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Continuous-time model identification of fractional-order models with time delays

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Abstract: Modelling of real physical systems having long memory transients and infinite dimensional structures using fractional-order dynamic models has significantly attracted interest over the last few years. For this reason, many identification techniques both in the frequency domain and time domain have been developed to model these fractional-order systems. However, in many processes time delays are also present and estimation of time delays along with continuous-time fractional-order model parameters have not been addressed anywhere. This study deals with the continuous-time model identification of fractional-order system models with time delays. In this study, a new linear filter is introduced for simultaneous estimation of all model parameters for commensurate fractional-order system models with time delays. The proposed method simultaneously estimates time delays along with other model parameters in an iterative manner by solving simple linear regression equations. For the case when the fractional order is unknown, we also propose a nested loop optimisation method where the time delay along with other model parameters are estimated iteratively in the inner loop and the fractional order is estimated in the non-linear outer loop. The applicability of the developed procedure is demonstrated by simulations on a fractional-order system model by doing Monte Carlo simulation analysis in the presence of white noise. The proposed algorithm has also been applied to identify a process of thermal diffusion in a wall in simulation, which are characterised by fractional-order behaviour.

1 Introduction

Fractional calculus is a generalisation of the traditional integer order integral and differential calculus to non-integer orders. With the growing power of computers, fractional calculus now has become an increasingly interesting topic of research in the scientific and industrial communities. In the last two decades, there has been a considerable development in the use of fractional operators in various fields. Before the 20th century, the theory of fractional calculus developed mainly as a pure theoretical field of mathematics useful only for mathematicians. A significant amount of discussions aimed at this subject has been presented by [1, 2]. However, recently it has been observed that many real-world physical systems are well characterised by fractional-order differential equations rather than using classical integer order models. In particular, materials having long memory and hereditary effects [3] and dynamical processes such as mass diffusion and heat conduction [4] in fractal porous media can be more adequately modelled by fractional-order (FO) models rather than integer-order models. Some of the other examples of fractal systems include transmission lines, electrochemical processes, dielectric polarisation and viscoelastic materials. Diffusive interfaces are particularly characterised by fractional-order dynamic behaviour, such as it appears in the case of an induction machine, with Foucault currents inside rotor bars [5] and heat transfer model relating flux and the temperature at the diffusive interface [6–8]. The special issue of signal processing [9] discusses in detail many applications of fractional calculus in different fields.

System identification has become a standard tool for modelling unknown systems. However, identifying a given system from data becomes more difficult when the physical systems are characterised by fractional-order differential equations instead of classical integer-order models. Thus, fractional models, using fractional differentiation, have been developed. The identified models for fractional-order systems can also be used to design controllers for these systems which may not be, just classical integer order controllers. For integer-order systems, the parameters of the model equation can be optimised directly once the maximum order of the system to be identified is fixed, while for fractional-order systems, identification requires the choice of the fractional powers (orders) of the operators, and also the coefficients of the operators. Thus, loss of integer order significantly complicates the identification process. Time-domain system identification using fractional differentiation models was initiated by Lay [10], Lin [11] and Cois [6], in their PhD thesis work in the late 1990s. The two model identification approaches developed were: equation error (ER)-based and output error-based approaches, both of which are very well studied in the literature for integer-order models. As in the case of continuous model identification for integer-order models, fractional differentiation of noisy signals also amplifies

noise. Hence, a linear transformation using low-pass filter can be applied to the model equation. As for the integer case, continuous-time model identification (CMI) using linear filter methods have also been proposed for FO models such as fractional integral filter, Poisson's state variable (SVFs) filters [12], and Refined Instrumental Variable for Continuous systems (RIVC) [13]. Using the iterative instrument variable (IV) approach it has been shown in [13] that the srivcf method is asymptotically unbiased, given the true system is in the right model class. They also proposed an algorithm for optimising the commensurate fractional order using a gradient-based method [14]. Many of these identification methods have already been applied to model some real physical processes for example see [5, 15, 16]. Use of fractional calculus theory for building parsimonious models can be found in [17, 18]. Also, identification methods based on orthogonal basis functions such as fractional Laguerre and Kautz basis functions, have been proposed [19]. The recent paper by [20] discusses briefly all these advances in time-domain system identification using fractional models.

However, none of these studies discuss methods for identification of fractional-order system models with time delays. Whenever material or energy is physically moved in a process or a plant, there is usually a time delay associated with the movement [21]. Time delay is also referred to as dead time, transportation lag or distance-velocity lag. Also, apparent time delays may result because of measurements or actuators in a process or in the identification exercise when a higher-order process is approximated by a lower-order model. It is reported in [22] that because of actuator limitations in some systems such as motion control, the system can be well modelled with a FO open-loop transfer function model with time delay. Malti *et al.* [15] noticed the time-lag in flux diffusion while modelling a thermal rod process (a fractional-order system) from experimental data. In this work, the focus is to develop a linear filter method based on the EE approaches for direct identification of continuous-time transfer function models. In this paper, we describe a scheme for continuous time identification of commensurate FO models with time delays. The proposed algorithm is a extension of the authors' previous work [23]. We propose two formulations based on the type of input signal excitation. The first formulation is based on step input excitation and the second one applies to more generic (RBS, PRBS, Sinusoidal, etc) kind of input signal excitation. In this scheme, the delay is estimated simultaneously with all other model parameters. The formulation as proposed by [24] for integer-order continuous-time systems is extended to identification of fractional system models. Wang and Zhang [25] proposed a method to estimate time delay along with other parameters for integer-order models using a step input. To the best knowledge of the authors no formulation for estimating all model parameters including delays has been proposed for fractional models. The formulation is based on the low-pass filtering operation where the filter is chosen as the combination of RIVC and a linear integral filter, to make the delay term appear as explicit parameter along with other parameters. The proposed method estimates the time delay along with constant model parameters in an iterative manner by solving simple linear regression equations. In the presence of noise, a modified scheme using instrument variable method where instruments are build based on auxiliary model is proposed. For fractional models the aim is also to identify fractional powers along with other model parameters. Here, we also propose a nested loop optimisation method where the time delay along with constant model parameters are estimated iteratively in the inner loop and the commensurate order is estimated in the non-linear outer loop for commensurate type transfer function models. The advantage of working with commensurate order models is that all fractional powers in the model are integer multiple of a single fractional order and therefore we need to estimate only one term in the outer loop. The proposed method is generic in the sense that it can also be applied to integer-order transfer function models.

The remainder of this paper is organised as follows. Section 2 presents a brief mathematical background of fractional calculus with an introduction to FO models. The CMI algorithm for commensurate fractional-order system models with time delays (CFOTDS) using step input and any other generic kind of input excitation is presented in Section 3. To study the efficacy of the proposed strategy developed in the Section 3, different examples of fractional-order models in the presence of noise are outlined in Section 4. Section 5 discusses the results for the proposed algorithm applied to a real process of thermal diffusion in a wall, followed by concluding remarks in Section 6.

2 Mathematical background

2.1 Definitions and FO models

Fractional calculus is a generalisation of integration and differentiation to non-integer orders. The two most popular definitions used to describe fractional differentiation and integration are the Grünwald–Letnikov (GL) discrete form of the definition and the Riemann–Liouville (RL) definition [1]. The GL definition for a function $f(t)$ is given as

$$\mathrm{D}^{\lambda} f(t) = \lim_{h\to\infty} \frac{1}{h^{\lambda}} \sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i} f(t - ih) \tag{1}$$

where

$$\binom{\lambda}{i} = \frac{\Gamma(\lambda+1)}{\Gamma(i+1)\Gamma(\lambda-i+1)} \tag{2}$$

and the operator D^{λ} defines fractional differentiation or integration depending on the sign of λ, $\Gamma(\cdot)$ being the well-known Euler's Gamma function and h is the finite sampling interval. This definition is particularly useful for digital implementation of the fractional operator. The RL definition is given as

$$\mathrm{D}^{\lambda} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\lambda)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\lambda+1-m}}\, \mathrm{d}\tau\right] \tag{3}$$

where m is an integer such that $(m-1 < \lambda < m)$ and $t > 0\ \forall\ \lambda \in \mathbb{R}_+$. For convenience, the Laplace domain notation is usually used to describe fractional differentiation–integration operation. When the initialisations are assumed to be zero,

$$L\{\mathrm{D}^{\lambda} f(t)\} = s^{\lambda} F(s) \quad (\lambda \in \mathbb{R}) \tag{4}$$

The generic single-input single-output (SISO) fractional-order

system representation in the Laplace domain is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\kappa_0 s^{\beta_0} + \kappa_1 s^{\beta_1} + \cdots + \kappa_m s^{\beta_m}}{1 + \mu_1 s^{\alpha_1} + \cdots + \mu_n s^{\alpha_n}} \tag{5}$$

where $\kappa_0, \kappa_1, \ldots, \kappa_m$ and $\mu_1, \mu_2, \ldots, \mu_n$ are constant model parameters or model coefficients, while $\beta_0 < \beta_1 < \cdots < \beta_m$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ are the fractional powers or fractional orders (real numbers). The transfer function (5) is called non-commensurate when β_j, α_i can take any arbitrary values.

