{\alpha^2}\sqrt{\dfrac{1+\alpha^2}{1+\beta^2}}. \end{cases} \tag{8}$$

The solution (α, β) is a constant pair corresponding to a specified GPM which can easily be solved using a numerical solver, e.g., the solver 'fsolve' in Matlab. The solution is unique, if there is any, since $\alpha > \tan\phi_m$ and $\beta > 0$ which ensure positive crossover frequencies and PI parameters. The initial guess of (α, β) for the numerical solver can be any pair of large enough positive numbers, e.g., $(2\tan\phi_m, 2\tan\phi_m)$, $(5, 5)$ (as used in the later numeric tests), etc.

Therefore (7) gives explicit expressions of the PI parameters (K_c, T_i) in terms of the process parameters (K_p, τ). For convenience, (7) is called *PI tuning formula*. Note that the crossover frequencies ω_p and ω_g are also explicitly given in (7).

As an inverse problem, we compute the GPM resulting from a given PI controller for an IPTD process. Still based on (3)–(6), the expression of GPM, namely *GPM-PI formula*, is obtained as follows:

$$\begin{cases} \omega_g = \alpha/T_i, \\ \omega_p = \beta/T_i, \\ A_m = \dfrac{\beta^2}{\alpha^2}\sqrt{\dfrac{1+\alpha^2}{1+\beta^2}}, \\ \phi_m = \arctan\alpha - \omega_g\tau, \end{cases} \tag{9}$$

where

$$\alpha = \sqrt{\frac{\gamma^2}{2}\left(1+\sqrt{1+\frac{4}{\gamma^2}}\right)}, \quad \text{with } \gamma := K_p K_c T_i, \tag{10}$$

(the negative α is omitted) and β is solved from

$$\arctan\beta = \theta\beta, \quad \text{with } \theta := \tau/T_i. \tag{11}$$

Solution (9) also gives expressions of the gain and phase crossover frequencies. As indicated by the above equations, the phase margin ϕ_m is explicitly expressed; however, deriving the gain margin A_m requires first solving (11) for β. Although a numerical solution can be used, for ease of application an approximate analytic solution is proposed. According to Appendix A.1, such a solution is

$$\begin{cases} \beta = \dfrac{\pi}{4\theta}\left(1+\sqrt{1-\dfrac{16\lambda_B\theta}{\pi^2}}\right), & \text{if } 0 < \theta \le \theta_B, \\ \beta = \dfrac{1}{2\theta}\sqrt{-5+\sqrt{\dfrac{120}{\theta}-95}}, & \text{if } \theta_B < \theta < 1, \end{cases} \tag{12}$$

where $\lambda_B = 0.917$ and $\theta_B = 0.582$. The constraint $0 < \theta < 1$ is imposed to ensure a positive solution for β. With β given in (12), both A_m and ω_p in (9) are then explicitly expressed. The above solution of (α, β) meanwhile justifies the uniqueness of the solution to (8).

To evaluate the accuracy of (12) as the solution of (11), numeric tests are carried out. Without loss of generality, let $K_p = 1$. For different (τ, A_m, ϕ_m), the PI parameters are first calculated by the PI tuning formula. With these PI parameters, the realized GPMs are then estimated by the GPM-PI formula, using β's estimated by (12). The estimated GPMs are compared with the originally specified GPMs correspondingly, so that the accuracy of the approximations is tested. In the computation, the parameters are chosen randomly as $\tau \in (0, 1]$ (which loses no generality since the PI tuning formula and GPM-PI formula both apply regardless of the process parameters), $A_m \in (1, 12]$ and $\phi_m \in (10°, 70°]$. The numerical

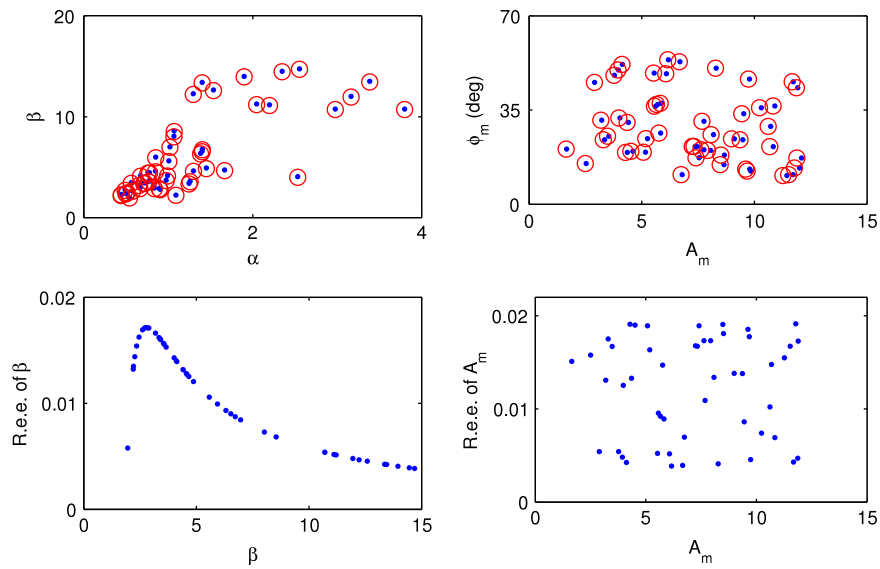

Fig. 2. GPMs estimated by GPM-PI formula versus true GPMs specified randomly a priori (50 tests), where the dots denote the estimated points and the circles denote the true points.

results are shown in Fig. 2, where the relative estimation error (R.e.e.) is defined as *R.e.e.* := *(the estimated value* − *the true value)/the true value*. Since α and ϕ_m are exactly derived by the GPM-PI formula, their estimation errors are omitted in the figure, as it remains the same for later discussions on PD and PID controls. The results indicate that the estimation errors of A_m's are normally within 2% and thus validate (9) adopting the approximate solution of β by (12).

