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Adaptive H_∞ Formation Control for Euler-Lagrange Systems

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***Abstract*— Design methods of adaptive H_∞ formation control of multi-agent systems composed of Euler-Lagrange systems are presented in this paper. The proposed control schemes are derived as solutions of certain H_∞ control problems, where estimation errors of tuning parameters and error terms in potential functions are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and that the desirable formations are achieved asymptotically via adaptation schemes.**

I. INTRODUCTION

Recently, formation control problems of multi-agent systems have attracted much attentions, and several formation control schemes were proposed based on various strategies (for example, leader-follower [1], behavior-based [2], virtual structure [3], and potential functions approaches [4], [5], [6]). Among those, the potential functions approaches seemed to be useful tools from the view points of flexibility of configurations of swarms, automatic avoidance of collisions of agents, and stability of maintaining formations. In those research works, adaptive control or sliding mode control methodologies were applied in order to deal with uncertainties of agents, and stability of control systems was assured via Lyapunov function analysis. Furthermore, robustness properties of the control schemes were also discussed in those works. However, so much attention has not been paid on control performance such as optimal property or transient performance in those approaches.

On the contrary, in recent decades, stable controller designs for nonlinear and adaptive control systems have been investigated from the view point of inverse optimality [7], [8]. In those research works, the resulting control systems are shown to be optimal to certain meaningful cost functionals, and stability of the overall systems is also assured. Those approaches are extended to the design of inverse optimal H_∞ adaptive control systems, and various adaptive control systems are derived from those strategies together with additional control performances such as robustness to uncertain time-varying elements of system parameters [9], [10].

The purpose of the present paper is to present design methods of adaptive formation control of multi-agent systems composed of Euler-Lagrange systems based on the notion of inverse optimality. The proposed control schemes are derived as solutions of certain H_∞ control problems, where estimation errors of tuning parameters and artificial error terms in potential functions are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and that the desirable formations are achieved asymptotically via adaptation schemes.

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II. PROBLEM STATEMENT

We consider a multi-agent system composed of N fully actuated mobile robots which are described as a class of Euler-Lagrange systems [4], [5] written as follows:

$$M_i(y_i)\ddot{y}_i + C_i(y_i, \dot{y}_i)\dot{y}_i = \tau_i, \quad (i = 1, \cdots, N), \qquad (1)$$

where $y_i \in \mathbf{R}^n$ is an output (a generalized coordinate), $\tau_i \in \mathbf{R}^n$ is a control input (a force vector), $M_i(y_i) \in \mathbf{R}^{n\times n}$ is an inertia matrix, and $C_i(y_i, \dot{y}_i) \in \mathbf{R}^{n\times n}$ is a matrix of Coriolis and centripetal forces. Each component has the following properties as a Euler-Lagrange system.

Properties of Euler-Lagrange Systems [11]

1) $M_i(y_i)$ is a bounded, positive definite, and symmetric matrix.
2) $\dot{M}_i(y_i) - 2C_i(y_i, \dot{y}_i)$ is a skew symmetric matrix.
3) The left-hand side of (1) can be written into
$$M_i(y_i)a_i + C(y_i, \dot{y}_i)b_i = -Y_i(y, \dot{y}_i, a_i, b_i)\theta_i, \qquad (2)$$
where $Y_i(y_i, \dot{y}_i, a_i, b_i)$ is a known function of y_i, $\dot{y}_i$, a_i, b_i (a regressor matrix), and θ_i is an unknown system parameter vector.

The control objective is to construct an adaptive formation control system for a swarm of mobile robots (1) in which desirable configurations are achieved asymptotically via adaptation schemes.

Remark More generalized Euler-Lagrange systems which include damping terms and gravitational forces, can be also considered in the present framework, since those are written in the similar form to (2). However, for simplicity of notations, the description (1) is to be employed hereafter.

III. FLOCKING CONTROL

First, we consider a particular flocking control problem [4] in which all agents stop at a desirable relative configuration defined by

$$\|y_i(t) - y_j(t)\| = d_{ij}, \quad (d_{ij} = d_{ji}, i \neq j), \qquad (3)$$
$$\dot{y}_i(t) = 0. \qquad (4)$$

A. Adaptive Flocking Control

We introduce a positive potential function $J(y) \in \mathbf{R}$ ($y = [y_1^\mathsf{T}, \cdots, y_N^\mathsf{T}]^\mathsf{T} \in \mathbf{R}^{nN}$) in order to handle the desired configuration (3), where the minimal point of $J(y)$ such as

$$J(y) \to \min, \quad \left(\frac{\partial J(y)}{\partial y_i} = 0, \quad (1 \le i \le N)\right), \qquad (5)$$

corresponds to the relative configuration (3). It is assumed that $J(y)$ is twice differentiable.