The transfer function as given by (5) can be classified as a commensurate transfer function. A transfer function $G(s)$ is commensurable of order γ if and only if it can be written as $G(s) = F(s^\gamma)$, where $F = T/R$ is a rational function, with T and R as two coprime polynomials. Assuming that $G(s)$ is commensurate transfer function of order γ, then it can be written as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\gamma}}{1 + \sum_{i=1}^{n} a_i s^{i\gamma}} \tag{6}$$

Therefore, for commensurate transfer function all fractional powers are integer multiple of a real number, γ. Commensurate transfer function models represent more generic class of polynomial-type transfer functions, where $\gamma = 1$ gives standard integer-order transfer function models. A commensurate transfer function of order γ for fractional-order time-delay system is given as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\gamma}}{1 + \sum_{i=1}^{n} a_i s^{i\gamma}} e^{-Ls} \tag{7}$$

where L is the time delay. In this work, we will be working only with commensurate transfer function models with delays as described in equation (7).

2.2 Stability condition

Stability condition for a class of transfer function of the form (6) has been established by [26]. The theorem is as follows:

Stability theorem: A commensurable γ-order transfer function $G(s) = F(s^\gamma) = T(s^\gamma)/R(s^\gamma)$, where $T(.)$ and $R(.)$ are two coprime polynomials, is BIBO stable if and only if

$$0 < \gamma < 2$$

and $\forall\ \sigma \in \mathcal{C}$ such that $R(\sigma) = 0$

$$|\arg(\sigma)| > \gamma\frac{\pi}{2}$$

2.3 Integer-order approximation

The modelling and simulation of fractional-order systems are complicated because of their long memory behaviour [27] and are based on the approximation (approximating the infinite dimensional nature) of the fractional derivative operator. For digital implementation of the fractional-order operator, the key step is numerical evaluation or discretisation of this operator. In most of the cases, it is not easy to obtain analytical expressions of the output for a given input excitation for a fractional-order transfer function model. Two classes of methods developed over the last few years to approximate the fractional derivative operator can be classified as: direct methods – based on the approximation of a fractional derivative operator by a rational discrete time one, and indirect methods – based on the approximation of a fractional derivative operator by a rational continuous-time one. Power series expansion and continuous fraction expansion (CFE) of the Euler's, Tustin and Al-Alaoui operators give different discrete approximations of the fractional operator. The power series expansion of Euler's operator gives numerical approximation of the GL definition as in (1). The details for the discretisation schemes can be found in [28] and [29]. One of the good continuous approximation for this fractional-order operator compared to GL definition is the Oustaloup continuous approximation [27], where it makes use of a recursive distribution of poles and zeros. In this paper, we will be using the Oustaloup continuous approximation for the simulation of fractional-order transfer functions. Additional details on this appear below.

Many real physical systems generally have bandlimited fractional behaviour and also because of the practical limitations of input and output signals (Shannon's cut-off frequency for the upper band and the spectrum of the input signal for the lower band), fractional operators are usually approximated by high-order rational models. As a result, a fractional model and its rational approximation have the same dynamics within a limited frequency band. The Oustaloup approximation of s^λ in the frequency band $[\omega_A, \omega_B]$ has been defined as

$$s^\lambda \to s^\lambda_{[\omega_A,\omega_B]} = C_0\left(\frac{1+\frac{s}{\omega_A}}{1+\frac{s}{\omega_B}}\right)^\lambda \simeq C_0 \prod_{k=1}^{N} \frac{1+\frac{s}{\acute{\omega}_k}}{1+\frac{s}{\omega_k}} \tag{8}$$

where

$$\omega_i = \alpha\acute{\omega}_i, \quad \acute{\omega}_{i+1} = \eta\acute{\omega}_i \quad \text{and} \quad \lambda = 1 - \frac{\log\alpha}{\log\alpha\eta} \tag{9}$$

where the parameter C_0 is chosen such that the approximation shall have a unit gain at 1 rad/s.

Note that the proposed method is independent from the way fractional differentiation and integration are simulated in the time domain. We have used the Oustaloup approximation in this paper.

3 Identification of CFOTDS

3.1 Identification formulation

The transfer function for CFOTDS of commensurate order α is given as

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{j\alpha}}{1 + \sum_{j=1}^{n} a_j s^{j\alpha}} e^{-Ls} \tag{10}$$

For integer order models, $\alpha = 1$ and only the model coefficients a_j, b_j and L are estimated. However, here we are interested in estimating α as well. For the present case, initial conditions are assumed zero and the model in the vector form can be represented as

$$\mathbf{a}_n \mathbf{s}^{n\alpha} Y(s) = \mathbf{b}_m \mathbf{s}^{m\alpha} U(s) e^{-Ls} + e(s) \tag{11}$$

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where

$$\mathbf{a}_n = [a_n\ a_{n-1} \ldots a_1\ 1] \in R^{1\times(n+1)} \tag{12}$$

$$\mathbf{b}_m = [b_m\ b_{m-1} \ldots b_1\ b_0] \in R^{1\times(m+1)} \tag{13}$$

$$\mathbf{s}^{n\alpha} = [s^{n\alpha}\ s^{(n-1)\alpha} \ldots s^{\alpha}\ s^0]^T \in R^{(n+1)\times 1} \tag{14}$$

$$\mathbf{s}^{m\alpha} = [s^{n\alpha}\ s^{(m-1)\alpha} \ldots s^{\alpha}\ s^0]^T \in R^{(m+1)\times 1} \tag{15}$$

and $Y(s)$, $U(s)$ and $e(s)$ are the Laplace transforms of output $y(t)$, input $u(t)$ and $e(t)$, respectively. The term $e(t)$ accounts for the noise. Note that here we are using an ER approach for estimating a continuous-time model.

Parameter estimation using a filtering approach has been very well established method available in the literature; however, to estimate only the parameters (a_n and b_m) but not the delay. Victor *et al.* [14] proposed a continuous-time identification method with optimal fractional differentiation order for fractional-order systems. The estimation of delay is mathematically different from the estimation of other parameters because the other parameters appear explicitly in the model while the delay appears implicitly as can be seen in (10). Next, we devise a linear filter method for estimation of model parameters. To obtain explicit appearance of the delay term in the estimation equation and have it appear as an element in the parameter vector, we introduce a linear filter method with a structure of the filter as a combination of RIVC and a linear integral filter. This structure of a filter has been introduced by [24] for rational order models. This low pass filter not only serves the purpose of removing noise amplification but it also makes the delay term appear as a explicit parameter to be estimated along with the other parameters. The filter transfer function is represented as

$$F(s^{\alpha}) = \frac{1}{sA(s^{\alpha})} \tag{16}$$

where $A(s^{\alpha})$ is the denominator of the model equation. Now applying the filtering operation on both sides of (11) yields

$$\mathbf{a}_n\mathbf{s}^{n\alpha}F(s^{\alpha})Y(s) = \mathbf{b}_m\mathbf{s}^{m\alpha}F(s^{\alpha})U(s)e^{-Ls} + F(s^{\alpha})e(s) \tag{17}$$

or

$$\mathbf{a}_n\mathbf{s}^{n\alpha}\frac{1}{sA(s^{\alpha})}Y(s) = \mathbf{b}_m\mathbf{s}^{m\alpha}\frac{1}{sA(s^{\alpha})}U(s)e^{-Ls} + \varsigma(s) \tag{18}$$

where $\varsigma(s) = F(s^{\alpha})e(s)$. Here $F(s^{\alpha})$ can be factored as

$$\frac{1}{sA(s^{\alpha})} = \frac{C(s^{\alpha})}{sA(s^{\alpha})} + \frac{1}{s} \tag{19}$$

where

$$C(s^{\alpha}) = -(a_n s^{n\alpha} + a_{n-1}s^{(n-1)\alpha} + \cdots + a_1 s^{\alpha}) \tag{20}$$

Also, $\mathbf{a_n}\mathbf{s}^{n\alpha}$ and $\mathbf{b_m}\mathbf{s}^{m\alpha}$ can be factored as

$$\mathbf{a}_n\mathbf{s}^{n\alpha} = (\bar{\mathbf{a}}_n\mathbf{s}^{(n-1)\alpha}s^{\alpha} + 1) \tag{21}$$

and

$$\mathbf{b}_m\mathbf{s}^{m\alpha} = (\bar{\mathbf{b}}_m\mathbf{s}^{(m-1)\alpha}s^{\alpha} + b_0) \tag{22}$$

where $\bar{\mathbf{a}}_n$ and $\bar{\mathbf{b}}_m$ are the $\mathbf{a}_n$ and $\mathbf{b}_m$ vectors, respectively, with the last element removed. Now defining the filtered output and input variables as

$$Y_f(s) = \frac{Y(s)}{sA(s^{\alpha})} \quad \text{and} \quad Y_{f_D}(s) = \frac{s^{\alpha}Y(s)}{sA(s^{\alpha})} \tag{23}$$

$$U_f(s) = \frac{U(s)}{sA(s^{\alpha})} \quad \text{and} \quad U_{f_D}(s) = \frac{s^{\alpha}U(s)}{sA(s^{\alpha})} \tag{24}$$