2.2. PD tuning formula and GPM-PD formula

Given a specified GPM (A_m, ϕ_m), an IPTD process in (1) and a PD controller in (2), the PD parameters (K_c, T_d) are to be solved. The definition of GPM leads to

$$\arg[G(j\omega_p)] = -\pi/2 + \arctan\omega_p T_d - \omega_p\tau = -\pi, \tag{13}$$

$$\frac{1}{A_m} = |G(j\omega_p)| = K_c K_p\sqrt{1+\omega_p^2 T_d^2}/\omega_p, \tag{14}$$

$$1 = |G(j\omega_g)| = K_c K_p\sqrt{1+\omega_g^2 T_d^2}/\omega_g, \tag{15}$$

$$\phi_m = \arg[G(j\omega_g)] + \pi = \pi/2 + \arctan\omega_g T_d - \omega_g\tau, \tag{16}$$

where the variables are defined the same as those in Section 2.1. By introducing two new variables $\alpha' := \omega_g T_d$ and $\beta' := \omega_p T_d$ in a similar way to that for the PI case, the parameters are solved from (13)–(16) such that

$$\begin{cases} \omega_g = \dfrac{1}{\tau}\left(\arctan\alpha' + \dfrac{\pi}{2} - \phi_m\right), \\ \omega_p = \dfrac{1}{\tau}\left(\arctan\beta' + \dfrac{\pi}{2}\right) = \dfrac{\beta'}{\alpha'}\omega_g, \\ K_c = \dfrac{\omega_g}{K_p\sqrt{1+\alpha'^2}}, \\ T_d = \alpha'/\omega_g, \end{cases} \tag{17}$$

where the constant pair (α', β') is solved from the equations

$$\begin{cases} \phi_m = \arctan\alpha' + \dfrac{\pi}{2} - \dfrac{\alpha'}{\beta'}\left(\arctan\beta' + \dfrac{\pi}{2}\right), \\ A_m = \dfrac{\beta'}{\alpha'}\sqrt{\dfrac{1+\alpha'^2}{1+\beta'^2}}. \end{cases} \tag{18}$$

The solution (α', β') is unique since $\alpha' > 0$ and $\beta' > 0$ which make sure positive crossover frequencies and PD parameters. The initial guess of (α', β') for the numerical solver can be any pair of large enough positive numbers, e.g., (5, 5), (10, 10), etc. Therefore, (17) gives the *PD tuning formula*.

Inversely, given an IPTD process in (1) and a PD controller in (2), the resultant GPM and crossover frequencies of the closed-loop system are derived from (13)–(16) as

$$\begin{cases} \omega_g = \alpha'/T_d, \\ \omega_p = \beta'/T_d, \\ A_m = \dfrac{\beta'}{\alpha'}\sqrt{\dfrac{1+\alpha'^2}{1+\beta'^2}}, \\ \phi_m = \arctan\alpha' - \omega_g\tau + \pi/2, \end{cases} \tag{19}$$

where

$$\alpha' = \sqrt{\gamma'^2/(1-\gamma'^2)}, \quad \text{with } \gamma' := K_p K_c T_d, \tag{20}$$

and β' is solved from

$$\arctan\beta' = \theta'\beta' - \pi/2, \quad \text{with } \theta' := \tau/T_d. \tag{21}$$

Since deriving the gain margin requires solving β' from (21), an approximate analytic solution is proposed for it. Divide the domain of β' into two: $0 < \beta' \le 1$ (β' being small) and $\beta' > 1$ (β' being large). In the former domain, use the approximation

$$\arctan\beta' \approx \lambda\beta', \quad \text{with } \lambda := \pi/4, \tag{22}$$

and in the latter domain use the approximation

$$\arctan\beta' = \frac{\pi}{2} - \arctan\frac{\lambda}{\beta'} \approx \frac{\pi}{2} - \frac{\lambda}{\beta'}. \tag{23}$$

Solve (22) and (23), respectively, and express the applicable domains in terms of θ', an approximate solution of (21) is derived as

$$\begin{cases} \beta' = \dfrac{\pi}{2\theta'}\left(1+\sqrt{1-\dfrac{4\theta'\lambda}{\pi^2}}\right), & \text{if } 0 < \theta' \le \theta'_B, \\ \beta' = \dfrac{\pi}{2(\theta'-\lambda)}, & \text{if } \theta' > \theta'_B, \text{ where } \theta'_B := \pi/2 + \lambda. \end{cases} \tag{24}$$

Therefore, (19) gives the *GPM-PD formula*, where the intermediate variables α' and β' are expressed in (20) and (24), respectively.

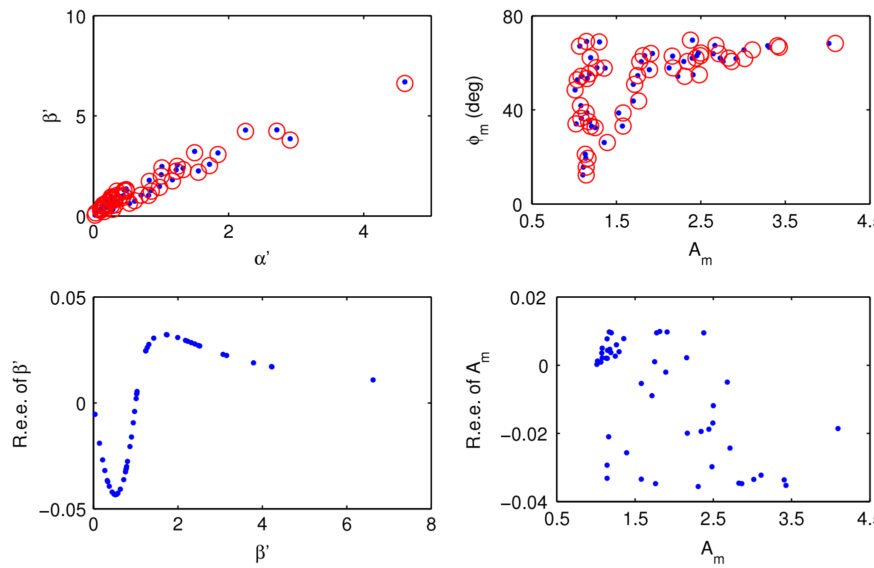

Fig. 3. GPMs estimated by GPM-PD formula versus true GPMs specified randomly a priori (50 tests), where the dots denote the estimated points and the circles denote the true points.