Define a control error s_i by

$$s_i = \dot{y}_i + g_i(y), \tag{6}$$

$$g_i(y) = \frac{\partial J(y)}{\partial y_i}. \tag{7}$$

Then, we obtain the next relation.

$$\begin{aligned} M_i\dot{s}_i + C_i s_i &= M_i(\ddot{y}_i + \dot{g}_i) + C_i(\dot{y}_i + g_i) \\ &= \tau_i - Y_i(y_i, \dot{y}_i, \dot{g}_i, g_i)\theta_i. \end{aligned} \tag{8}$$

The control law and adaptation law are determined such as

$$\tau_i = Y_i(y_i, \dot{y}_i, \dot{g}_i, g_i)\hat{\theta}_i - K_{pi}s_i - k_g g_i, \tag{9}$$

$$\frac{d}{dt}\hat{\theta}_i = -\Gamma_i Y_i(y_i, \dot{y}_i, \dot{g}_i, g_i)^{\mathsf{T}} s_i, \tag{10}$$

$$(K_{pi} = K_{pi}^{\mathsf{T}} > 0,\ k_g > 0,\ \Gamma_i = \Gamma_i^{\mathsf{T}} > 0),$$

where $\hat{\theta}_i$ is a current estimate of θ_i, and is tuned by the adaptation law (10). For stability analysis, we introduce a positive function V

$$V = \frac{1}{2}\sum_{i=1}^{N}\left(s_i^{\mathsf{T}} M_i s_i + \tilde{\theta}_i^{\mathsf{T}}\Gamma_i^{-1}\tilde{\theta}_i\right) + k_g J(y), \tag{11}$$

$$\tilde{\theta}_i = \hat{\theta}_i - \theta_i, \tag{12}$$

and take the time derivative of V along the trajectory of s_i, $\hat{\theta}_i$ and y.

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N}\left(-s_i^{\mathsf{T}} K_{pi}s_i - k_g s_i^{\mathsf{T}} g_i + k_g g_i^{\mathsf{T}}\dot{y}_i\right) \\ &= \sum_{i=1}^{N}\left\{-s_i^{\mathsf{T}} K_{pi}s_i - k_g s_i^{\mathsf{T}} g_i + k_g g_i^{\mathsf{T}}(s_i - g_i)\right\} \\ &= -\sum_{i=1}^{N}\left(s_i^{\mathsf{T}} K_{pi}s_i + k_g g_i^{\mathsf{T}} g_i\right) \le 0. \end{aligned} \tag{13}$$

Then, it follows that $s_i \in \mathcal{L}^\infty \cap \mathcal{L}^2$, $\hat{\theta}_i \in \mathcal{L}^\infty$, $J(y) \in \mathcal{L}^\infty$, and that $g_i \in \mathcal{L}^\infty \cap \mathcal{L}^2$ by considering twice differentiability of $J(y)$. Hence it is shown that $\dot{y}_i \in \mathcal{L}^\infty$, $\dot{g}_i \in \mathcal{L}^\infty$, and that $\tau_i \in \mathcal{L}^\infty$, and finally it holds that $\dot{s}_i \in \mathcal{L}^\infty$. Therefore, the following relation is deduced from Barbalat Lemma

$$\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} g_i(t) = 0, \tag{14}$$

and furthermore it follows that

$$\lim_{t\to\infty} \dot{y}_i(t) = 0. \tag{15}$$

It is also shown that the relative configuration (3) is achieved asymptotically, since $g_i \to 0$ (14).

The present control scheme (9) is similar to the ones discussed in [4], if we choose $k_g = 0$ in (9).

B. Adaptive H_∞ Flocking Control

Next, we propose an adaptive H_∞ control scheme where an estimation error of the tuning parameter and an artificial additive error to the potential function $J(y)$ are regarded as external disturbances to the processes. We consider the following positive function V_0.

$$V_0 = \frac{1}{2}\sum_{i=1}^{N} s_i^{\mathsf{T}} M_i s_i + (k_g + \delta)J(y), \tag{16}$$

where k_g, $\delta > 0$, and δ is an artificial error added to $J(y)$. We determine the control law such as

$$\tau_i = Y_i(y_i, \dot{y}_i, \dot{g}_i, g_i)\hat{\theta}_i - k_g g_i + v_i, \tag{17}$$

where v_i is a stabilizing signal to be determined later based on an H_∞ criterion. We take the time derivative of V_0 along the trajectory of s_i and y.

$$\begin{aligned} \dot{V}_0 &= \sum_{i=1}^{N}\left\{s_i^{\mathsf{T}}(\tau_i - Y_i^{(\dot{g},g)}\theta_i) + (k_g + \delta)g_i^{\mathsf{T}}\dot{y}_i\right\} \\ &= \sum_{i=1}^{N}\left\{s_i^{\mathsf{T}}(v_i + Y_i^{(\dot{g},g)}\tilde{\theta}_i)\right. \\ &\qquad \left. -(k_g + \delta)g_i^{\mathsf{T}} g_i + \delta g_i^{\mathsf{T}} s_i)\right\}, \end{aligned} \tag{18}$$

$$Y_i^{(\dot{g},g)} \equiv Y_i(y_i, \dot{y}_i, \dot{g}_i, g_i). \tag{19}$$

From the evaluation of $\dot{V}_0$, we introduce the following virtual system.

