Simulink Model of a Full State Observer for a DC Motor Position, Speed, and Current

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***Abstract*— In this paper we develop a state observer model for the armature of a DC motor based on the well-known equivalent circuit model, and torque and speed equations. Towards this end, and as a first step, we derive a state space representation for the circuit model, and demonstrate its controllability and observability properties. Using the Luenberger full state observer technique, we derive and implement the latter in MATLAB/Simulink for position control of the motor, and verify its operation.**

***Keywords*—Circuit Model, State Space, DC Motors, Armature, Rotor, Luenberger Sate Observer, Simulink.**

I. Introduction

DC motors are classified into two categories: the permanent magnet type and the electromagnet type, based on how the magnetic field is created. The latter category is further subdivided into self-excited and separately excited, depending on whether there is a physical connection or not between the field windings and the armature windings. If the two windings are connected in series, this is referred to as a series motor. These are known for their variable speed and high starting torque. Applications include cranes, conveyors, elevators, and electric locomotives. If the two windings are connected in parallel, this gives rise to a shunt DC motor, which has a fairly constant speed and a medium starting torque [1]. These are used in fans, pumps, controlled fabrication machines, automated equipment such as industrial robots, and smart printers and plotters. In such applications, it is imperative that the predetermined position be acquired from the preceding position within a short period of time. Hence it becomes necessary to control the input voltage supplied to the motor by continuously detecting the position and speed of the rotor shaft.

An observer is a dynamic system that is used to estimate the state of a system or some of the states of a system. A full-state observer is used to estimate all the states of the system. The observer can be designed as either a continuous-time system or a discrete-time system. The characteristics are the same, and the design processes are at least very similar and in some cases identical. The purpose of the observer is to generate an estimate of the state based on measurements of the system output and the system input. The input and output signals are assumed to be exactly measurable. Also, the observer uses a mathematical model of the state space realization of the system, and is software implemented [2]-[4].

In this paper a full state observer is designed for a DC motor, based on the actual electrical equivalent circuit of the armature winding and the relationship between position and voltage. The observer is simulated via MATLAB/Simulink and the results and performance are compared with those of the actual system.

The paper is organized as follows. First the theory for the full observer is presented in section two. In sections three the armature electrical circuit is presented and the state space representation is derived. In section four, the design of the observer for the position is carried out. In section five we present a Simulink implementation of the system, as well as the simulation results.

II. State Observer Theory

A. The Physical System

The assumptions here are that the real system is a deterministic, linear time-invariant (LTI), continuous control system that is observable and controllable, and whose internal states may not be determined by direct observation. Its dynamics are described by the following state space equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + Du(t) \quad (2)$$

Where x is the state vector, u the control input, y the output, and A, B, C, D are constant system matrices of appropriate dimensions.

B. System Controllability and Observability

A system is said to be controllable if there exists a control input that transfers any state of the system to zero in finite time. It can be shown that a LTI system is controllable if and only if its controllability matrix, given in (3), has full rank, i.e. its rank is equal to the number of states [5]. Note that the rank of the controllability matrix of an LTI system can be readily determined in MATLAB using the commands *rank(ctrb(A,B))* or *rank(ctrb(sys)).*

$$CO = [B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B] \qquad (3)$$

All the state variables of the system may not be directly measurable if, for instance, one or more components of the system is in an inaccessible location. In these cases it is necessary to estimate the values of the unknown internal state variables using only the available system output.

A system is said to be observable if the initial state, $x(t_0)$, can be determined from the system output, $y(t)$, over some finite time $t_0 \le t \le t_f$. Mathematically, a LTI system is observable if and only if the observability matrix, given in (4), has full rank, i.e. its rank is equal to the number of states [5]. Note here also that in MATLAB this can easily be checked by the command *rank(obsv(A,C))* or *rank(obsv(sys)).* Also, it is worth mentioning that controllability and observability are dual concepts. A system (A, B, C, D) is controllable if and only if the system (A^T, C, B^T, D) is observable. Here A^T and B^T are the transpose matrices of A and B, respectively. This fact will be useful when designing an Observer [5][6].