Meanwhile the solution of (α', β') justifies the uniqueness of the solution to (18) for a given GPM.

To evaluate the accuracy of (24) as a solution of (21), numeric computations are carried out to test it. The IPTD process parameters and the GPMs are specified in a similar way to those for the PI case (refer to Section 2.1). Analogously, the results are obtained and shown in Fig. 3, which demonstrate the accuracy of the GPM-PD formula adopting β' estimated by (24).

2.3. PID tuning formula and GPM-PID formula

Given a specified GPM (A_m, ϕ_m), an IPTD process in (1) and a PID controller in (2), the PID parameters (K_c, T_i, T_d) can be solved. The definition of GPM leads to

$$\begin{aligned} -\pi &= \arg[G(j\omega_p)] \\ &= -\pi + \arctan\frac{\omega_p T_i}{1-\omega_p^2 T_i T_d} + \mathcal{H}(1-\omega_p^2 T_i T_d)\pi - \omega_p\tau, \end{aligned} \tag{25}$$

$$\frac{1}{A_m} = |G(j\omega_p)| = \frac{K_c K_p\sqrt{(1-\omega_p^2 T_i T_d)^2 + \omega_p^2 T_i^2}}{\omega_p^2 T_i}, \tag{26}$$

$$1 = |G(j\omega_g)| = \frac{K_c K_p\sqrt{(1-\omega_g^2 T_i T_d)^2 + \omega_g^2 T_i^2}}{\omega_g^2 T_i}, \tag{27}$$

$$\begin{aligned} \phi_m &= \arg[G(j\omega_g)] + \pi \\ &= \arctan\frac{\omega_g T_i}{1-\omega_g^2 T_i T_d} + \mathcal{H}(1-\omega_g^2 T_i T_d)\pi - \omega_g\tau, \end{aligned} \tag{28}$$

where the function $\mathcal{H}(\bullet)$ is defined as

$$\mathcal{H}(t) := \begin{cases} 0, & \text{if } t \ge 0, \\ 1, & \text{if } t < 0. \end{cases} \tag{29}$$

Since there are five unknowns $(\omega_g, \omega_p, K_c, T_i, T_d)$, but only four equations, one additional condition is required for a unique solution. In the literature, normally it assumes that $T_d = kT_i$ and $k \in (0, 0.5]$ [1,2]. By defining α and β the same as those in Section 2.1, the parameters are solved from (25)–(28) such that

$$\begin{cases} \omega_g = \dfrac{1}{\tau}\left(\arctan\dfrac{\alpha}{1-k\alpha^2} + \mathcal{H}(1-k\alpha^2)\pi - \phi_m\right), \\ \omega_p = \dfrac{1}{\tau}\left(\arctan\dfrac{\beta}{1-k\beta^2} + \mathcal{H}(1-k\beta^2)\pi\right) = \dfrac{\beta}{\alpha}\omega_g, \\ K_c = \dfrac{\alpha\omega_g}{K_p\sqrt{(1-k\alpha^2)^2+\alpha^2}}, \\ T_i = \alpha/\omega_g, \\ T_d = kT_i \end{cases} \tag{30}$$

where (α, β) is solved from the following equations

$$\begin{cases} \phi_m = \arctan\dfrac{\alpha}{1-k\alpha^2} + \mathcal{H}(1-k\alpha^2)\pi \\ \quad - \dfrac{\alpha}{\beta}\left(\arctan\dfrac{\beta}{1-k\beta^2} + \mathcal{H}(1-k\beta^2)\pi\right), \\ A_m = \dfrac{\beta^2}{\alpha^2}\sqrt{\dfrac{(1-k\alpha^2)^2+\alpha^2}{(1-k\beta^2)^2+\beta^2}}. \end{cases} \tag{31}$$

The solution (α, β) is unique for ensuring positive crossover frequencies and PID parameters subject to a given k. This is justified by an explicit solution of (α, β) in terms of the PID parameters as presented later. The initial guess of (α, β) for a numerical solver to solve (31) can be any pair of large enough positive numbers, e.g., (5, 5), (10, 10), etc.

Eq. (30) is the *PID tuning formula*. Note that when solving (31), depending on the value of k, four different cases need to be considered: (1) $1-k\alpha^2 > 0$, $1-k\beta^2 > 0$; (2) $1-k\alpha^2 > 0$, $1-k\beta^2 < 0$; (3) $1-k\alpha^2 < 0$, $1-k\beta^2 > 0$; and (4) $1-k\alpha^2 < 0$, $1-k\beta^2 < 0$. If none of these cases gives a solution, we may take (31) as having no solution for (α, β) and the GPM should be re-specified to other values; or an alternative solution can be obtained such that the attained GPM is in certain sense (e.g., the least square sense) closest to the specified one.

Inversely, given an IPTD process in (1) and a PID controller in (2), the resultant GPM and crossover frequencies of the closed-loop system are derived from (25)–(28) as

$$\begin{cases} \omega_g = \alpha/T_i, \\ \omega_p = \beta/T_i, \\ A_m = \dfrac{\beta^2}{\alpha^2}\sqrt{\dfrac{(1-k\alpha^2)^2+\alpha^2}{(1-k\beta^2)^2+\beta^2}}, \\ \phi_m = \arctan\dfrac{\alpha}{1-k\alpha^2} + \mathcal{H}(1-k\alpha^2)\pi - \omega_g\tau, \end{cases} \tag{32}$$