$$\dot{s}_i = f_i + g_{i1}\tilde{\theta}_i + g_{i2}\delta + g_{i3}v_i, \tag{20}$$

$$f_i = 0,\ g_{i1} = Y_i^{(\dot{g},g)},\ g_{i2} = g_i,\ g_{i3} = 1. \tag{21}$$

We are to stabilize the virtual system via a control input v_i by utilizing H_∞ criterion, where $\tilde{\theta}_i$ and δ are regarded as external disturbances to the process [9], [10]. For that purpose, we introduce the next Hamilton-Jacobi-Isaacs (HJI) equation and its solution V_i.

$$\begin{aligned} \mathcal{L}_{f_i}V_i + \frac{1}{4}&\left\{\frac{\|\mathcal{L}_{g_{i1}}V_i\|^2}{\gamma_{i1}^2} + \frac{\|\mathcal{L}_{g_{i2}}V_i\|^2}{\gamma_{i2}^2}\right. \\ &\left. -(\mathcal{L}_{g_{i3}}V_i)R_i^{-1}(\mathcal{L}_{g_{i3}}V_i)^{\mathsf{T}}\right\} + q_i = 0, \end{aligned} \tag{22}$$

$$V_i = \frac{1}{2}\|s_i\|^2, \tag{23}$$

where q_i and R_i are a positive function and a positive definite matrix, respectively, and those are derived from HJI equation based on inverse optimality [7], [8], [9], [10] for the given solution V_i and the positive constants γ_{i1}, γ_{i2}. The substitution of the solution V_i (23) into HJI equation (22) yields

$$\begin{aligned} \frac{1}{4}&\left\{\frac{s_i^{\mathsf{T}} Y_i^{(\dot{g},g)} Y_i^{(\dot{g},g)\mathsf{T}} s_i}{\gamma_{i1}^2}\right. \\ &\left. + \frac{s_i^{\mathsf{T}} g_i g_i^{\mathsf{T}} s_i}{\gamma_{i2}^2} - s_i^{\mathsf{T}} R_i^{-1} s_i\right\} + q_i = 0. \end{aligned} \tag{24}$$

Then, q_i and R_i are given as follows:

$$q_i = \frac{1}{4}s_i^{\mathsf{T}} K_i s_i, \tag{25}$$

$$R_i = \left(\frac{Y_i^{(\dot{g},g)} Y_i^{(\dot{g},g)\mathsf{T}}}{\gamma_{i1}^2} + \frac{g_i g_i^{\mathsf{T}}}{\gamma_{i2}^2} + K_i\right)^{-1}, \tag{26}$$

$$K_i = K_i^{\mathsf{T}} > 0,\ \ (K_i \in \mathbf{R}^{n\times n}), \tag{27}$$

where $K_i (= K_i^{\mathsf{T}} > 0)$ is a free parameter. By utilizing R_i, v_i is deduced as a solution for the corresponding H_∞ control problem.

$$
\begin{aligned}
v_i &= -\frac{1}{2}R_i^{-1}\mathcal{L}_{g_{i2}}V_i = -\frac{1}{2}R_i^{-1}s_i \\
&= -\frac{1}{2}\left(\frac{Y_i^{(\dot{g},g)}Y_i^{(\dot{g},g)\mathsf{T}}}{\gamma_{i1}^2} + \frac{g_i g_i^{\mathsf{T}}}{\gamma_{i2}^2} + K_i\right)s_i. \qquad (28)
\end{aligned}
$$

Then, we obtain the following theorems for the multi-agent system (1).

Theorem 1 *The H_∞ flocking control system composed of (1), (17) and (28) is uniformly bounded for an arbitrary bounded design parameter $\hat{\theta}_i$. Furthermore, v_i is an optimal control solution which minimizes the following cost functional J_{cost}.*

$$
\begin{aligned}
J_{cost} = \sup_{\tilde{\theta}_1,\cdots,\tilde{\theta}_N,\delta\in\mathcal{L}^2} &\left\{\sum_{i=1}^N \int_0^t (q_i + v_i^{\mathsf{T}}R_i v_i)d\tau + V_0(t)\right. \\
&\left. - \sum_{i=1}^N \gamma_{i1}^2 \int_0^t \|\tilde{\theta}_i\|^2 d\tau - \sum_{i=1}^N \gamma_{i2}^2 \int_0^t \delta^2 d\tau\right\}. \qquad (29)
\end{aligned}
$$

Additionally, the next inequality holds for any finite t.

$$
\begin{aligned}
&\sum_{i=1}^N \int_0^t (q_i + v_i^{\mathsf{T}}R_i v_i)d\tau + V_0(t) \\
&\le \sum_{i=1}^N \left\{\gamma_{i1}^2 \int_0^t \|\tilde{\theta}_i\|^2 d\tau + \gamma_{i2}^2 \int_0^t \delta^2 d\tau\right\} + V_0(0). \qquad (30)
\end{aligned}
$$

Theorem 2 *The adaptive H_∞ flocking control system composed of (1), (17), (28) and the adaptation law (10) is uniformly bounded, and the following relation holds*

$$
\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} \dot{y}_i(t) = \lim_{t\to\infty} g_i(t) = 0, \qquad (31)
$$

and the desirable relative configuration (3) is achieved asymptotically.