Thus (18) becomes

$$Y_f(s) = -\bar{\mathbf{a}}_n\mathbf{s}^{(n-1)\alpha}Y_{f_D}(s) + \bar{\mathbf{b}}_m\mathbf{s}^{(m-1)\alpha}U_{f_D}(s)e^{-Ls} + b_0\left(\frac{C(s^{\alpha})}{sA(s^{\alpha})} + \frac{1}{s}\right)U(s)e^{-Ls} + \varsigma(s) \tag{25}$$

Defining additional filtered variables as

$$U_{f_C}(s) = C(s^{\alpha})U_f(s) = \frac{C(s^{\alpha})}{sA(s^{\alpha})}U(s) \tag{26}$$

$$U_I(s) = \frac{U(s)}{s} \tag{27}$$

Then (25) can be written as

$$Y_f(s) = -\bar{\mathbf{a}}_n\mathbf{s}^{(n-1)\alpha}Y_{f_D}(s) + \bar{\mathbf{b}}_m\mathbf{s}^{(m-1)\alpha}U_{f_D}(s)e^{-Ls} + b_0U_{f_C}(s)e^{-Ls} + b_0U_I(s)e^{-Ls} + \varsigma(s) \tag{28}$$

Before taking the laplace inverse on both sides, we define the laplace inverse for various terms

$$\mathcal{L}^{-1}(Y_f(s)) = y_f(t) \tag{29}$$

$$\mathcal{L}^{-1}(Y_{fD}(s)) = y_{f_D}(t) \tag{30}$$

$$\mathcal{L}^{-1}(U_{fD}(s)) = u_{f_D}(t) \tag{31}$$

$$\mathcal{L}^{-1}(\varsigma(s)) = \varsigma(t) \tag{32}$$

$$\mathcal{L}^{-1}(\mathbf{s}^{(n-1)\alpha}Y_{f_D}(s)) = y_{f_D}^{(n-1)\alpha}(t) \tag{33}$$

$$\mathcal{L}^{-1}(\mathbf{s}^{(m-1)\alpha}U_{f_D}(s)e^{-Ls}) = u_{f_D}^{(m-1)\alpha}(t-L) \tag{34}$$

$$\mathcal{L}^{-1}(U_{f_C}(s)) = u_{f_C}(t) \tag{35}$$

$$\mathcal{L}^{-1}(U_{f_C}(s)e^{-Ls}) = u_{f_C}(t-L) \tag{36}$$

and

$$\mathcal{L}^{-1}(U_I(s)) = \int_0^t u(t)\,\mathrm{d}t \tag{37}$$

$$\mathcal{L}^{-1}(U_I(s)e^{-Ls}) = u_I(t-L) \tag{38}$$

Now, depending upon the type of input signal used for perturbation we propose two different formulations for estimating the model parameters.

3.1.1 Using step input signal: In continuous model identification, by using a step input signal we can estimate

all the parameters of a higher dimensional model. If we have a step input signal of step size h, then

$$U(s) = \frac{h}{s} \tag{39}$$

then (38) becomes

$$\mathcal{L}^{-1}(U_I(s)e^{-Ls}) = \mathcal{L}^{-1}\left(\frac{1}{s}\frac{h}{s}e^{-Ls}\right) = h(t-L)\ \forall\ t > L \tag{40}$$

and (36) can be written as

$$\mathcal{L}^{-1}(U_{f_C}(s)e^{-Ls}) = \mathcal{L}^{-1}\left(\frac{C(s^\alpha)}{sA(s^\alpha)}\frac{h}{s}e^{-Ls}\right) = u_{f_C}(t-L) \tag{41}$$

Now taking inverse Laplace transform of (28) we have

$$\begin{aligned} y_f(t) = &-\bar{\mathbf{a}}_n \mathbf{y}_{f_D}^{(n-1)\alpha}(t) + \bar{\mathbf{b}}_m \mathbf{u}_{f_D}^{(m-1)\alpha}(t-L) \\ &+ b_0 h u_{f_C}(t-L) + b_0 h(t-L) + \varsigma(t) \quad \text{for } t > L \end{aligned} \tag{42}$$

If we define $\mathbf{G}_{f1}$ as

$$\mathbf{G}_{f1} = \begin{bmatrix} \mathbf{u}_{f_D}^{(m-1)\alpha}(t-L) \\ hu_{f_C}(t-L) + ht \end{bmatrix}^{\mathrm{T}} \tag{43}$$

then

$$y_f(t) = \begin{bmatrix} -\mathbf{y}_{f_D}^{(n-1)\alpha}(t) & \mathbf{G}_{f1} & -h \end{bmatrix} \begin{bmatrix} \bar{\mathbf{a}}_n \\ \mathbf{b}_m \\ b_0 L \end{bmatrix} + \varsigma(t) \tag{44}$$

or equivalently

$$\psi(t) = \phi(t)\theta + \varsigma(t) \tag{45}$$

where $\theta = \begin{bmatrix} \bar{\mathbf{a}}_n \\ \mathbf{b}_m \\ b_0 L \end{bmatrix}$. Note that $a_0 = 1$. Similarly, we can consider model (45) for all $t = t_k$, where $k = t, t+1, \ldots, N$, such that $t > L$, N being the total number of data points. The stacked terms in this equation then yield the following estimation equation

$$\Psi = \Phi\theta + \Delta \tag{46}$$

which is linear in parameter equation and can be solved using linear least squares. However, the delay, L appears as $b_0 L$ but since b_0 is estimated simultaneously in θ, we can estimate delay using this fact. In practice, the selection of the output $y(t)$ after $t > L$ can be made as follows [30]. When the process output is stationary, the process output will be monitored for a period, the 'listening' period, during which the noise band B_n can be found. Then, $y(t)$ satisfying

$$\arg(y(t)) > 2B_n \tag{47}$$

can be treated as the process response after $t > L$, and thus can be used for the model (46).

3.1.2 Generic input signal: For any other kind of input signal, we use graphical information from the time series input curve. From Fig. 1 it can be seen that, for the input signal $U(t)$, the $U_I(t-L)$ term corresponds to area under the $U(t)$ curve over the time instant $(t-L)$. This can be written as the sum of three terms (or representing areas on the curve) as

$$u_I(t-L) = u_I(t) - \overbrace{\int_{t-L}^{t} [u(t_k) - u(t)]dt_k}^{\text{area I}} - \underbrace{u(t)L}_{\text{area II}} \tag{48}$$

Here, for demonstration purpose we have used a sinusoidal input but the above relationship can be used for any type of input signal excitation. Using the above defined relation (48) and taking laplace inverse of (28), we have

$$\begin{aligned} y_f(t) = &-\bar{\mathbf{a}}_n \mathbf{y}_{f_D}^{(n-1)\alpha}(t) + \bar{\mathbf{b}}_m \mathbf{u}_{f_D}^{(m-1)\alpha}(t-L) + b_0 u_{f_C}(t-L) \\ &+ b_0\left[u_I(t) - \int_{t-L}^{t} [u(t_k) - u(t)]\mathrm{d}t_k - u(t)L\right] + \varsigma(t) \end{aligned} \tag{49}$$

Now if we again define the augmented $\mathbf{G}_{f2}(t)$ as

$$\mathbf{G}_{f2}(t) = \begin{bmatrix} \mathbf{u}_{f_D}^{(m-1)\alpha}(t-L) \\ u_{f_C}(t-L) + u_I(t) - \int_{t-L}^{t} [u(t_k) - u(t)]\mathrm{d}t_k \end{bmatrix} \tag{50}$$

then

$$y_f(t) = \begin{bmatrix} -\mathbf{y}_{f_D}^{(n-1)\alpha}(t) & \mathbf{G}_{f2}(t) & -u(t) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{a}}_n \\ \mathbf{b}_m \\ b_0 L \end{bmatrix} + \varsigma(t) \tag{51}$$

The model (51) can be written as a linear regression equation of the form

$$\psi(t) = \phi(t)\theta + \varsigma(t) \tag{52}$$

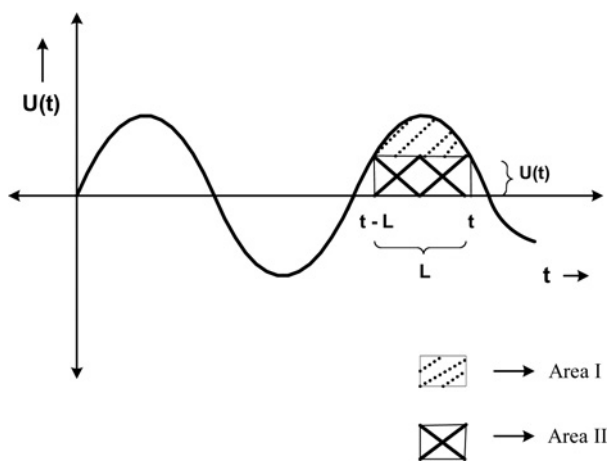

Fig. 1 *Time series process input curve*

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where

$$\theta = \begin{bmatrix} \bar{\mathbf{a}}_{\mathbf{n}} \\ \mathbf{b}_m \\ b_0 L \end{bmatrix} \tag{53}$$

Similarly, we can consider model (52) for all $t = t_k$ where $k = 1, 2, \ldots, N$. The stacked terms in this equation for different times then yield the following estimation equation

$$\Psi = \Phi\theta + \Delta \tag{54}$$

Thus, using all the filtered variables and approximating the area for the input curve, we are able to make time delay term appear as an explicit term in the form of a parameter vector in the regression model. Now given any input–output data, we can formulate this identification problem as given above and estimate all the parameters using linear least square method.