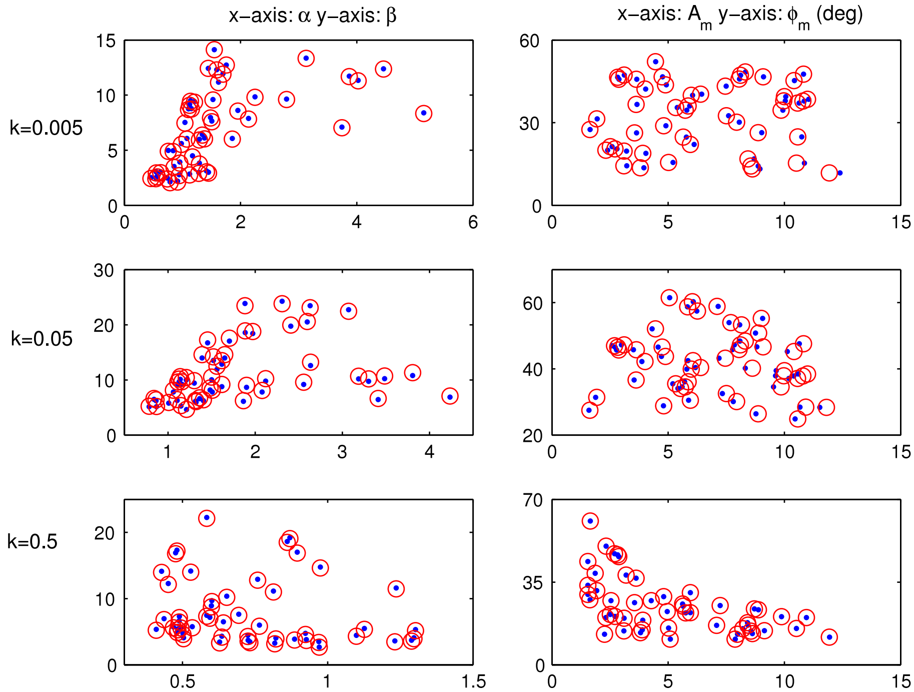

Fig. 4. GPMs estimated by GPM-PID formula versus true GPMs specified randomly a priori (50 tests), where the dots denote the estimated points and the circles denote the true points.

where α and β are the respective solutions of the two equations:

$$(\gamma^2k^2-1)\alpha^4+\gamma^2(1-2k)\alpha^2+\gamma^2=0,\quad \text{and} \tag{33}$$

$$\arctan\frac{\beta}{1-k\beta^2}+\mathcal{H}(1-k\beta^2)\pi=\theta\beta, \tag{34}$$

where γ and θ are defined in (10) and (11). Eqs. (33)–(34) can be solved numerically. Alternatively, their approximate solutions can be obtained as below.

For (33), noticing the common conditions that $k \le 0.5$ and $\gamma k < 1$ as adopted by a large number of existing rules [1], its unique solution (the negative solution is omitted) is obtained as

$$\alpha=\sqrt{\frac{\gamma^2}{2(1-\gamma^2k^2)}\left(1-2k+\sqrt{1-4k+\frac{4}{\gamma^2}}\right)}. \tag{35}$$

When $k=0$, this solution reduces to (10), namely the solution for the case of PI control.

For (34), according to Appendix A.2, an approximate solution is obtained as

$$\beta=\begin{cases}\sqrt{\frac{1}{2}\left[\frac{1}{k}-\frac{3}{\theta^2}+\sqrt{\left(\frac{1}{k}+\frac{3}{\theta^2}\right)^2-\frac{12}{k\theta^3}}\right]}, & \text{if } \beta<\beta_B;\\ \frac{\pi}{4(\theta-\lambda_Bk)}\left(1+\sqrt{1-\frac{16\lambda_B(\theta-\lambda_Bk)}{\pi^2}}\right), & \\ \quad\text{if } \beta_B\le\beta<1/\sqrt{k}; & \\ \frac{\pi}{4(\theta-\lambda'_Bk)}\left(1+\sqrt{1-\frac{16\lambda'_B(\theta-\lambda'_Bk)}{\pi^2}}\right), & \\ \quad\text{if } 1/\sqrt{k}<\beta\le\beta'_B; & \\ -a_2/3+U, \quad \text{if } \beta>\beta'_B, & \end{cases} \tag{36}$$

where

$$\lambda_B:=\lambda(1/x_B),\qquad \beta_B:=(\sqrt{1+4kx_B^2}-1)/(2kx_B),$$
$$\lambda'_B:=\lambda(1/x'_B),\qquad \beta'_B:=(\sqrt{1+4kx'^2_B}+1)/(2kx'_B), \tag{37}$$

with $x_B:=1.5$, $x'_B:=1.0$ and $\lambda(t):=(\arctan t)/t$; and

$$\begin{cases}U:=\sqrt[3]{R+\sqrt{D}}+\sqrt[3]{R-\sqrt{D}}, \quad \text{if } D\ge 0;\\ U:=2\sqrt[6]{R^2-D}\cos(\varphi/3),\\ \quad\text{with } \varphi:=\arctan(\sqrt{-D}/R)+\mathcal{H}(R)\pi,\ \text{if } D<0,\end{cases} \tag{38}$$

with

$$\begin{aligned}&D:=Q^3+R^2,\qquad Q:=(3a_1-a_2^2)/9,\\ &R:=(9a_2a_1-27a_0-2a_2^3)/54,\\ &a_0:=\pi/(k\theta),\qquad a_1:=(\lambda'_B-\theta)/(k\theta),\qquad a_2:=-\pi/\theta.\end{aligned} \tag{39}$$

To summarize, (32) gives the *GPM-PID formula*, with the intermediate variables α and β being expressed by (35) and (36), respectively. By the way, the solution of (α,β) justifies the uniqueness of the solution to (31) for a given GPM.

Remark 1. (a) Since the boundary conditions in (36) are implicit, the candidate solutions are calculated in turn until a valid one is obtained. (b) Refer to the end of Appendix A.2 for a less accurate yet simpler approximate solution of (34). □

Numerical computations are carried out to evaluate the accuracy of (36) as the solution of (34). The IPTD process parameters and the GPMs are specified in a similar way to those for the PI case (see Section 2.1). Numerical results are obtained for different values of k as shown in Figs. 4 and 5. Since the estimation errors are normally within 5%, the results validate the calculation of A_m in the GPM-PID formula based on the β approximated by (36).