Proof: By considering HJI equation, we take the time derivative of $V_0(t)$ (16) along the trajectories of the multu-agent system (1) and the H_∞ flocking control scheme.

$$
\begin{aligned}
\dot{V}_0 &= \sum_{i=1}^N \left\{s_i^{\mathsf{T}}(v_i + Y_i^{(\dot{g},g)}\tilde{\theta}_i) - (k_g+\delta)g_i^{\mathsf{T}}g_i + \delta g_i^{\mathsf{T}}s_i)\right\} \\
&\quad - \sum_{i=1}^N \left[\frac{1}{4}\left\{\frac{s_i^{\mathsf{T}}Y_i^{(\dot{g},g)}Y_i^{(\dot{g},g)\mathsf{T}}s_i}{\gamma_{i1}^2} + \frac{s_i^{\mathsf{T}}g_i g_i^{\mathsf{T}}s_i}{\gamma_{i2}^2} - s_i^{\mathsf{T}}R_i^{-1}s_i\right\} + q_i\right] \\
&= \sum_{i=1}^N \left(v_i + \frac{1}{2}R_i^{-1}s_i\right)^{\mathsf{T}} R_i \left(v_i + \frac{1}{2}R_i^{-1}s_i\right) \\
&\quad - \sum_{i=1}^N v_i^{\mathsf{T}}R_i v_i - \sum_{i=1}^N q_i - \sum_{i=1}^N (k_g+\delta)g_i^{\mathsf{T}}g_i \\
&\quad - \sum_{i=1}^N \gamma_{i1}^2 \left\|\tilde{\theta}_i - \frac{Y_i^{(\dot{g},g)\mathsf{T}}s_i}{2\gamma_{i1}^2}\right\|^2 + \sum_{i=1}^N \gamma_{i1}^2\|\tilde{\theta}_i\|^2 \\
&\quad - \sum_{i=1}^N \gamma_{i2}^2 \left|\delta - \frac{g_i^{\mathsf{T}}s_i}{2\gamma_{i2}^2}\right|^2 + \sum_{i=1}^N \gamma_{i2}^2\delta^2. \qquad (32)
\end{aligned}
$$

Similarly, we take the time derivative of $V(t)$ (11) along the trajectory of the multu-agent system (1) and the adaptive H_∞ flocking control scheme.

$$
\dot{V}(t) = -\sum_{i=1}^N \left(s_i^{\mathsf{T}}R_i^{-1}s_i + k_g g_i^{\mathsf{T}}g_i\right) \le 0. \qquad (33)
$$

Then, Theorem 1 and Theorem 2 are derived from the evaluations of $\dot{V}_0(t)$ (32) and $\dot{V}(t)$ (33). ■

Remark In the proposed adaptive control system, it is also shown that $J(y)$ is uniformly bounded. Therefore, the collision of agents ($y_i = y_j$ $(i \neq j)$) is avoided automatically, if we choose $J(y)$ with the property such that $J(y) \to \infty$ as $y_i \to y_j$ [4], [5], [6].

IV. FORMATION CONTROL I

Secondly, we consider a formation control problem [4], [6] in which all agents continue to move with a desired velocity $\dot{y}_r$ (34) and with a desired relative configuration defined by (35).

$$
\dot{y}_i(t) = \dot{y}_r(t), \qquad (34)
$$

$$
\|y_i(t) - y_j(t)\| = d_{ij}, \qquad (d_{ij} = d_{ji},\ i \neq j), \qquad (35)
$$

where y_r is a reference point of the agents.

A. Adaptive Formation Control I

We introduce a positive potential function $J(y) \in \mathbf{R}$, which has the same property as the previous one (5), in order to handle the desired relative configuration (35). Define a control error s_i by

$$
s_i = \Delta\dot{y}_i + g_i(y), \qquad (36)
$$

$$
\Delta y_i = y_i - y_r, \qquad (37)
$$

where g_i is defined by (7). Then, we obtain the next relation.

$$
\begin{aligned}
M_i\dot{s}_i + C_i s_i &= M_i(\ddot{y}_i - \ddot{y}_r + \dot{g}_i) + C_i(\dot{y}_i - \dot{y}_r + g_i) \\
&= \tau_i - Y_i(y_i, \dot{y}_i, a_i, b_i)\theta_i, \qquad (38)
\end{aligned}
$$

where a_i and b_i are determined such as

$$
a_i = \dot{g}_i - \ddot{y}_r, \quad b_i = g_i - \dot{y}_r. \qquad (39)
$$

The control law and adaptation law are determined such as

$$
\tau_i = Y_i(y_i, \dot{y}_i, a_i, b_i)\hat{\theta}_i - K_{pi}s_i - k_g g_i, \qquad (40)
$$

$$
\frac{d}{dt}\hat{\theta}_i = -\Gamma_i Y_i(y_i, \dot{y}_i, a_i, b_i)^{\mathsf{T}}s_i, \qquad (41)
$$

$$
(K_{pi} = K_{pi}^{\mathsf{T}} > 0,\ k_g > 0,\ \Gamma_i = \Gamma_i^{\mathsf{T}} > 0).
$$

We take the time derivative of the positive function V (11) along the trajectories of s_i and $\hat{\theta}_i$, where s_i is defined by (36).