3.2 Parameter estimation

3.2.1 When α is known: For the case when the commensurate order α is known, we only need to estimate $\mathbf{a}_n$, $\mathbf{b}_n$ and L. Since the filter itself involves the coefficients $\mathbf{a}_n$ and we need L in order to formulate the above linear regression equation, we start with some initial values of $\mathbf{a}_n$ and $\mathbf{L}$, then solving the linear model developed in the previous section using linear least squares we can obtain a new estimate of the parameter vector θ. This parameter vector also gives us updated estimates of $\mathbf{a}_n$ (note that $a_0 = 1$) and $\mathbf{L}$. The updated values are again used to get the new estimates. The proposed algorithm is similar to the RIVC algorithm except the proposed algorithm formulates an iterative procedure to simultaneously estimate the parameters and the delay, L. However, the delay L appears as $b_0 L$ but since b_0 is estimated separately in θ, we can estimate the delay using this and do it iteratively until until the convergence is achieved for all the parameters. Note that we still have L term coupled with the b_0 term, so any error in estimating one term translates to another.

3.2.2 Instrument variable method: For the cases when the data are corrupted with white noise, the filtering operation converts the white noise signal to colored noise and this algorithm gives biased estimates in the presence of coloured noise. Therefore in order to get unbiased estimates of the parameters, we use the bootstrap instrumental variable (IV) algorithm [31] where the instruments are built based on the auxiliary model (using predicted $y(\hat{y})$ instead of measured y values). The instrument variable for the first formulation is then defined as

$$\phi_{\mathrm{IV}}(t) = \begin{bmatrix} -\hat{\mathbf{y}}_{f_D}^{(n-1)\alpha}(t) \\ G_{f1}(t) \\ -h \end{bmatrix}^{\mathrm{T}} \tag{55}$$

and for the second formulation it is defined as

$$\phi_{\mathrm{IV}}(t) = \begin{bmatrix} -\hat{\mathbf{y}}_{f_D}^{(n-1)\alpha}(t) \\ G_{f2}(t) \\ -U(t) \end{bmatrix}^{\mathrm{T}} \tag{56}$$

Using this, we can construct the instrumental variable matrix as $\Phi_{\mathrm{IV}}(t)$ and we add this IV scheme within the iteration steps of our proposed method thus requiring no additional steps, and the parameter estimation step is then given by

$$\hat{\theta}_{\mathrm{IV}}^{(i)} = \left(\Phi_{\mathrm{IV}}(\hat{\theta}_{\mathrm{IV}}^{(i-1)})^{\mathrm{T}}\Phi(\hat{\theta}_{\mathrm{IV}}^{(i-1)})\right)^{-1}\Phi_{\mathrm{IV}}(\hat{\theta}_{\mathrm{IV}}^{(i)})^{\mathrm{T}}\ \Psi(\hat{\theta}_{\mathrm{IV}}^{(i-1)}) \tag{57}$$

where (i) gives the iteration count, and $\Phi_{\mathrm{IV}}(\hat{\theta}_{\mathrm{IV}}^{(i-1)})$, $\Phi(\hat{\theta}_{\mathrm{IV}}^{(i-1)})$ and $\Psi(\hat{\theta}_{\mathrm{IV}}^{(i-1)})$ constructs Φ_{IV}, Φ and Ψ, respectively, for the parameter vector $\hat{\theta}_{\mathrm{IV}}^{(i-1)}$.

3.2.3 When α is unknown: For cases when the commensurate order α needs to be estimated along with other parameters, we can get an estimate of α by posing the problem as a nested loop optimization problem. We start with an initial value of α in the outer loop and in the inner loop we iteratively estimate the model parameters ($\mathbf{a}_n$, $\mathbf{b}_m$) and the delay term (L), as discussed in the previous section. Once convergence is achieved in the inner loop for a fixed α, we update α in the outer loop in a non-linear fashion.

3.3 Summary of the proposed algorithm

The iterative procedure for the parameter estimation for both the formulations can be summarised as

Step 1: Outer loop: Initialisation 1: Initialise the algorithm with some initial value for α.
Step 2: Inner loop: Initialisation 2: Initialise the inner loop with some initial values for $\hat{\mathbf{a}}_n^{(0)}$ and $\hat{L}^{(0)}$.
1. LS step: $i = 1$: Construct Ψ and Φ by replacing $\mathbf{a}_n$ and L with the estimates, as $\hat{\mathbf{a}}_n^{(0)}$ and $\hat{L}^{(0)}$ and get new estimates of the parameters as

$$\hat{\theta}^{(1)} = (\Phi^{\mathrm{T}}\Phi)^{-1}(\Phi^{\mathrm{T}}\Psi)$$

Get values of $\hat{\mathbf{a}}_n^{(1)}$, $\hat{\mathbf{b}}_m^{(1)}$ and $\hat{L}^{(1)}$ from $\hat{\theta}^{(1)}$.
2. IV step: $i = i + 1$ to convergence: Construct Ψ, Φ and Φ_{IV} by replacing $\mathbf{a}_n$, $\mathbf{b}_m$ and L with estimates as $\hat{\mathbf{a}}_n^{(i-1)}$, $\hat{\mathbf{b}}_m^{(i-1)}$ and $\hat{L}^{(i-1)}$ and obtain new $\hat{\theta}^{(i)}$ estimates as

$$\hat{\theta}^{(i)} = (\Phi_{\mathrm{IV}}^{\mathrm{T}}\Phi)^{-1}(\Phi_{\mathrm{IV}}^{\mathrm{T}}\Psi)$$

Obtain the values of $\hat{\mathbf{a}}_n^{(i)}$, $\hat{\mathbf{b}}_m^{(i)}$ and $\hat{L}^{(i)}$ from $\hat{\theta}^{(i)}$ and repeat this step till convergence.
Step 3: Update value of α based on the minimisation of the objective function (i.e. repeat steps 1 and 2 till this objective function is minimised).

$$\hat{\alpha} = \arg\min_{\alpha}(\Delta^{\mathrm{T}}\Delta)$$

The algorithm stops when

$$\|\hat{\alpha}^{(i)} - \hat{\alpha}^{(i-1)}\|_2 < \epsilon$$

the norm of the difference in the parameter vector for two consecutive iterations is less than, ϵ which is chosen as equal to 10^{-4}.

For the cases when α is known, we will only have the inner loop where the model parameters $\mathbf{a}_n$, $\mathbf{b}_m$ and L are estimated iteratively. Usually if an integer-order approximation is available for a fractional-order transfer function model, the values of parameters from these models are used as initial guesses for this algorithm. Also, since we are only dealing with SISO models, a finite impulse response for the data set can provide a good initial guess for the time delay. The algorithm for the proposed scheme is sketched in Fig. 2. The optimisation toolbox in MATLAB is used for solving the outer non-linear loop.

3.4 Convergence issues for the proposed method

The initialisation of the inner loop involves choices of $\mathbf{a}_n$, $\mathbf{b}_m$ and L. It is very difficult to prove theoretically the convergence of the proposed algorithm and this is beyond the scope of this paper. In practice, any initial choice is good except that the filter should not be unstable. As the filter is updated in every step, the final estimate of the parameters is not found to be much sensitive to the initial choice. However, for the outer loop some knowledge on the commensurate fractional order is necessary. This is the limitation of the proposed algorithm that if the outer loop is initialised with a poor initial guess, the convergence of inner loop is not always guaranteed. For the case when the fractional order is known, extensive simulation study shows that the parameter estimates obtained in the inner iterative loop converges to the true parameter values.