$$OB = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad (4)$$

C. Full Observer Model

There are several ways to derive the state equations for the full-state observer. One approach is to model the observer state equations as a model of the actual system plus a correction term based on the measured output and the estimate of what that output is expected to be. With the actual system described by (1) and (2), the observer is modeled as [2][6]

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L[y(t) - \hat{y}(t)] \qquad (5)$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t) \qquad (6)$$

where L is the n × m gain matrix for the observer. The state equation in (5) is seen to model the actual state equation (1), with the true state, $x(t)$, replaced by the estimate, $\hat{x}(t)$, and a correction term which is the difference between the actual measured output $y(t)$ and its estimate $\hat{y}(t)$. Similarly, the output equation in (6) is also seen to be a model of the system's output equation, with $x(t)$ replaced by $\hat{x}(t)$.

Substituting (6) in (5) yields the following alternative form for the observer:

$$\dot{\hat{x}}(t) = (A - LC)\hat{x}(t) + (B - LD)u(t) + Ly(t) \qquad (7)$$

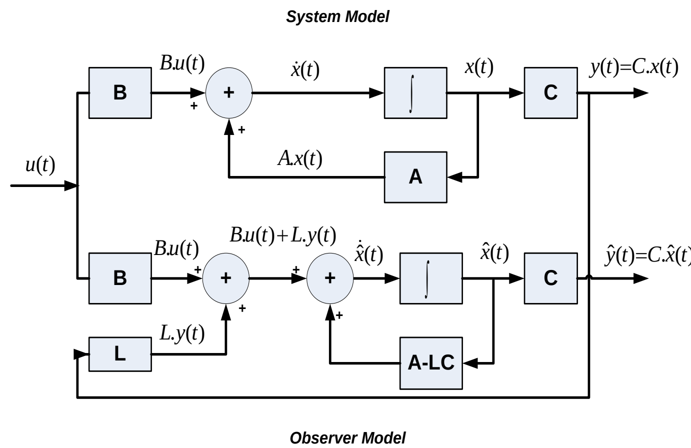


Figure 1: Block Diagram of System and Observer Models.

Note that although the matrix D explicitly appears in (7), it has no bearing on the state estimate produced by the observer. The reason is because in (5) the term $Du(t)$ cancels out in $y(t) - \hat{y}(t)$. The block diagram for the system described by (1) and (20), and its corresponding observer described by (6) and (7) are shown in Figure 1 for the case D=0.

D. Error Estimation

The purpose of the observer is to produce an estimate of the true state $x(t)$ of the real system. It is reasonable to assume that there will be some error in the estimate at the initial time, but it is hoped that the error would decrease over time. The estimation error is defined as

$$e(t) = x(t) - \hat{x}(t) \qquad (8)$$

Using (1) and (7), it can easily be shown that this estimation error signal satisfies the differential equation

$$\dot{e}(t) = (A - LC)e(t) \qquad (9)$$

Thus, the state equation for the estimation error is a homogeneous differential equation governed by the n×n matrix $A - LC$. The solution to this equation is

$$e(t) = e(0)\,\mathrm{e}^{-(A-LC)t} \qquad (10)$$

The eigenvalues of the matrix (A – LC) can be made arbitrary by appropriate choice of the observer gain, L, when the pair (A,C) is observable. So the observer error e(t) goes to zero as t goes t infinity. If the gain matrix L is chosen so that the eigenvalues of $A - LC$ are strictly in the left-half of the complex plane, then the error equation is asymptotically stable, and therefore the estimation error will decay to zero over time. Also, if the system (A, C) is completely observable, then L can be chosen so that the eigenvalues of $A - LC$ are placed at arbitrary locations in the plane, provided that complex eigenvalues occur in complex conjugate pairs.

E. *Computation of Gain Matrix L*

The gain matrix L of the full-state observer can be computed using any of the methods used to compute the control gain matrix K for a control system [5][7]. For the control problem with full-state feedback, the closed-loop system matrix of interest is $A-BK$. Comparing that with the observer problem, the closed-loop system matrix is $A-LC$. The structure of those two matrices is similar; only the order of the unknown matrix differs between BK and LC. Since the eigenvalues of a matrix and its transpose are the same, the observer problem can be formulated the same way as the control problem by considering the transpose matrix $(A-LC)^T = A^T - C^T L^T$. Therefore, the gain matrix L can be computed using the Row-Reduced Echelon (RRE) method, Singular Value Decomposition (SVD), or the MATLAB *place* function in the same way as the control gain matrix K by replacing (A, B) by (A^T, C^T). By doing this, the result from any of these methods will give the matrix L^T.