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^N \left(-s_i^{\mathsf{T}}K_{pi}s_i - k_g s_i^{\mathsf{T}}g_i + k_g g_i^{\mathsf{T}}\dot{y}_i\right) \\
&= \sum_{i=1}^N \left\{-s_i^{\mathsf{T}}K_{pi}s_i - k_g s_i^{\mathsf{T}}g_i + k_g g_i^{\mathsf{T}}(s_i - g_i + \dot{y}_r)\right\} \\
&= -\sum_{i=1}^N \left(s_i^{\mathsf{T}}K_{pi}s_i + k_g g_i^{\mathsf{T}}g_i\right) + \sum_{i=1}^N g_i^{\mathsf{T}}\dot{y}_r. \qquad (42)
\end{aligned}
$$

Here, we assume that

$$\sum_{i=1}^{N} g_i = 0. \tag{43}$$

It should be noted that the potential function $J(y)$ satisfying (43), is realized by choosing $d_{ij} = d_{ji}$ and by adjusting other parameters. Then, the following relation holds,

$$\dot{V}(t) = -\sum_{i=1}^{N} \left(s_i^{\mathsf{T}} K_{pi} s_i + k_g g_i^{\mathsf{T}} g_i \right) \leq 0, \tag{44}$$

and similarly to the previous case, it is shown that the control system is uniformly bounded, and that

$$\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} g_i(t) = 0, \tag{45}$$

and furthermore it follows that

$$\lim_{t\to\infty} \Delta\dot{y}_i(t) = 0, \tag{46}$$

and the tracking of the velocity (34) is achieved asymptotically. Additionally, since $g_i \to 0$ (45), the desired relative configuration (35) is also attained asymptotically.

B. Adaptive H_∞ Formation Control I

Next, we propose an adaptive H_∞ control scheme where an estimation error of the tuning parameter and an artificial additive error to the potential function $J(y)$ are regarded as external disturbances to the processes. We consider the positive function V_0 (16), where s_i is defined by (36). We determine the control law such as

$$\tau_i = Y_i(y_i, \dot{y}_i, a_i, b_i)\hat{\theta}_i - k_g g_i + v_i, \tag{47}$$

where v_i is a stabilizing signal to be determined later based on an H_∞ criterion. We take the time derivative of V_0 along the trajectory of s_i and y.

$$\begin{aligned} \dot{V}_0 &= \sum_{i=1}^{N} \left\{ s_i^{\mathsf{T}}(\tau_i - Y_i^{(a,b)}\theta_i) + (k_g + \delta) g_i^{\mathsf{T}} \dot{y}_i \right\} \\ &= \sum_{i=1}^{N} \Big\{ s_i^{\mathsf{T}}(v_i + Y_i^{(a,b)}\tilde{\theta}_i) \\ &\qquad -(k_g+\delta) g_i^{\mathsf{T}} g_i + \delta g_i^{\mathsf{T}} s_i) \Big\}, \end{aligned} \tag{48}$$

$$Y_i^{(a,b)} \equiv Y_i(y_i, \dot{y}_i, a_i, b_i), \tag{49}$$

where it is assumed that g_i satisfies the condition (43). From the evaluation of $\dot{V}_0$, we introduce the following virtual system.

$$\dot{s}_i = f_i + g_{i1}\tilde{\theta}_i + g_{i2}\delta + g_{i3} v_i, \tag{50}$$

$$f_i = 0,\ g_{i1} = Y_i^{(a,b)},\ g_{i2} = g_i,\ g_{i3} = 1. \tag{51}$$

We are to stabilize the virtual system via a control input v_i by utilizing H_∞ criterion, where $\tilde{\theta}_i$ and δ are regarded as external disturbances to the process [9], [10]. For that purpose, we introduce the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution V_i.

$$\begin{aligned} \mathcal{L}_{f_i} V_i + \frac{1}{4} \Bigg\{ & \frac{\|\mathcal{L}_{g_{i1}} V_i\|^2}{\gamma_{i1}^2} + \frac{\|\mathcal{L}_{g_{i2}} V_i\|^2}{\gamma_{i2}^2} \\ & -(\mathcal{L}_{g_{i3}} V_i) R_i^{-1} (\mathcal{L}_{g_{i3}} V_i)^{\mathsf{T}} \Bigg\} + q_i = 0, \end{aligned} \tag{52}$$

$$V_i = \frac{1}{2}\|s_i\|^2, \tag{53}$$

where q_i and R_i are a positive function and a positive definite matrix, respectively, and those are derived from HJI equation based on inverse optimality [7], [8], [9], [10] for the given solution V_i and the positive constants γ_{i1}, γ_{i2}. The substitution of the solution V_i (53) into HJI equation (52) yields

$$\begin{aligned} \frac{1}{4} \Bigg\{ & \frac{s_i^{\mathsf{T}} Y_i^{(a,b)} Y_i^{(a,b)\mathsf{T}} s_i}{\gamma_{i1}^2} \\ & + \frac{s_i^{\mathsf{T}} g_i g_i^{\mathsf{T}} s_i}{\gamma_{i2}^2} - s_i^{\mathsf{T}} R_i^{-1} s_i \Bigg\} + q_i = 0. \end{aligned} \tag{54}$$