3. Application to unifying the existing tuning rules

Rules of tuning PI/PD/PID controllers for an IPTD process have been accumulated in the past decades. These rules are based on various requirements and specifications on performance and robustness of the closed-loop system and were derived with various methods [1]. However, most of them can be unified by the tuning formulas presented above. From the PI, PD, PID tuning

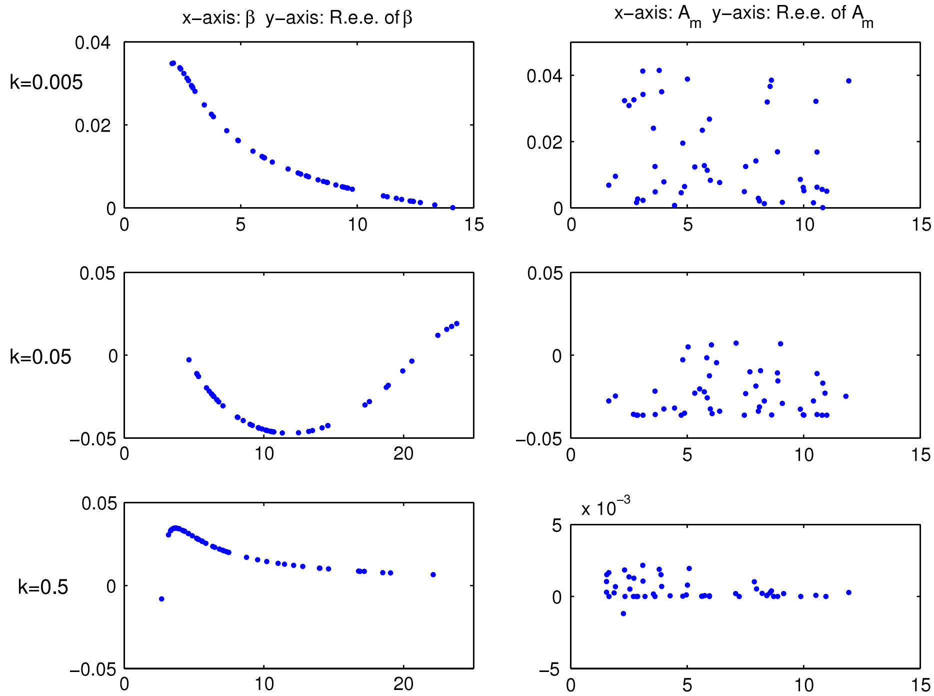

Fig. 5. Relative estimation errors of the results in Fig. 4.

formulas respectively in (7), (17), and (30), we see that the PID parameters have a common form of

$$K_c = \frac{k_1}{K_p \tau}, \qquad T_i = k_2 \tau, \qquad T_d = k_3 \tau, \tag{40}$$

where the parameters k_1, k_2, k_3 are specifically

$$\begin{aligned}
&\text{PI controller: } k_1 = \frac{\alpha(\arctan\alpha - \phi_m)}{\sqrt{1+\alpha^2}}, \\
&k_2 = \frac{\alpha}{\arctan\alpha - \phi_m}, \qquad k_3 = 0; \\
&\text{PD controller: } k_1 = \frac{\arctan\alpha' + \pi/2 - \phi_m}{\sqrt{1+\alpha'^2}}, \qquad k_2 = \infty, \\
&k_3 = \frac{\alpha'}{\arctan\alpha' + \pi/2 - \phi_m}; \\
&\text{PID controller: } k_1 = \frac{\alpha\varphi_\alpha}{\sqrt{(1-k\alpha^2)^2+\alpha^2}}, \qquad k_2 = \frac{\alpha}{\varphi_\alpha}, \\
&k_3 = kk_2.
\end{aligned} \tag{41}$$

Here $\varphi_\alpha := \arctan\frac{\alpha}{1-k\alpha^2} + \mathcal{H}(1-k\alpha^2)\pi - \phi_m$, and α, α' and α for the PI, PD and PID controllers are determined from (8), (18) and (31), respectively.

The common form of PI/PD/PID parameters in (40) indicates that different rules employing different values of (k_1, k_2, k_3) are realizing different GPMs which consequently lead to various closed-loop performances. This gives a unified interpretation to the vast variety of PI/PD/PID tuning rules accumulated in the literature [1]. From this viewpoint, PI/PD/PID control design on an IPTD process is essentially choosing a proper GPM or parameter set (k_1, k_2, k_3). The GPM or parameter set can be selected via performance optimization subject to design constraints. Depending on the specific performance index and design constraints, the solution may differ from case to case and particular studies are required. A summary of various designs can be found in [1]. In particular, the well-known SIMC rule [12] uses a GPM of about (3.0, 46.9°) and the improved SIMC rule (with enhanced disturbance rejection) [18] about (2.9, 42.5°) for an IPTD process, when the recommended settings are adopted for both methods.

Finally, we apply the GPM-PI/PD/PID formulas derived in the last section to estimate the GPMs realized by relevant PI/PD/PID tuning rules as collected in [1]. The GPM-PI/PD/PID formulas indicate that any PI/PD/PID controllers with the same (k_1, k_2, k_3) in (40) result in the same GPM, regardless of the process parameters. This enables numeric computation of the exact GPM realized by each rule in the form of (40). To compare, GPM attained by each rule is computed by using both the GPM-PI/PD/PID formula and the numeric approach. The results are documented in the link [24], which take more than four pages to present and hence are omitted here. The results show that various GPMs are achieved by the existing tuning rules. Note that the larger the gain margin or the smaller the phase margin is, the more aggressive yet less robust the closed-loop performance will be. The summary of such GPMs thus provides a rich reference for control engineers to tune PID controllers. Meanwhile the results verify that the GPM-PI/PD/PID formulas are accurate for GPM estimations.