III. ARMATURE CIRCUIT MODEL

A common actuator in control systems is the DC motor. It directly provides rotary motion and, coupled with wheels or drums and cables, can cause translational motion of another machine. In such a motor (separately excited DC motor), the field windings are excited by a DC current in order to create a magnetic field. In turn, the armature windings receive current from a separate DC source which results in the creation of a torque by Lenz's Law and a back electromotive force (EMF) by Faraday's law [1].

A. *Motor Equations*

The electric equivalent circuit of the armature and the free-body diagram of the rotor are shown in the Figure 2, where R_A and L_A are the equivalent resistance and inductance, respectively, of the armature winding, i_A the armature current, V the input voltage, E_A the induced back electro-motive force (emf) created as a result of injecting a current into a magnetized coil.

It is assumed that the input of the system is the voltage *V* applied to the motor's armature, while the output is the position θ of the shaft. It is further assumed a viscous friction model, that is, the friction torque is proportional to shaft angular speed. Referring to Figure 2, the corresponding governing Kirchoff's voltage law and Newton's second law equations are given by (11) [1].

In general, the developed torque generated by a DC motor is proportional to the armature current and the strength of the magnetic field. Here we assume that the magnetic field is

Symbol	Unit	Definition
V	*Volts (V)*	Input voltage
i_A	*Ampere (A)*	Armature current
E_A	*Volts (V)*	Back EMF
R_A	*Ohm (Ω)*	Armature Winding Resistance
L_A	*Henry (H)*	Inductance of Armature Windings
K_m	*Volts/radians/s*	Machine Constant
T_d	$N.m$	Developed Torque
θ	*Radians*	Shaft angular position
$\omega = \dot{\theta}$	*Radians/s*	Angular speed
$\ddot{\theta}$	$Radians/s^2$	Angular acceleration
J_m	$kg.m^2$	Moment of Inertia
B_m	$N.m.s$	Viscous Frictional Constant
T_L	$N.m$	Load Torque

Table1: Motor Parameters and Constants.

constant and, therefore, the motor torque is proportional to the armature current i_A as shown in (12). This is referred to as an armature-controlled motor.

$$T_d = K_m i_A \tag{11}$$

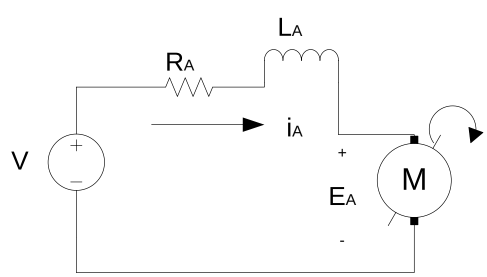


Figure 2: Electrical Equivalent Circuit of Motor Armature.

Here K_m is the machine constant. Also, the back emf, E_A, is proportional to the angular velocity of the shaft

$$E_A = K_m \omega_m = K_m \dot{\theta} \tag{12}$$

Referring to Figure 2, the corresponding governing Kirchoff's voltage law and Newton's second law equations are given by

$$V = i_A R_A + L_A \frac{di_A(t)}{dt} + K_m \dot{\theta} \tag{13}$$

$$J_m \ddot{\theta} = -B_m \dot{\theta} + K_m i_A - T_L \tag{14}$$

Where J_m is the moment of inertia of the rotor and B_m the motor viscous frictional constant [6][9].