4 Simulation study

To illustrate the utility of the proposed algorithm, the identification exercise is carried out on some simulation examples. For the formulation using step input-type excitation, we present the result for the case when all model parameters including commensurate order are unknown. Using the second formulation, we present the results for the following two cases: when the commensurate order is known and when it is unknown. The identification exercise is carried out using two transfer functions of the form given below

$$G(s) = \frac{b_0}{a_1 s^{\alpha} + 1} e^{-Ls} \tag{58}$$

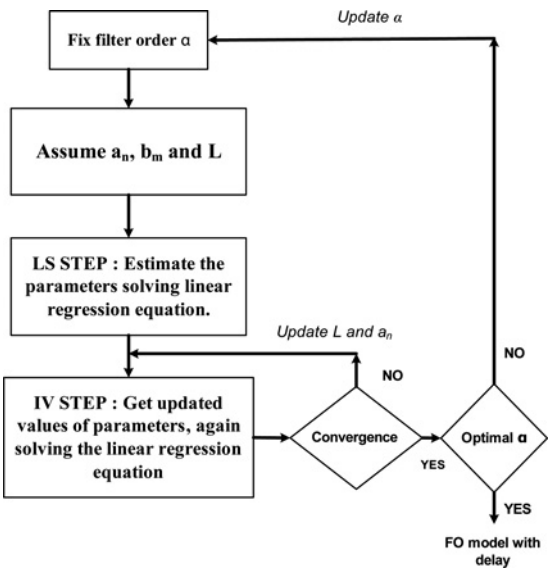

Fig. 2 *Algorithm for estimating parameters for CFOTDS*

$$G(s) = \frac{b_0}{a_2 s^{2\alpha} + a_1 s^{\alpha} + 1} e^{-Ls} \tag{59}$$

where α is the commensurate fractional order for this model. A zero initial condition is assumed for all cases. The sampled noise-free outputs generated from simulations for a given input excitation are corrupted by discrete-time white noise sequences with a signal-to-noise ratio (SNR) given as

$$\mathrm{SNR} = \frac{\mathrm{var(signal)}}{\mathrm{var(noise)}} \tag{60}$$

In presence of noise, Monte Carlo (MC) simulation analysis (number of runs is more for less computational expensive case) is done to evaluate the efficacy of the proposed algorithm. The integral in (50) is evaluated numerically. A fast sampling rate is chosen to reduce the estimation error because of the approximations involved in using fractional operator and a continuous-time model in general assumes that a sufficiently fast sampling rate data is used for parameter estimation.

4.1 Example 1

For this case, we considered the following fractional-order system described in (61), where we are also estimating the commensurate order α along with a_1, b_0 and L.

$$G_{\mathrm{FO}_1}(s) = \frac{1}{s^{0.5} + 1} e^{-0.5s} \tag{61}$$

Thus, the true parameters are $b_0 = 1$, $a_1 = 1$, $L = 0.5$ and $\alpha = 0.50$. The integer-order model approximation for (61) is available for this process and is given as

$$G_{\mathrm{IO}_1}(s) = \frac{1}{1.5s + 1} e^{-0.1s} \tag{62}$$

The fractional-order term is approximated with Oustaloup's approximation with $N = 15$ in the frequency interval $[10^{-3}, 10^{3}]$.

4.1.1 Step input excitation: The sampled data are generated by simulating the system using a unit step input with sampling time of 0.1 s. Fig. 3 shows the response of

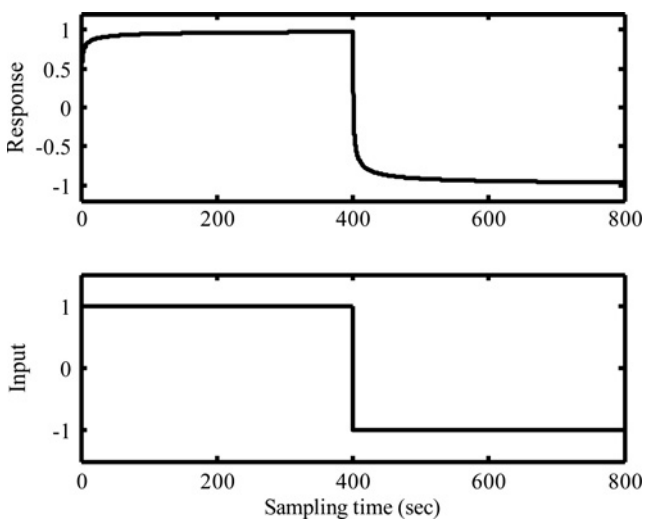

Fig. 3 *Step response for G_{FO_1}*

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Table 1 Step input: estimated parameters for process $G_{FO_1}(s)$

	True	SNR = ∞	SNR = 10		SNR = 5	
			$\bar{\hat{\theta}}$	$s(\hat{\theta})$	$\bar{\hat{\theta}}$	$s(\hat{\theta})$
α	0.50	0.500	0.498	0.027	0.492	0.039
a_1	1.00	1.001	1.000	0.025	1.001	0.06
b_0	1.00	1.000	1.003	0.009	1.01	0.026
L	0.50	0.501	0.490	0.023	0.488	0.05

the process to two successive steps. The output $y(t)$ is corrupted with noise having the following values of SNR: ∞(deterministic case), 10 and 5; and for each SNR, for different noise realisations we perform 200 MC simulations. For each case, we estimated the fractional order (α) as well as other model parameters (a_1, b_0, L) simultaneously using the proposed nested loop optimisation algorithm. Table 1 gives the the average ($\bar{\hat{\theta}}$) and the sample standard deviation ($s(\hat{\theta})$) of each parameter for these MC simulations. We started with an initial guess of $\alpha = 0.4$ for all the cases. As can be seen the estimated parameters including the fractional order α are quite close to the true values, thus indicating that the proposed algorithm gives unbiased estimates of all the parameters in the presence of noise and the uncertainty associated with each parameter is more for lower SNR. However, there are some computational issues with the outer non-linear loop, as for some other guess value of α, the inner loop does not always converge. Therefore having some process knowledge regarding the fractional order α is important.

4.1.2 Generic input excitation: The input excitation is chosen to be a pseudo-random binary sequence (PRBS) generated using the idinput function in MATLAB with levels [−1, 1]. As we do not have any rule of thumb available for fractional order processes, here a rule of thumb, which is commonly used in many identification techniques is used: the frequency band for PRBS perturbation is used according to the following rule, Frequency band $\simeq \left[0, \frac{30}{Ts}\right]$, where Ts is the settling time. The frequency band for PRBS excitation is chosen as [0, 0.02] and the sampled data are generated using a sampling time of 0.1 s. Fig. 4 shows the input–output data used for identification for this type of excitation. For this example we performed the identification exercise for these two cases:

- When α is known and
- When α is unknown.

The same three values of SNR (∞, 10, 5) are chosen and for each SNR, MC simulations are performed for different noise realisations. The algorithm is initialised with values of a_1, b_0, L from the integer-order model and for the case with unknown α it is initialised with a value of 0.55 and the inner loop with the parameters from the integer-order model.

Case 1 – When α is known: For this case, we estimated the model parameters (a_1, b_0, L) assuming the commensurate order α is known and equal to 0.5. Table 2 gives the average and the sample standard deviation of each parameter for 200 MC simulations. As can be seen, the estimated parameters are quite close to the true values, thus indicating that the proposed algorithm gives unbiased estimates even in the presence of noise.

Case 2 – When α is unknown: For this case, we estimated the fractional order (α) as well as other model parameters (a_1, b_0, L) simultaneously using the proposed nested loop optimisation algorithm. Table 3 gives the average and the sample standard deviation of each parameter for 100 MC simulations. Some error in estimating fractional order in the outer loop is present because of the approximations involved. Fig. 5 presents the ratios of estimated to true parameters along with their confidence intervals (average ± one standard deviation) in a graphical form. As can be seen, the estimated parameters are quite close to the true values, and the scaled confidence intervals include the ratio of one, thus indicating that the proposed algorithm gives unbiased estimates in the presence of noise.

Fig. 6 presents the Bode plots for parameter estimates from all the 200 realisations. As can be seen the 200 models fit the Bode diagram of the simulated system (61) really well.

4.2 Example 2

The process with the following transfer function is used

$$G_{FO_2}(s) = \frac{1}{8s^{2\times 0.75} + 5s^{0.75} + 1} e^{-4.8s} \tag{63}$$

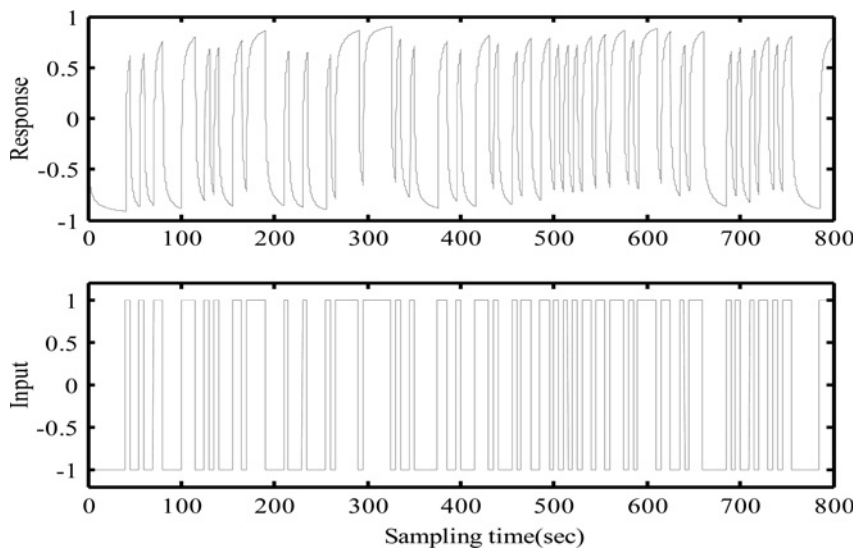

Fig. 4 *Generic input excitation data for G_{FO_1}*

Table 2 PRBS input: estimated parameters for process $G_{FO_1}(s)$

	True	SNR = ∞	SNR = 10		SNR = 5	
			$\bar{\hat{\theta}}$	$s(\hat{\theta})$	$\bar{\hat{\theta}}$	$s(\hat{\theta})$
a_1	1.00	1.001	1.011	0.017	1.007	0.044
b_0	1.00	1.000	1.001	0.006	1.000	0.015
L	0.50	0.500	0.502	0.012	0.490	0.039