4. Conclusion

For an IPTD process, PI/PD/PID tuning formulas with specified GPM were obtained and so were GPM-PI/PD/PID formulas for estimating GPM resulting from a given PI/PD/PID controller. The tuning formulas indicate a common form of the PID parameters and unify a large number of tuning rules as PI/PD/PID controller tuning with various GPM specifications. The GPM formulas accurately estimate the GPM realized by each relevant PI/PD/PID tuning rule as collected in [1] and the results are summarized in the link [24]. The results show that a variety of GPMs are attained by the existing rules. Since the rules were developed based on various criteria and methods, the summary of their resulting GPMs provides a rich reference for control engineers to tune PID controllers, helping to select a rule or GPM for a specific design.

Acknowledgements

The authors deeply appreciate A. O'Dwyer for sharing the editable electronic tables presented in his book [1] on pages 76–82 and 259–261, which contributed a lot to the supplemental material documented in the link [24]. The authors also thank S. Skogestad for the insightful comments and helpful discussions on the draft of this paper. And comments from anonymous reviewers are acknowledged for having helped improve the quality of this paper.

Appendix. Approximate analytic solutions of β's for (11) and (34)

To solve (11) and (34) for approximate solutions, first consider approximating the following equation.

$$x = \tan y, \quad y \in (-\pi/2, \pi/2). \tag{42}$$

Divide the domain of y into two parts:

$$\begin{aligned} &\mathcal{D}_1 := (-\arctan x_b, \arctan x_b), \quad \text{and} \\ &\mathcal{D}_2 := (-\pi/2, -\arctan x_b] \cup [\arctan x_b, \pi/2), \end{aligned} \tag{43}$$

where $x_b \geq 1$ is a boundary value. Since (42) has odd solutions, it is sufficient to consider solving it in the domain consisting of $\mathcal{D}_1^r := [0, \arctan x_b)$ and $\mathcal{D}_2^r := [\arctan x_b, \pi/2)$.

In $\mathcal{D}_1^r$, approximate (42) by the Taylor expansion of $\tan y$ to the fifth order, giving

$$x = \tan y \approx y + y^3/3 + 2y^5/15, \tag{44}$$

of which the relative approximation error is

$$e_1(y) := (y + y^3/3 + 2y^5/15)/\tan y - 1. \tag{45}$$

In $\mathcal{D}_2^r$, first convert (42) into the arctangent form and then approximate it by

$$y = \arctan x = \pi/2 - \arctan z \approx \pi/2 - \lambda_b z, \tag{46}$$

where $z := x^{-1}$ and $\lambda_b := \lambda(1/x_b)$ and $\lambda(\bullet)$ is a function defined as

$$\lambda(t) := (\arctan t)/t, \quad t \in (0, +\infty). \tag{47}$$

The corresponding relative approximation error is

$$e_2(z) := \tan(\pi/2 - \lambda_b z)/z^{-1} - 1 = z/\tan(\lambda_b z) - 1. \tag{48}$$

Note that (i) to be consistent with $e_1(y)$, the tangents of both sides of (46) are taken to calculate $e_2(z)$; and (ii) the Taylor expansion is not used in $\mathcal{D}_2^r$ since it is hard to attain high accuracy; and (iii) in $\mathcal{D}_2^r$ it has $z \in (0, x_b^{-1}]$.

From (45) and (48), it can be easily proved that $e_1(y) < 0$, $\mathrm{d}e_1(y)/\mathrm{d}y \leq 0$, $e_2(z) > 0$ and $\mathrm{d}e_2(z)/\mathrm{d}z \leq 0$. Thus the maximum absolute values of $e_1(y)$ and $e_2(y)$ are, respectively,

$$\begin{cases} \|e_1(y)\|_\infty = -e_1(\arctan x_b), \quad \text{and,} \\ \|e_2(z)\|_\infty = \lim\limits_{z\to 0} e_2(z) = 1/\lambda - 1. \end{cases} \tag{49}$$

Here $\|e_1(y)\|_\infty$ and $\|e_2(z)\|_\infty$ are both functions of x_b, as shown in Fig. 6, where the intersection point is numerically obtained as $x_B := x_b \approx 1.848$. At this point, the maximum absolute values of the relative errors by the two different approximations equal each other at 9.10%, and $\lambda_B := \lambda_b = \lambda(1/x_B) \approx 0.917$.

For y being an explicit function of x, e.g., $y = 2x$, by taking x_B and λ_B as the boundary parameters for the above two approximations, an approximate solution of (42) can be obtained by solving either (44) or (46) for x.

In addition, notice that in some cases where y is an explicit function of x, (44) may prevent an analytic solution of x. As a compromised solution, a lower-order Taylor expansion of $\tan y$ may be adopted. Consider the third-order Taylor expansion case where (44) and (45) are replaced respectively by

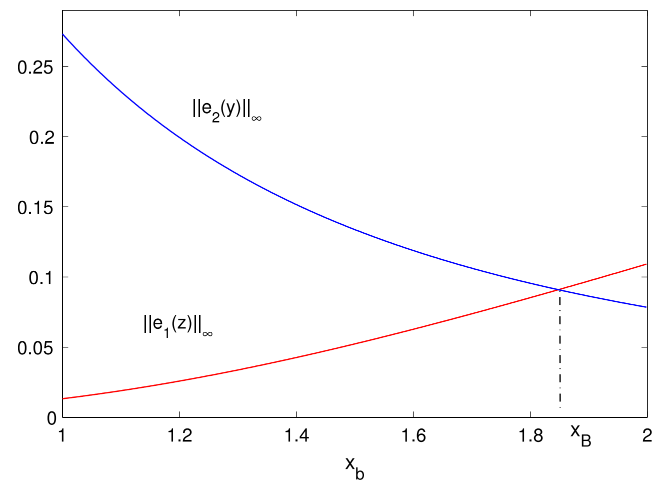

Fig. 6. The maximal absolute values of the relative errors of the approximate solutions, as functions of the boundary point x_b.

$$x = \tan y \approx y + y^3/3, \quad \text{and} \tag{50}$$

$$e_1(y) = (y + y^3/3)/\tan y - 1. \tag{51}$$

Keep (46) unchanged. By deducting similarly as above, the approximation boundaries are obtained as $x_B \approx 1.500$ and $\lambda_B \approx 0.882$, at which the maximum absolute values of the relative errors by the two different approximations equal each other at 13.38%.