B. *Transfer Function*

We take equations (13) and (14) as a basis for deriving two transfer functions for the motor under no load conditions, i. e, $T_L = 0$, with input being the voltage and the output the angular speed for the first one and the position for the second. Taking the Laplace transforms of (13) and (14) [8] gives:

$$I_A(s)(R_A + L_A s) + [V(s) - sK_m\theta] = 0 \qquad (15)$$

$$s^2 J\theta + sB_m\theta = K_m I_A(s) \qquad (16)$$

Solving for $I_A(s)$ from (15) and substituting in (16) yields

$$\frac{\theta(s)}{V(s)} = \frac{K_m}{s[(sJ_m + B_m)(L_A s + R_A) + K_m^2]} \qquad (17)$$

and

$$\frac{\dot{\theta}(s)}{V(s)} = \frac{K_m}{(sJ_m + B_m)(L_A s + R_A) + K_m^2} \qquad (18)$$

C. *State Space Representation*

By defining the state vector x, the output y and the input u as follows:

$$x = \begin{bmatrix} i_A \\ \theta \\ \dot{\theta} \end{bmatrix}, y = \theta, u = V \qquad (19)$$

the state space equations for the motor are derived using (13) and (14) as follows:

$$\frac{d}{dt}\begin{bmatrix} i_A \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{R_A}{L_A} & 0 & -\frac{K_m}{L_A} \\ 0 & 0 & 1 \\ \frac{K_m}{J_m} & 0 & -\frac{B_m}{J_m} \end{bmatrix}\begin{bmatrix} i_A \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_A} \\ 0 \\ 0 \end{bmatrix} V \qquad (20)$$

and

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} i_A \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad (21)$$

IV. OBSERVER DESIGN FOR DC MOTOR POSITION

For the Observer design, we consider a motor with the following parameters:

R_A	L_A	K_m	J_m	B_m
1	10^{-3}	0.1	5.10^{-3}	10^{-4}

Table 1: Values of Motor Constants and Parameters

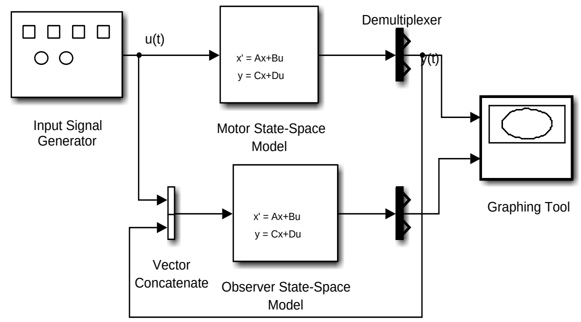


Figure 3: Simulink Block Diagram for the DC Motor Model and the Full Observer Model.

Then the systems matrices A, B, C (D=0) are constructed using (20 and (21), and observability and controllability are checked using MATLAB. The Observer is designed by calculating the matrix L such that the eigenvalues of the matrix (A-LC) are placed at -500+j250, -500-j250, and -200 respectively, and calculating the corresponding matrices “A”, “B”, and “C” for the observer model using (5) and (6).

V. SIMULINK SIMULATION RESULTS

The Simulink block diagram for the system and the Observer is shown in Figure 3. The input signal generator block generates the signal u(t) which serves as the excitation voltage for the motor, modeled by the motor state-space block which produces the state x and y as its output. The system output y, being equal to the second component of the state vector, is extracted from x using a de-multiplexer block. Both the motor input u(t) and the output y serve as inputs to the observer system, as shown in Figure 3. Note that, since the Simulink state space block requires the system it simulates to be in the form of equations (1) and (2), the observer equation (7) needs to be reformulated to match (1), as follows:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \qquad (22)$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t) \qquad (23)$$

Where the observer matrices can be calculated from the system matrices, as given below:

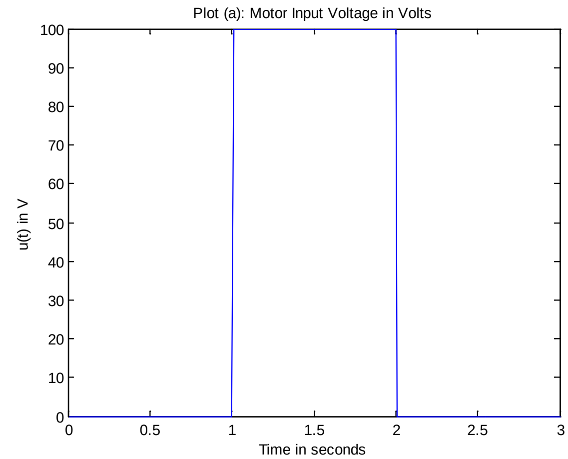
$$\begin{aligned} \hat{A} &= A - LC \qquad & \hat{B} &= [B \vdots L] \\ \hat{C} &= C \qquad & \hat{D} &= [D \vdots D] \end{aligned} \qquad (24)$$