Table 3 PRBS input: estimated parameters for process $G_{FO_1}(s)$

	True	SNR = ∞	SNR = 10		SNR = 5	
			$\bar{\hat{\theta}}$	$s(\hat{\theta})$	$\bar{\hat{\theta}}$	$s(\hat{\theta})$
α	0.50	0.499	0.502	0.019	0.509	0.036
a_1	1.00	1.000	1.01	0.013	1.018	0.029
b_0	1.00	1.001	1.003	0.009	0.998	0.018
L	0.50	0.499	0.506	0.01	0.515	0.029

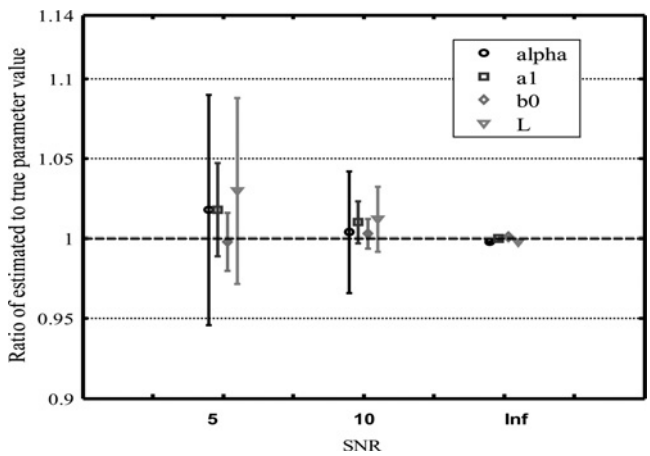

Fig. 5 *Effect of noise level on parameter estimates*

Thus, the true parameters are $b_0 = 1$, $a_1 = 5$, $a_2 = 8$, $L = 4.8$ and $\alpha = 0.75$. The input excitation is chosen to be a PRBS generated using the idinput function in MATLAB with levels $[-1, 1]$. The frequency band for PRBS excitation is chosen as [0, 0.015] and the sampled data is generated using a sampling time of 0.1 s. Fig. 7 shows the input–output data used for identification for this type of excitation.

Table 4 gives the average and the sample standard deviation of each parameter for 100 MC simulations. As can be seen, the average ± one standard deviation of the

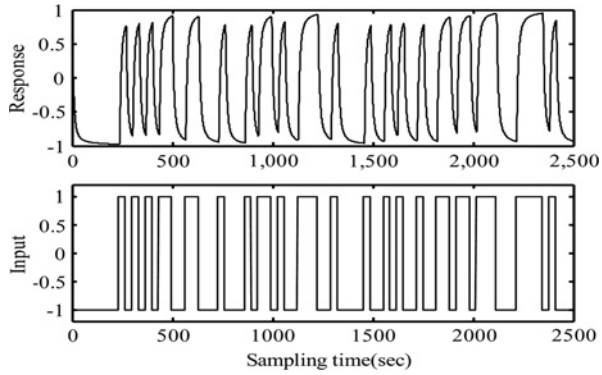

Fig. 7 *Generic input excitation data for G_{FO_2}*

Table 4 PRBS input: estimated parameters for process $G_{FO_2}(s)$

	True	SNR = ∞	SNR = 20		SNR = 10	
			$\bar{\hat{\theta}}$	$s(\hat{\theta})$	$\bar{\hat{\theta}}$	$s(\hat{\theta})$
α	0.75	0.750	0.744	0.022	0.740	0.025
a_1	5.00	5.010	4.896	0.254	4.821	0.511
a_2	8.00	8.004	8.149	0.331	8.110	0.415
b_0	1.00	0.999	1.008	0.002	1.000	0.031
L	4.80	4.793	4.701	0.108	4.689	0.151

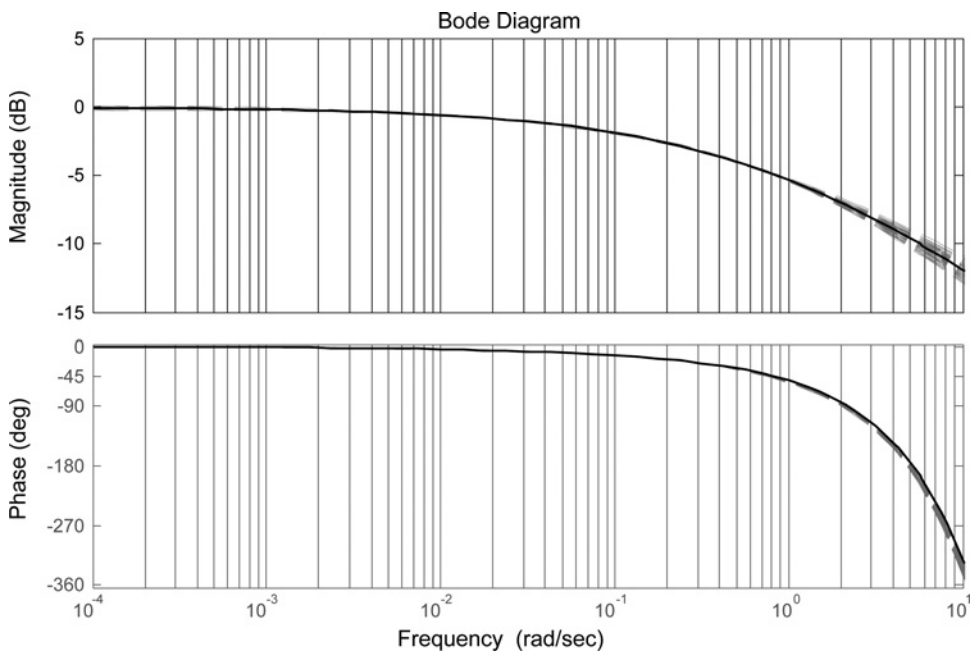

Fig. 6 *Bode plots for 200 MC simulation runs: (-) true process, (- -) 200 FO models*

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parameters include the true value of parameters. The algorithm is initialised with the value of α as 0.5 for all cases.

5 Application to thermal diffusion in a wall

For many real processes, fractional differentiation appears naturally when the system transients are governed by a diffusion equation, and particularly between the variables governing the functioning of the interface. Benchellal *et al.* [7] have shown that the transfer function $H(s)$ relating heat flux and the temperature, on the front face of the heated wall (which is governed by classical heat conduction equation), is a fractional order transfer function with half-integer order. This classical wall problem is considered as a process for this simulation study to illustrate the importance of our proposed algorithm on a real physical system. Fig. 8 represents the classical wall problem used to analyse heat transfer. The governing equation relating heat flux, $\Phi(x, t)$ and the temperature, $T(x, t)$ for this process is given by the heat diffusion equation as

$$\frac{\partial T(x, t)}{\partial t} = \zeta \frac{\partial^2 T(x, t)}{\partial x^2} \tag{64}$$

$$\Phi(x, t) = -\lambda \frac{\partial T(x, t)}{\partial x} \tag{65}$$

where ζ is the thermal diffusivity $(= \frac{\lambda}{\rho c})$, λ is the thermal conductivity, ρ is the mass density and c is the specific heat. The boundary conditions are such that the temperature at face B is kept constant and equal to zero during the overall heating experiment and the external heat is added at face A, that is

$$T(l, t) = 0 \tag{66}$$

$$\Phi(0, t) = \Phi_{\text{in}}(t) \tag{67}$$

where 'l' is the distance between two walls.

5.1 Transfer function for the wall problem

If S_A is the cross-sectional area of the wall and we define $y(t) = T(x^*, t)$ and heat input (not flux) as, $u(t) = \Phi_{\text{in}}(t)S_A$, then an analytical expression for the transfer function, $H(s)$ relating $Y(s)$ and $U(s)$ can be formulated as

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{S_A \lambda \sqrt{\frac{s}{\zeta}}} \frac{\exp\left((2l - x^*)\sqrt{\frac{s}{\zeta}}\right) - \exp\left(x^*\sqrt{\frac{s}{\zeta}}\right)}{1 + \exp\left(2l\sqrt{\frac{s}{\zeta}}\right)} \tag{68}$$

Once the exponential series expansion is done, the transfer function relating temperature and heat input is a commensurate FO model with commensurate order 0.5. For the case when $x^* = 0$, the transfer function as given by [8] is

$$\frac{T(0, s)}{\Phi_{in}(s)} = \frac{1}{\lambda\sqrt{\frac{s}{\zeta}}} \frac{\exp\left(2l\sqrt{\frac{s}{\zeta}}\right) - 1}{\exp\left(2l\sqrt{\frac{s}{\zeta}}\right) + 1} \tag{69}$$

Thus, it would require an infinite number of terms in both the numerator and denominator to model this process accurately, and working with a reduced model structure will always result in some modelling errors. A truncated fractional-order transfer function for model (69) has been presented by [8].