A.1. An approximate solution of (11)

In particular, let $x := \beta > 0$ and $y := \theta\beta > 0$ in (42). From (44) and (46), an approximate solution of (42) can be obtained as follows:

$$\begin{cases} \beta = \dfrac{1}{2\theta}\sqrt{-5 + \sqrt{\dfrac{120}{\theta} - 95}}, & 0 < \beta < \beta_B, \\ \beta = \dfrac{\pi}{4\theta}\left(1 + \sqrt{1 - \dfrac{16\lambda_B\theta}{\pi^2}}\right), & \beta \geq \beta_B, \end{cases} \tag{52}$$

where $\lambda_B = 0.917$ and $\beta_B = 1.848$. Alternatively, by specifying the conditions of θ, the solution (52) can be re-expressed as

$$\begin{cases} \beta = \dfrac{\pi}{4\theta}\left(1 + \sqrt{1 - \dfrac{16\lambda_B\theta}{\pi^2}}\right), & \text{if } 0 < \theta \leq \theta_B, \\ \beta = \dfrac{1}{2\theta}\sqrt{-5 + \sqrt{\dfrac{120}{\theta} - 95}}, & \text{if } \theta_B < \theta < 1, \end{cases} \tag{53}$$

where

$$\theta_B := \min\left\{\frac{\pi^2}{16\lambda_B}, \frac{1}{\beta_B}\left(\frac{\pi}{2} - \frac{\lambda_B}{\beta_B}\right)\right\} \approx 0.582. \tag{54}$$

Note that for (52), as the boundaries of the applying regions of θ do not coincide, for simplicity θ_B is taken as the one calculated from the second equation of (52). The validity of the approximate solution of (42) by (53) is demonstrated by the exemplary results shown in Fig. 2.

A.2. An approximate solution of (34)

To solve (34), two different cases are considered separately as follows (The point $\beta = 1/\sqrt{k}$ is undefined in the equations and is therefore omitted.):

$$\arctan \frac{\beta}{1-k\beta^2} = \theta\beta, \quad \text{if } 1-k\beta^2 > 0; \tag{55}$$

$$\arctan \frac{\beta}{1-k\beta^2} = \theta\beta - \pi, \quad \text{if } 1-k\beta^2 < 0. \tag{56}$$

Let $x := \beta/(1-k\beta^2)$ and $y := \theta\beta$ in (42). From (46) and (50) an approximate solution of (55) is derived as

$$\begin{cases} \beta = \sqrt{\dfrac{1}{2}\left[\dfrac{1}{k} - \dfrac{3}{\theta^2} + \sqrt{\left(\dfrac{1}{k}+\dfrac{3}{\theta^2}\right)^2 - \dfrac{12}{k\theta^3}}\right]}, \\ \quad \text{if } 0 < \beta < \beta_B, \\ \beta = \dfrac{\pi}{4(\theta-\lambda_B k)}\left(1+\sqrt{1-\dfrac{16\lambda_B(\theta-\lambda_B k)}{\pi^2}}\right), \\ \quad \text{if } \beta_B \le \beta < 1/\sqrt{k}, \end{cases} \tag{57}$$

where

$$\lambda_B := \lambda(1/x_B), \qquad \beta_B := (\sqrt{1+4kx_B^2} - 1)/(2kx_B), \tag{58}$$

with $x_B := 1.5$ and $\lambda(\bullet)$ being defined in (47).

To solve (56), the approximation skills used in (22)–(23) are adopted. Specifically, by applying the skill used in (23), an approximate solution of (56) is obtained as

$$\beta = \frac{\pi}{4(\theta-\lambda'_B k)}\left(1+\sqrt{1-\frac{16\lambda'_B(\theta-\lambda'_B k)}{\pi^2}}\right), \quad \text{if } 1/\sqrt{k} < \beta \le \beta'_B, \tag{59}$$

where

$$\lambda'_B := \lambda(1/x'_B), \qquad \beta'_B := (\sqrt{1+4kx'^2_B} - 1)/(2kx'_B), \tag{60}$$

with $x'_B := 1.0$ and $\lambda(\bullet)$ being defined in (47). And for the case where $\beta > \beta'_B$, by applying the skill used in (22) the following equation of β is obtained:

$$\beta^3 + a_2\beta^2 + a_1\beta + a_0 = 0, \tag{61}$$

where

$$a_2 := -\pi/\theta, \qquad a_1 := (\lambda'_B - \theta)/(k\theta), \qquad a_0 := \pi/(k\theta). \tag{62}$$

Eq. (61) is a standard cubic equation with real coefficients, and its feasible solution (being real and positive) is obtained as

$$\beta = -a_2/3 + S + T, \tag{63}$$

where

$$S := \sqrt[3]{R+\sqrt{D}}, \qquad T := \sqrt[3]{R-\sqrt{D}}, \tag{64}$$

with

$$\begin{aligned} &D := Q^3 + R^2, \qquad Q := (3a_1 - a_2^2)/9, \\ &R := (9a_2a_1 - 27a_0 - 2a_2^3)/54. \end{aligned} \tag{65}$$

Since D in (64) may be negative, leading to complex numbers in the calculations which should be avoided in applications, the solution (63) is expressed in an alternative way such that

$$\beta = -a_2/3 + U, \tag{66}$$

where

$$\begin{cases} U := \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}, \quad \text{if } D \ge 0; \\ U := 2\sqrt[6]{R^2 - D}\cos(\varphi/3), \\ \quad \text{with } \varphi := \arctan(\sqrt{-D}/R) + \mathcal{H}(R)\pi, \quad \text{if } D < 0. \end{cases} \tag{67}$$

Here $\mathcal{H}(\bullet)$ is the function defined in (29), and D and R keep the same as those in (65).