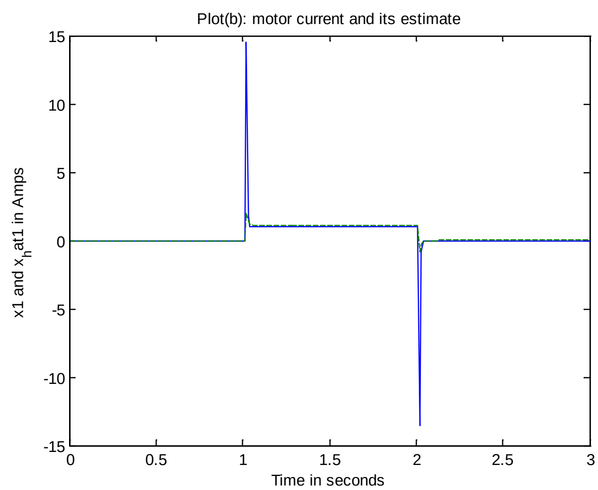
and $\hat{u}(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$.

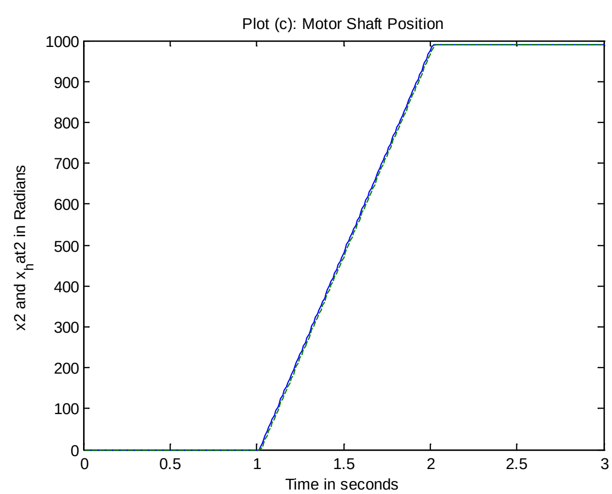
Two simulations were run for two different input voltages: a pulse of amplitude 100V, and a sinusoid of 100V amplitude and 60Hz frequency, with an initial state of [0, 0, 0]. The actual states, namely the current i_A, the angular position θ, the angular speed $\dot{\theta}$, and their estimates are plotted in Figure 4 for a time frame of 5*sec* for the pulse, and in Figure 5 for the sinusoid. The plots of Figure 5 show that the state estimates almost match the actual states, whereas Figure 5 indicate that in the sinusoidal case, though initially and up to about 15 *msec* after the start of the simulation, the estimates diverged from the actual quantities being estimated, they did converge very quickly after the initial 15 *msec*, thus verifying the design.

VI. CONCLUSION

An asymptotic algebraic state estimation method known as Luenberger Observer model has been successfully applied to estimate the current, position, and angular speed of a motor. Further, examining the performance of such an observer shows that this method provides satisfactory estimates even in the presence of noise levels, and different initial conditions.