Based on the first-principles model, a time delay or dead time will not appear in a process transfer function until and unless there is mass or energy flow. Since this process involves energy flow, and depending on the location of the process measurement device (which is represented by x^* here), a dead time may appear in the process. Malti *et al.* [15] noticed the time lag in flux diffusion while modelling thermal rod process from experimental data. Thus, apparent time delay may be present in this process. Using this fact into consideration, and the fact that the process dynamics involve non-integer behaviour, it is assumed that this system can be approximated by FO model with a time delay using a fewer number of parameters. So, we are trying to model this process as a parsimonious in parameter model using fractional-order dynamic model with a delay term.

Sampled data are generated by performing numerical simulations using the finite-difference method. The following properties of Brass are used for simulations: $\rho = 8.522 \times 10^3$ kg/m^3, $c = 385$ J/kg °C, $\lambda = 111$ W/m °C. The distance between the walls, l is chosen as 5 cm and surface area of the wall(S_A) is 100 cm^2. Temperature is measured at a distance of 2 cm from the front face of the wall (x^*). 300 discretisation points are chosen. For a step input of 10 KW in heat input, if

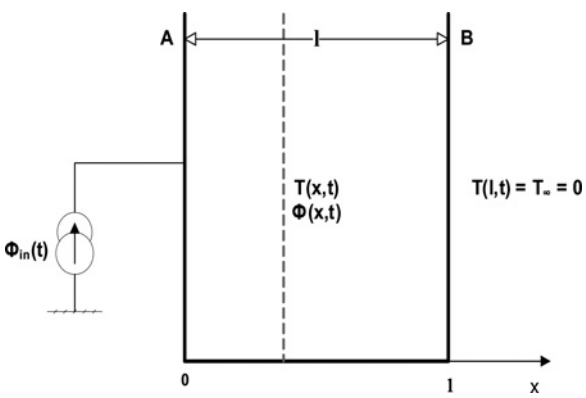

Fig. 8 *Classical wall problem*

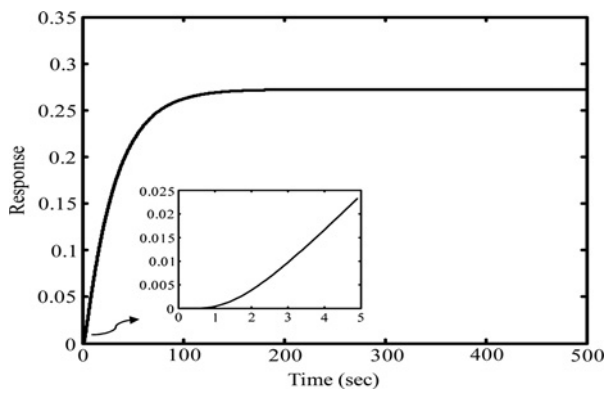

Fig. 9 *Step response of the process*

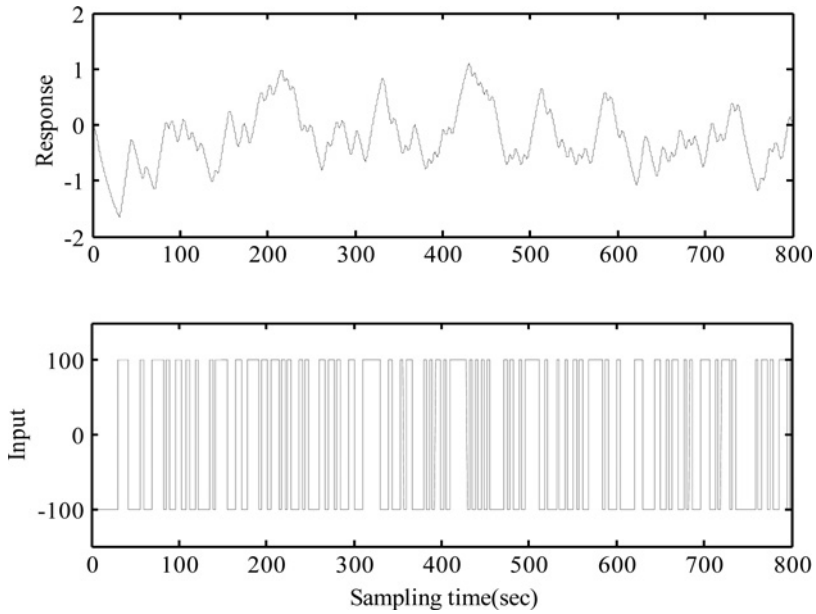

Fig. 10 *Time response for the wall process*

we see the zoomed in Fig. 9 for the initial time, it shows why we are trying to model this as FO model with delay.

5.2 Identification results

The sampled data (15 000 data points) for the identification exercise is generated using sampling time of 0.1 s with PRBS-type input excitation with levels of [−0.1, 0.1] KW in the frequency band of [0, 0.02]. A Gaussian white noise signal with SNR = 20 is added into the simulated noise free output sequence. This process is known to exhibit fractional order dynamics for frequencies less than 10^3 rad/s [7]. Here, we use the Oustaloup approximation with $N = 15$ in the frequency interval $[10^{-3}, 10^3]$ to approximate the fractional differential operator. The overall data were partitioned into two parts: (a) an identification set: first 8000 data points and (b) the validation data set: next 7000 data points. The identification data set is shown in Fig. 10. Next, the proposed algorithm is used to fit a fractional order model with time delay to the identification data. The model structure of the commensurate model (7) is varied to find a model that gives the best predictions. The estimated continuous time FO model (using average parameter value) along with sampled standard deviation for all parameters using 50 MC simulations is given as (70) (see (70))

It is not possible to show the predictions from all the 50 models, so only predictions from the average model has been presented in Fig. 11. It shows the model predictions of $G_{FO}(s)$ (infinite step ahead predictions) and the process output for all 50 realisations on the validation data set. As can be seen, FO model (G_{FO}) fits the measured output quite well. Next we compare the step response (for a step change of 10 KW in heat input) of all the 50 estimated models, to the step response from the actual process as shown in Fig. 12. As can be seen from Fig. 12, G_{FO} estimates both gain and delay very accurately. The Bode plot for the model

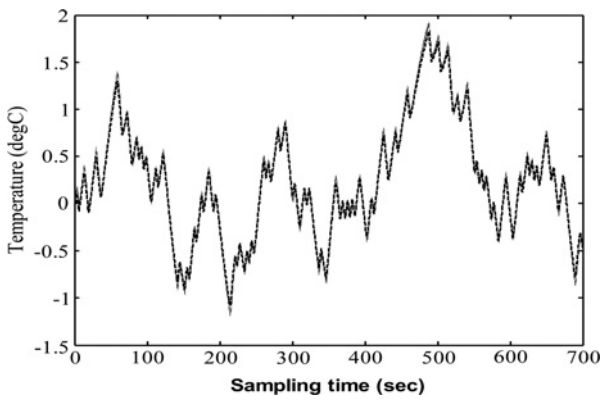

Fig. 11 *Model predictions on the validation set: (- -) black dotted line is model prediction)*

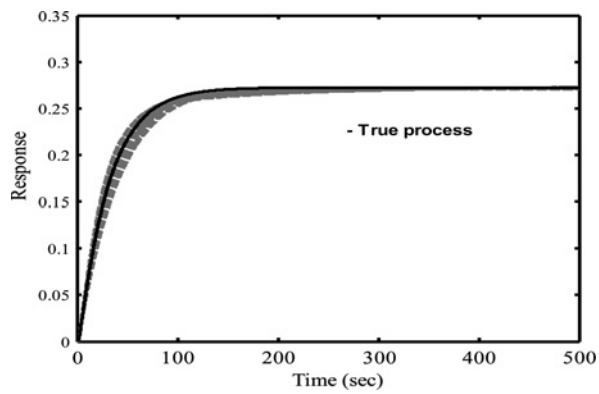

Fig. 12 *Step response of the estimated models (- -) and true process (-)*

$$G_{FO}(s) = \frac{0.0273(\pm 0.003)}{46.239(\pm 6.211)s^{2\times(0.585\pm 0.035)} + 2.263(\pm 1.265)s^{0.585(\pm 0.035)} + 1} e^{-0.899(\pm 0.058)s} \tag{70}$$