With (57), (59) and (66), the approximate solution of (34) is thus obtained as follows

$$\beta = \begin{cases} \sqrt{\dfrac{1}{2}\left[\dfrac{1}{k} - \dfrac{3}{\theta^2} + \sqrt{\left(\dfrac{1}{k}+\dfrac{3}{\theta^2}\right)^2 - \dfrac{12}{k\theta^3}}\right]}, \quad \text{if } \beta < \beta_B; \\ \dfrac{\pi}{4(\theta-\lambda_B k)}\left(1+\sqrt{1-\dfrac{16\lambda_B(\theta-\lambda_B k)}{\pi^2}}\right), \\ \quad \text{if } \beta_B \le \beta < 1/; \\ \dfrac{\pi}{4(\theta-\lambda'_B k)}\left(1+\sqrt{1-\dfrac{16\lambda'_B(\theta-\lambda'_B k)}{\pi^2}}\right), \\ \quad \text{if } 1/\sqrt{k} < \beta \le \beta'_B; \\ -a_2/3 + U, \quad \text{if } \beta > \beta'_B, \end{cases} \tag{68}$$

where the intermediate variables, λ_B and β_B, λ'_B and β'_B, a_2 and U, are defined in (58), (60), and {(62), (65), (67)}, respectively. Since it is hard to give the piecewise conditions of (68) in terms of θ as that in (53), the candidate solutions are calculated in a top-down sequence until a feasible β is obtained; if no feasible solution is achieved, (34) will be taken as having no solution, or a numerical solution to it has to be tried.

Additionally, another simpler yet less accurate approximate solution for (34) can be derived. The main idea is as follows. For the case of (55) and the case of (56) with $1/\sqrt{k} < \beta \le \beta'_B$ (Here β'_B is of a different value from that in (68).), the approximate solutions remain the same as those in (57) and (59), respectively; and for the case of (56) with $\beta > \beta'_B$, first (56) is approximated by replacing "$1-k\beta^2$" with $-k\beta^2$ (requiring that $k\beta'^2_B \gg 1$—here $k\beta'^2_B = 10$ is used, by selecting a proper boundary point x'_B). Then by applying the same skill as that in (22), a less accurate yet simpler approximate solution of (34) can be obtained. Specifically, it is as follows:

$$\beta = \begin{cases} \sqrt{\dfrac{1}{2}\left[\dfrac{1}{k} - \dfrac{3}{\theta^2} + \sqrt{\left(\dfrac{1}{k}+\dfrac{3}{\theta^2}\right)^2 - \dfrac{12}{k\theta^3}}\right]}, \quad \text{if } \beta < \beta_B; \\ \dfrac{\pi}{4(\theta-\lambda_B k)}\left(1+\sqrt{1-\dfrac{16\lambda_B(\theta-\lambda_B k)}{\pi^2}}\right), \\ \quad \text{if } \beta_B \le \beta < 1/\sqrt{k}; \\ \dfrac{\pi}{4(\theta-\lambda'_B k)}\left(1+\sqrt{1-\dfrac{16\lambda'_B(\theta-\lambda'_B k)}{\pi^2}}\right), \\ \quad \text{if } 1/\sqrt{k} < \beta \le \beta'_B; \\ \dfrac{\pi}{2\theta}\left(1+\sqrt{1-\dfrac{4\lambda''_B\theta}{k\pi^2}}\right), \quad \text{if } \beta > \beta'_B, \end{cases} \tag{69}$$

where $\lambda_B := \lambda(1/x_B)$, $\lambda'_B := \lambda(1/x'_B)$, $\lambda''_B := \lambda(x'_B)$, $\beta_B := (\sqrt{1+4kx_B^2} - 1)/(2kx_B)$ and $\beta'_B := \sqrt{10/k}$, with $x_B := 1.5$, $x'_B := \beta'_B/(k\beta'^2_B - 1)$ and $\lambda(\bullet)$ being defined in (47). As expected, the estimated β may not be accurate when $\beta > 1/\sqrt{k}$, but it is found to be able to achieve the final goal of estimating the gain margin A_m with satisfactory accuracy. The relative estimation errors are mostly within 7%. Exemplary results are shown in Fig. 7.

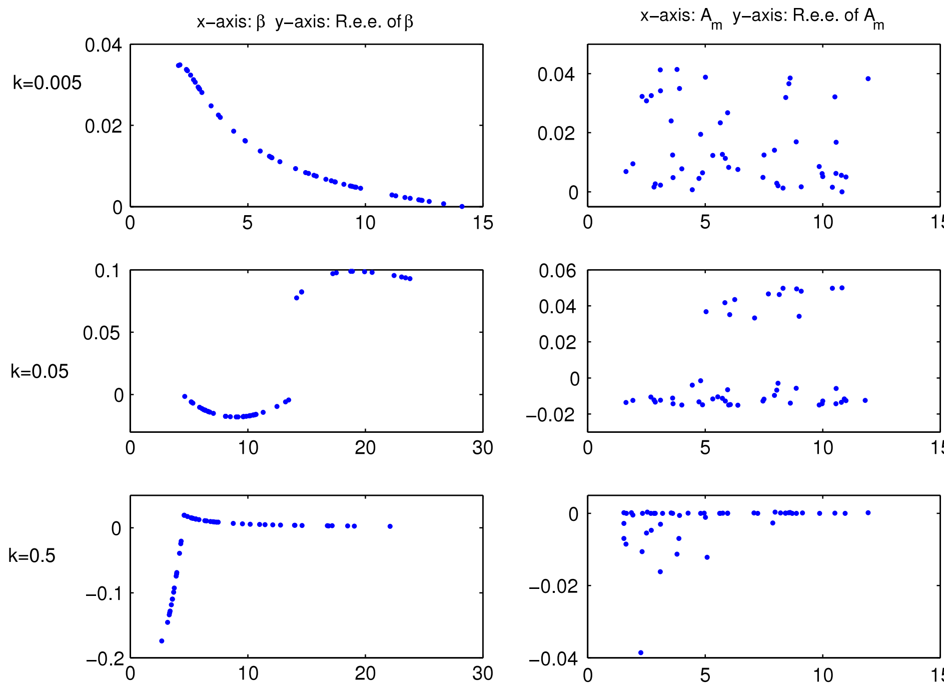

Fig. 7. Typical relative estimation errors of β and A_m, with β being estimated by (69).

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