Then, q_i and R_i are given as follows:

$$q_i = \frac{1}{4} s_i^{\mathsf{T}} K_i s_i, \tag{55}$$

$$R_i = \left(\frac{Y_i^{(a,b)} Y_i^{(a,b)\mathsf{T}}}{\gamma_{i1}^2} + \frac{g_i g_i^{\mathsf{T}}}{\gamma_{i2}^2} + K_i \right)^{-1}, \tag{56}$$

$$K_i = K_i^{\mathsf{T}} > 0,\ \ (K_i \in \mathbf{R}^{n\times n}), \tag{57}$$

where $K_i (= K_i^{\mathsf{T}} > 0)$ is a free parameter. By utilizing R_i, v_i is deduced as a solution for the corresponding H_∞ control problem.

$$\begin{aligned} v_i &= -\frac{1}{2} R_i^{-1} \mathcal{L}_{g_{i2}} V_i = -\frac{1}{2} R_i^{-1} s_i \\ &= -\frac{1}{2} \left(\frac{Y_i^{(a,b)} Y_i^{(a,b)\mathsf{T}}}{\gamma_{i1}^2} + \frac{g_i g_i^{\mathsf{T}}}{\gamma_{i2}^2} + K_i \right) s_i. \end{aligned} \tag{58}$$

Then, we obtain the following theorems for the multi-agent system (1).

Theorem 3 *It is assumed that $J(y)$ satisfies the condition (43). Then, the H_∞ formation control system composed of (1), (47) and (58) is uniformly bounded for an arbitrary bounded design parameter $\hat{\theta}_i$. Furthermore, v_i is an optimal control solution which minimizes the following cost functional J_{cost}.*

$$\begin{aligned} J_{cost} = \sup_{\tilde{\theta}_1,\cdots,\tilde{\theta}_N,\delta\in\mathcal{L}^2} & \Bigg\{ \sum_{i=1}^{N} \int_0^t (q_i + v_i^{\mathsf{T}} R_i v_i) d\tau + V_0(t) \\ & - \sum_{i=1}^{N} \gamma_{i1}^2 \int_0^t \|\tilde{\theta}_i\|^2 d\tau - \sum_{i=1}^{N} \gamma_{i2}^2 \int_0^t \delta^2 d\tau \Bigg\}. \end{aligned} \tag{59}$$

Additionally, the next inequality holds for any finite t.

$$\begin{aligned} & \sum_{i=1}^{N} \int_0^t (q_i + v_i^{\mathsf{T}} R_i v_i) d\tau + V_0(t) \\ & \leq \sum_{i=1}^{N} \left\{ \gamma_{i1}^2 \int_0^t \|\tilde{\theta}_i\|^2 d\tau + \gamma_{i2}^2 \int_0^t \delta^2 d\tau \right\} + V_0(0). \end{aligned} \tag{60}$$

Theorem 4 *On the same assumption (43), the adaptive H_∞ formation control system composed of (1), (47), (58) and the adaptation law (41) is uniformly bounded, and the following relation holds*

$$\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} \Delta\dot{y}_i(t) = \lim_{t\to\infty} g_i(t) = 0, \tag{61}$$

and the desirable configuration (35) together with the desired velocity tracking (34) is achieved asymptotically.

Proof: Similarly to the previous flocking control case, Theorem 3 and Theorem 4 are deduced from the evaluation of $\dot{V}_0(t)$ and $\dot{V}(t)$ along the trajectory of the multi-agent system (1), where HJI equation (52) is also considered. ■

V. FORMATION CONTROL II

Finally, we generalize the previous two cases (flocking control and formation control I), and consider a formation control problem of the leader-follower type [5], where all agents continue to move with a desired velocity $\dot{y}_r$

$$\dot{y}_i(t) = \dot{y}_r(t), \tag{62}$$

and also satisfy the formation constraints on the maximum distance from the reference point y_r and on the minimum relative distance from other agents written as below:

$$\|y_i - y_r\| \le r_i, \quad (r_i > 0,\ 1 \le i \le N), \tag{63}$$

$$\|y_i - y_j\| \ge d_{ij}, \quad (d_{ij} = d_{ji} > 0,\ 1 \le i \ne j \le N). \tag{64}$$

Instead of (64), the relative configuration (3), (35) can be also adopted as a specified case of the constraint on relative distances from other agents.

A. Adaptive Formation Control II

We introduce a positive potential function $J_G(\Delta y) \in \mathbf{R}$ ($\Delta y = [\Delta y_1^\mathsf{T}, \cdots, \Delta y_N^\mathsf{T}]^\mathsf{T}$) in order to handle the formation constraint on the maximum distance from the reference point y_r (63), and introduce another positive potential function $J_L(y)$ to handle the formation constraint on the minimum relative distance from other agents (64). It is assumed that $J_G(\Delta y)$ and $J_L(y)$ are twice differentiable, and that the desired total configurations (63), (64) correspond to the minimal points of $J_G(\Delta y)$ and $J_L(y)$ such as

$$J_G(\Delta y) \to \min, \quad \left(\frac{\partial J_G(\Delta y)}{\partial \Delta y_i} = 0 \ \ (1 \le i \le N)\right), \tag{65}$$

$$J_L(y) \to \min, \quad \left(\frac{\partial J_L(y)}{\partial y_i} = 0 \ \ (1 \le i \le N)\right). \tag{66}$$

Or equivalently, (65), (66) hold uniformly in the appropriate region defined by (63), (64).