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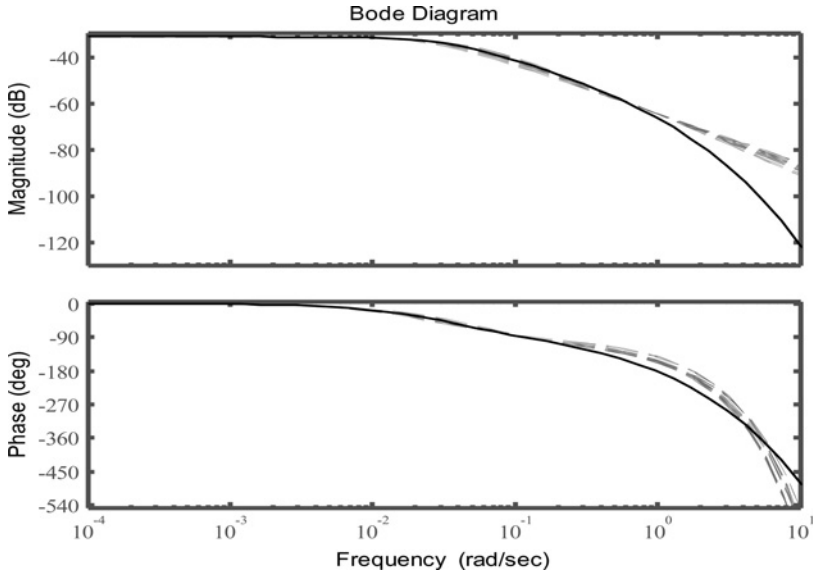

Fig. 13 *Frequency responses of the estimated models (- -) and the true process (-)*

G_{FO} along with the actual frequency response of the process is shown in Fig. 13. It can be seen that the frequency response of the model G_{FO} is nearly the same as the true process at low frequencies; however, at high frequencies the delay term in the model starts to dominate and there is mismatch between the true and model behaviour. The frequency response plot indicates that our proposed modelling scheme is able to capture the deterministic part of the process quite well at low as well as moderate frequency regions. Thus, the proposed algorithm can be used to model the low-frequency behaviour of this process. This process that is described by half-integer order model behaviour is discussed here to emphasise the importance of the developed algorithm to model fractional-order processes without requiring an integer-order approximation of the process. The developed FO model can then further be used to design rational or fractional-order controllers.

6 Conclusion

In this paper, a continuous-time identification method for commensurate FO models with time delay is proposed. The proposed method works with any kind of input signal excitation. It is based on a linear filter method where the filter is chosen as a combination of RIVC and a linear integral filter. Using this kind of filter, we can make the delay term appear as explicit parameter similar to other constant model parameters and can form a linear regression model to estimate the parameters in an iterative manner. For the case when the commensurate order α is unknown, a nested loop optimisation method is proposed to estimate the time delay along with constant model parameters in an iterative way in the inner loop and the fractional order in the outer loop. The applicability of the developed procedure is demonstrated on a CFOTDS for the cases when α is known and when it is unknown. In the presence of noise, MC simulation analysis for different noise realisations has been carried out to demonstrate that the proposed algorithm gives unbiased estimates even in the presence of noise. The proposed algorithm is also applied on a fractional order system of classical wall heat transfer problem, which is described by fractional behaviour. Future work proposed is to extend this algorithm for non-commensurate models.

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8 References

1 Oldham, K.B., Spanier, J.: 'The fractional calculus' (Academic Press, San Diego, 1974)
2 Podlubny, I.: 'Fractional differential equations' (Academic Press, San Diego, 1999)
3 Bagley, R.L., Torvik, P.: 'On the appearance of the fractional derivative in the behavior of real materials', *J. Appl. Mech.*, 1984, **51**, (1), pp. 294–298
4 Jenson, V.G., Jeffreys, G.V.: 'Mathematical methods in chemical engineering' (Academic, New York, 1977, 2nd edn.)
5 Benchellal, A., Bachir, S., Poinot, T., Trigeassou, J.C.: 'Identification of a non-integer model of induction machines'. IFAC Workshop on Fractional Differentiation and its Applications (FDA'04), Bordeaux, France, 2004, pp. 400–407
6 Cois, O.: 'Systèmes linéaires non entiers et identification par modèle non entier: application en thermique'. PhD thesis, Université Bordeaux, Talence, France, 2002
7 Benchellal, A., Poinot, T., Trigeassou, C.: 'Approximation and identification of diffusive interfaces by fractional systems', *Signal Process.*, 2006, **86**, (10), pp. 2712–2727
8 Gabano, J.D., Poinot, T.: 'Fractional identification algorithms applied to thermal parameter estimation'. 15th IFAC Symp. on System Identification, Saint-Malo, France, 2009, pp. 1316–1321
9 Ortigueira, M., Tenreiro, J.: 'Signal processing special issue: fractional calculus applications in signals and systems', *Signal Process.*, 2006, **86**, pp. 2285–2480
10 Le Lay, L.: 'Identification fréquentielle et temporelle par modèle non entier'. PhD thesis, Université Bordeaux, Talence, France, 1998
11 Lin, J.: 'Modélisation et identification de systémes d'ordre non entier'. PhD thesis, Université de Poitiers, Poitiers, France, 2001
12 Cois, O., Oustaloup, A., Poinot, T., Battaglia, J.L.: 'Fractional state variable filter for system identification by fractional model'. IEEE Sixth European Control Conf. (ECC'2001), Portugal, 2001
13 Malti, R., Victor, S., Oustaloup, A., Garnier, H.: 'An optimal instrumental variable method for continuous-time fractional model identification'. Proc. 17th IFAC World Congress, Korea, 2008, pp. 14379–14384
14 Victor, S., Malti, R., Oustaloup, A.: 'Instrumental variable method with optimal fractional differentiation order for continuous-time system identification'. 15th IFAC Symp. on System Identification, Saint-Malo, France, 2009, pp. 904–909

15 Malti, R., Sabatier, J., Akçay, H.: 'Thermal modeling and identification of an aluminum rod using fractional calculus'. 15th IFAC Symp. on System Identification, Saint-Malo, France, 2009, pp. 958–963
16 Sabatier, J., Aoun, M., Oustaloup, A., Grégoire, G., Ragot, F., Roy, P.: 'Fractional system identification for lead acid battery state of charge estimation', *Signal Process.*, 2006, **86**, pp. 2645–2657
17 Muddu, M., Narang, A., Patwardhan, S.C.: 'Development of ARX models for predictive control using fractional order and orthonormal basis filter parametrization', *Ind. Eng. Chem. Res.*, 2009, **48**, (19), pp. 8966–8979
18 Muddu, M., Narang, A., Patwardhan, S.C.: 'Reparametrized ARX models for predictive control of staged and packed bed distillation columns', *Control Eng. Pract.*, 2010, **18**, (2), pp. 114–130
19 Aoun, M., Malti, R., Levron, F., Oustaloup, A.: 'Synthesis of fractional Laguerre basis for system approximation', *Automatica*, 2007, **43**, pp. 1640–1648
20 Malti, R., Victor, S., Oustaloup, A.: 'Advances in system identification using fractional models', *J. Comput. Nonlinear Dyn.*, 2008, **3**, pp. 021401 (1–7)
21 Seborg, D., Edgar, T., Mellichamp, D.: 'Process dynamics and control' (John Wiley & Sons, New York, 1989)
22 Manabe, S.: 'Early development of fractional order control'. Proc. ASME 2003 Design Engineering and Technology Conf., Chicago, 2003, pp. 609–616
23 Narang, A., Shah, S.L., Chen, T.: 'Continuous time model identification of fractional order models with time delays'. 15th IFAC Symp. on System Identification, Saint-Malo, France, 2009, pp. 916–921
24 Ahmed, S., Huang, B., Shah, S.L.: 'Novel identification method from step response', *Control Eng. Pract.*, 2007, **15**, pp. 545–556
25 Wang, Q.G., Zhang, Y.: 'Robust identification of continuous systems with dead-time from step responses', *Automatica*, 2001, **37**, pp. 377–390
26 Matignon, D.: 'Stability properties for generalized fractional differential systems'. ESAIM Proc., 1998, pp. 145–158
27 Oustaloup, A.: 'La Dérivation non entiére' (Hermés, Paris, 1995)
28 Vinagre, B.M., Podlubny, I., Hernandez, A., Feliu, V.: 'Some approximations of fractional order operators used in control theory and applications', *Fractional Calculus & Applied Analysis*, 2000, **3**, (3), pp. 231–248
29 Chen, Y.Q., Moore, K.L.: 'Discretization schemes for fractional-order differentiators and integrators', *IEEE Trans. Circuits Syst., I*, 2002, **49**, (3), pp. 363–367
30 Bi, Q., Cai, W.J., Lee, E.L., Wang, Q.G., Hang, C.C., Zhang, Y.: 'Robust identification of first-order plus dead-time model from step response', *Control Eng. Pract.*, 1999, **7**, pp. 71–77
31 Young, P.C.: 'An instrumental variable method for real time identification of noisy process', *Automatica*, 1970, **6**, pp. 271–287

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