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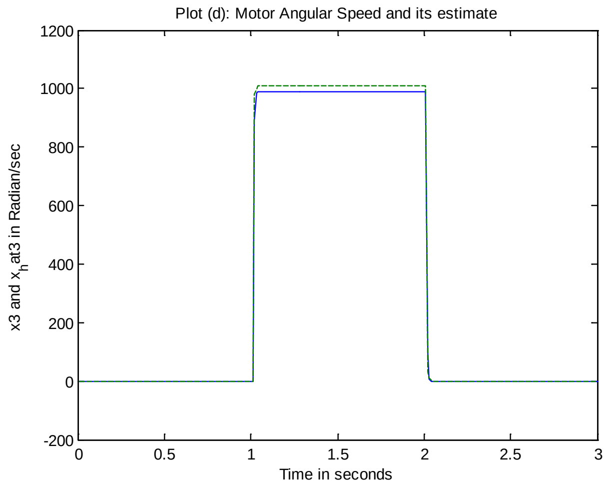
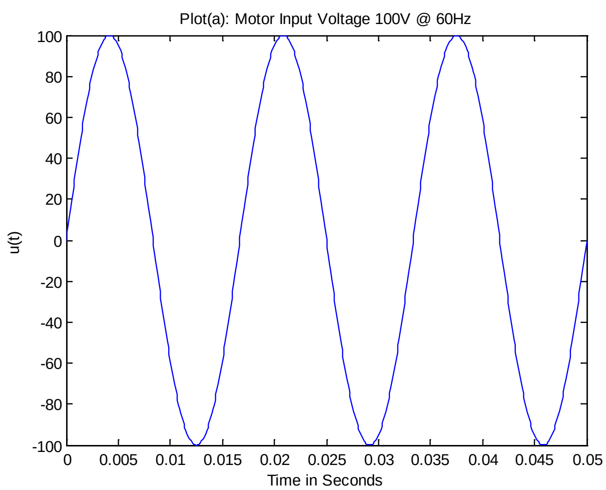
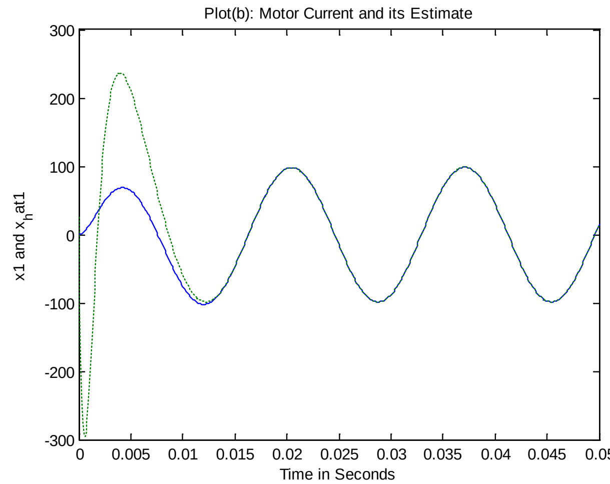
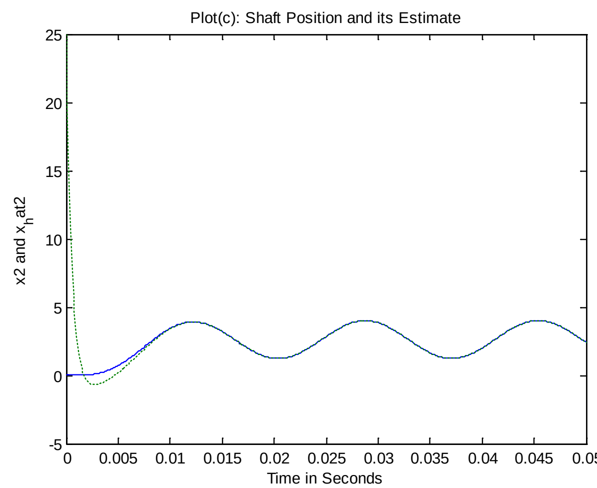


Figure 4: (a): Input pulse Voltage, (b)-(d): Plots of armature current, shaft position , and motor angular speed and their respective estimates, as produced by the observer system..







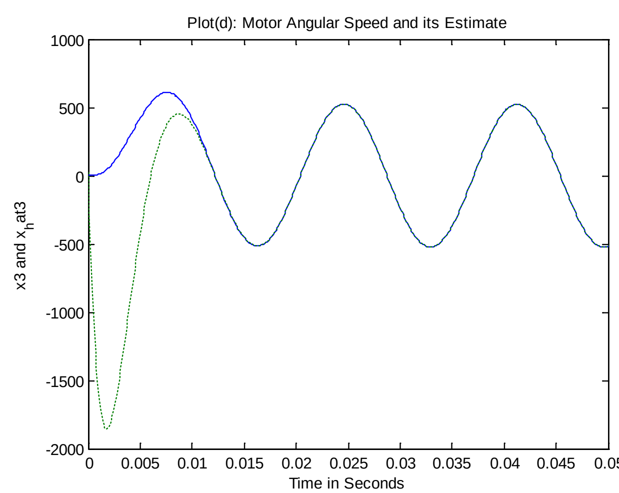


Figure 5: (a) Sinusoidal input, (b)-(d): Plots of armature current, shaft position, and motor angular speed and their respective estimates, as produced by the observer system.