Define a control error s_i by (36), (37) and

$$g_i(y) = \xi_i + \rho_i, \tag{67}$$

$$\xi_i(y) = \frac{\partial J_G(\Delta y)}{\partial \Delta y_i}, \quad \rho_i(y) = \frac{\partial J_L(y)}{\partial y_i}. \tag{68}$$

Then, we obtain the following relation.

$$\begin{aligned} M_i \dot{s}_i + C_i s_i &= M_i(\ddot{y}_i - \ddot{y}_r + \dot{g}_i) + C_i(\dot{y}_i - \dot{y}_r + g_i) \\ &= \tau_i - Y_i(y_i, \dot{y}_i, a_i, b_i)\theta_i, \end{aligned} \tag{69}$$

where a_i and b_i are defined by (39) and (67). We utilize the control law (40) and the adaptive law (41), where g_i is defined by (67). We introduce a positive function V

$$\begin{aligned} V = \frac{1}{2}\sum_{i=1}^{N}&\left(s_i^\mathsf{T} M_i s_i + \tilde{\theta}_i^\mathsf{T}\Gamma_i^{-1}\tilde{\theta}_i\right) \\ &+ k_g\{J_G(\Delta y) + J_L(y)\}, \end{aligned} \tag{70}$$

and take the time derivative of V along the trajectory of s_i and $\hat{\theta}_i$.

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N}\left(-s_i^\mathsf{T} K_{pi} s_i - k_g s_i^\mathsf{T} g_i + k_g \xi_i^\mathsf{T}\Delta\dot{y}_i + k_g \rho_i^\mathsf{T}\dot{y}_i\right) \\ &= \sum_{i=1}^{N}\{-s_i^\mathsf{T} K_{pi} s_i - k_g s_i^\mathsf{T} g_i + k_g \xi_i^\mathsf{T}(s_i - g_i) \\ &\quad + k_g \rho_i^\mathsf{T}(s_i - g_i + \dot{y}_r)\} \\ &= -\sum_{i=1}^{N}\left(s_i^\mathsf{T} K_{pi} s_i + k_g g_i^\mathsf{T} g_i\right) + \sum_{i=1}^{N}\rho_i^\mathsf{T}\dot{y}_r. \end{aligned} \tag{71}$$

Here, we assume that

$$\sum_{i=1}^{N}\rho_i = 0. \tag{72}$$

Similarly to (43), the potential function $J_L(y)$ satisfying (72), is realized by choosing $d_{ij} = d_{ji}$ and by adjusting other parameters. Then, the following relation holds,

$$\dot{V}(t) = -\sum_{i=1}^{N}\left(s_i^\mathsf{T} K_{pi} s_i + k_g g_i^\mathsf{T} g_i\right) \le 0, \tag{73}$$

and it is shown that the control system is uniformly bounded, and that

$$\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} g_i(t) = 0, \tag{74}$$

and furthermore it follows that

$$\lim_{t\to\infty}\Delta\dot{y}_i(t) = 0, \tag{75}$$

and the tracking of the velocity (62) is achieved asymptotically. On the contrary, since $g_i = \xi_i + \rho_i \to 0$, it follows that

$$\sum_{i=1}^{N} g_i = \sum_{i=1}^{N}\xi_i + \sum_{i=1}^{N}\rho_i \to 0, \tag{76}$$

and the next relation is derived from the assumption (72).

$$\sum_{i=1}^{N}\xi_i \to 0. \tag{77}$$

Here we consider the case where all agents do not satisfy the formation constraint related to $J_G(\Delta)$, and several agents are outside the desired region defined by (63). It should be noted that $\xi_i = 0$ for the agents inside the desired region. If those agents outside the desired region are on the one side of the region, then the corresponding ξ_i have the same sign along one axis, and this shows that (77) means the relation $\xi_i \to 0$ $(1 \le i \le N)$. Therefore, it follows that $\rho_i \to 0$ $(1 \le i \le N)$. Next, we consider the case where several agents are on the opposite sides outside the desired region. If we choose a sufficient large region related to $J_G(\Delta)$, then it follows that $\rho_i \to 0$ for the agents outside the region. Hence, $\xi_i \to 0$ holds for the corresponding ξ_i. In the end, by choosing appropriate formation constraints, such as an appropriate desired region related to $J_G(\Delta y)$ and appropriate relative distances related to $J_L(y)$, the next equation holds for all agents

$$\lim_{t\to\infty}\xi_i(t) = \lim_{t\to\infty}\rho_i(t) = 0, \tag{78}$$

and the desired formation of the leader-follower type is achieved asymptotically [5].

B. Adaptive H_∞ Formation Control II

Next, we propose an adaptive H_∞ control scheme where an estimation error of the tuning parameter and an artificial additive error to the potential functions $J_G(\Delta y)$, $J_L(y)$ are regarded as external disturbances to the processes. We consider the following positive function V_0.

$$V_0 = \frac{1}{2}\sum_{i=1}^{N} s_i^\top M_i s_i + (k_g+\delta)J_G(\Delta y) \\ +(k_g+\delta)J_L(y), \tag{79}$$

where k_g, $\delta > 0$, and δ is an artificial error added to $J_G(\Delta y)$ and $J_L(y)$. We utilize the control law (47) where g_i is defined by (67), and take the time derivative of V_0 along the trajectory of s_i, Δy_i and y.

$$\begin{aligned}\dot{V}_0 &= \sum_{i=1}^{N}\Big\{ s_i^\top(\tau_i - Y_i^{(a,b)}\theta_i) + (k_g+\delta)\xi_i^\top \Delta\dot{y}_i \\ &\quad +(k_g+\delta)\rho_i^\top \dot{y}_i\Big\} \\ &= \sum_{i=1}^{N}\Big\{ s_i^\top(v_i + Y_i^{(a,b)}\tilde{\theta}_i) \\ &\quad -(k_g+\delta)g_i^\top g_i + \delta g_i^\top s_i)\Big\},\end{aligned} \tag{80}$$

where it is assumed that ρ_i satisfies the condition (72). From the evaluation of $\dot{V}_0$, we introduce the virtual system (50), (51) in which g_i is defined by (67), and are to stabilize the virtual system via a control input v_i by utilizing H_∞ criterion, where $\tilde{\theta}_i$ and δ are regarded as external disturbances to the process [9], [10]. Then, by repeating the same discussion as the previous case, for q_i, R_i and v_i determined such as

$$q_i = \frac{1}{4}s_i^\top K_i s_i, \tag{81}$$

$$R_i = \left(\frac{Y_i^{(a,b)}Y_i^{(a,b)\top}}{\gamma_{i1}^2} + \frac{g_i g_i^\top}{\gamma_{i2}^2} + K_i\right)^{-1}, \tag{82}$$

$$K_i = K_i^\top > 0,\ \ (K_i \in \mathbf{R}^{n\times n}), \tag{83}$$

$$\begin{aligned} v_i &= -\frac{1}{2}R_i^{-1}\mathcal{L}_{g_{i2}}V_i = -\frac{1}{2}R_i^{-1}s_i \\ &= -\frac{1}{2}\left(\frac{Y_i^{(a,b)}Y_i^{(a,b)\top}}{\gamma_{i1}^2} + \frac{g_i g_i^\top}{\gamma_{i2}^2} + K_i\right)s_i.\end{aligned} \tag{84}$$

we obtain the last theorems of the present manuscript.

Theorem 5 *It is assumed that $J_L(y)$ satisfies the condition (72). Then, the H_∞ formation control system composed of (1), (47), (67) and (84) is uniformly bounded for an arbitrary bounded design parameter $\hat{\theta}_i$. Furthermore, v_i is an optimal control solution which minimizes the following cost functional J_{cost}.*

$$J_{cost} = \sup_{\tilde{\theta}_1,\cdots,\tilde{\theta}_N,\delta\in\mathcal{L}^2}\left\{\sum_{i=1}^{N}\int_0^t (q_i + v_i^\top R_i v_i)d\tau + V_0(t) \right. \\ \left. -\sum_{i=1}^{N}\gamma_{i1}^2\int_0^t \|\tilde{\theta}_i\|^2 d\tau. - \sum_{i=1}^{N}\gamma_{i2}^2\int_0^t \delta^2 d\tau\right\}. \tag{85}$$

Additionally, the next inequality holds for any finite t.

$$\sum_{i=1}^{N}\int_0^t (q_i + v_i^\top R_i v_i)d\tau + V_0(t) \\ \leq \sum_{i=1}^{N}\left\{\gamma_{i1}^2\int_0^t \|\tilde{\theta}_i\|^2 d\tau + \gamma_{i2}^2\int_0^t \delta^2 d\tau\right\} + V_0(0). \tag{86}$$

Theorem 6 *On the same assumption (72), the adaptive H_∞ formation control system composed of (1), (47), (67), (84) and the adaptation law (41) is uniformly bounded, and the following relations hold*

$$\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty}\Delta\dot{y}_i(t) = \lim_{t\to\infty} g_i(t) = 0, \tag{87}$$

and the desired velocity tracking (62) is achieved asymptotically. Furthermore, by choosing appropriate formation constraints, such as an appropriate desirable region related to $J_G(\Delta y)$ and appropriate relative distances related to $J_L(y)$, the desired formation of the leader-follower type is achieved asymptotically ($\xi_i \to 0$, $\rho_i \to 0$).

Proof: The proof is carried out similarly to Theorem 3 and Theorem 4. ■

VI. CONCLUDING REMARKS

Design methodologies of adaptive H_∞ formation control of multi-agent systems composed of Euler-Lagrange systems have been proposed in the present paper. The resulting control strategies are derived as solutions of certain H_∞ control problems, where estimation errors of tuning parameters and error terms in potential functions are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and that the desirable formations are achieved asymptotically via adaptation schemes.

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