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H_∞ sliding mode based scaled consensus control for linear multi-agent systems with disturbances

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ABSTRACT

This paper studies the scaled consensus control problem of networked multi-agent systems with linear coupling dynamics and external disturbances. A state feedback based distributed H_∞ sliding mode control (SMC) approach is firstly established by designing integral-type sliding function, and a linear matrix inequality (LMI) based sufficient condition is given, which can guarantee the states of all agents achieving scaled consensus with H_∞ disturbance attenuation index on sliding surface. A distributed adaptive SMC law with adaptive updated law is proposed such that the sliding surface is reachable. Then, the output feedback based distributed H_∞ SMC is considered by designing distributed observer, and a SMC law is synthesized for the reaching motion based on the state estimates. A LMI based sufficient condition for the scaled consensus with H_∞ disturbance attenuation index of the overall closed-loop system is derived. At last, the proposed distributed H_∞ SMC is further extended to solve the scaled consensus control problem of networked multi-agent systems under switching topology. An example is included to show the effectiveness of the proposed methods.

1. Introduction

Over the last few years, cooperative control of multi-agent systems has been paid to much attention due to its potential applications in satellite formation flying, cooperative unmanned air vehicles, distributed sensor networks, and so forth [1–3]. One of the most considered coordination control issues for multi-agent systems is to design a protocol based only on local relative information between neighboring agents to make the states of all agents reach an agreement on a state, which is known as the consensus problem. Du et al. [4–24] investigated the distributed consensus control problems for multi-agent systems by combining the graph theory with system control theories.

Many new classes of consensus problems have been studied in recent years, such as cluster consensus [25], bipartite consensus [26,27] and scaled consensus [28–30]. In particular, the newly proposed scaled consensus is more attractive, since in many physical-world networks, the states of agents converge to an initial-condition-dependent equilibrium, but do not reach a common value, such as in the closed queueing networks [32], compartmental mass-action systems and water distribution systems [28]. Likewise, many distributed algorithms seek to assign diverse values across agents,

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such as task-allocation and web-page-ranking algorithms [28]. There is also a typical example is the transcale coordination control systems of rendezvous and docking tasks of space spacecrafts and their simulators on ground [31], in which the motions of spacecrafts and ground simulators possess different scales. Because of it is hard to exactly reconstruct the motions of spacecrafts, for which the simulation tests are usually implemented in constrained regions on ground, so how to reach the multi-scale coordination control between spacecrafts and their simulators is remain an open problem [29,30]. The aim of scaled consensus is to design distributed protocol via local interactions to guarantee the states of agents reach assigned proportions, rather than a common value, in equilibrium [28], so it may effective solve the above problems. Moreover, the scale consensus not only can possess the same bipartite and cluster consensus behaviors through employing suitable scales [29]. Roy [28] and Meng and Jia [29] investigated the scaled consensus for multi-agent systems with single-integrator dynamics under fixed topology and switching topology, respectively. However, many real physical systems are modeled as multi-agent systems with second-order or higher order dynamics, such as distributed spacecrafts and coupled manipulators, so the distributed scaled consensus control problems for multi-agent systems with second-order or higher order dynamics should be further addressed, which forms one of the two motivations for our research.

In a real physical system network, the evolutions of multi-agent systems are unavoidably affected by disturbances and noises [33–35]. Meng and Jia [30] addressed the robust scaled consensus for multi-agent systems with single-integrator dynamics and disturbances. For systems with persistent disturbances, H_∞ control has been widely used since it provides explicit performance index in the sense of L_2 gain, and many researchers devoted to transfer robust consensus into H_∞ control problem [34–37]. Sliding mode control (SMC) is also an effective robust control approach for linear or nonlinear systems with external disturbances, because of the sliding mode dynamics have intrinsic adaptiveness for matched uncertain nonlinear dynamics and external disturbances [38–40]. However, for systems with unmatched disturbances, the traditional SMC can not attenuate mismatched disturbances effectively on sliding mode dynamics. To solve this problem, the H_∞ SMC is proposed to attenuate mismatched disturbances on sliding mode dynamics by combining H_∞ control and SMC, which has been extensively applied to design robust controllers for time-delay systems [41,42], stochastic systems [43,44], and switching systems [45,46] and shows the preferable disturbances attention ability. Although the H_∞ SMC has many advantages for uncertain systems with disturbances, the above results only consider H_∞ sliding mode controllers design for traditional single systems. A natural question is: How to extend the H_∞ SMC into the distributed framework to solve robust consensus problems for multi-agent systems with disturbances? This is the second motivation of our research.

In order to response to the above discussion, we will study the robust scaled consensus control problem for networked linear multi-agent systems with external disturbances by a novel distributed H_∞ SMC approach in this paper. The main contributions can be summarized as follows: (a) The robust scaled consensus for multi-agent systems with linear coupling dynamics is considered, where the second-order and high-order multi-agent systems can be seen as the special case of the discussed systems; (b) We expand the traditional H_∞ SMC design to the distributed H_∞ SMC design for a multi-agent system under directed topology, where the distributed state feedback design and distributed output feedback design are both established. By using the Lyapunov stability theory and linear matrix inequality (LMI) technique, the integral sliding function is established and the distributed SMC law is constructed, which can guarantee the states of all agents can be driven onto sliding surface and achieving scaled consensus if disturbances satisfy matching condition and scaled consensus with H_∞ index if disturbances do not satisfy matching condition in there. Finally, a numerical simulation example is included to show the effectiveness of the proposed methods.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2. The H_∞ SMC based scaled consensus problems are presented in Section 3. Simulations are provided in Section 4. Conclusions are given in Section 5.

Notation: $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_1$ denotes the 1-norm, and $\|\cdot\|_2$ denotes the $L_2[0,\infty)$ norm. $\mathbf{1}_n \in \Re^n$ denotes the column vector that all elements are ones, $I_n \in \Re^{n\times n}$ denotes the identity matrix. $\otimes$ is the Kronecker product. $X > 0$ means the matrix X is symmetric and positive definite.

2. Problem formulation

2.1. Algebraic graph theory

Denote $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ as a weighted directed graph with the set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighboring set of node i is defined by $N_i = \{j \in V : (j, i) \in \mathcal{E}\}$. The adjacency matrix of the graph $\mathcal{G}$ is defined by $\mathcal{A} = [a_{ij}] \in \Re^{n\times n}$, and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$; otherwise $a_{ii} = 0$. The Laplacian matrix of the graph $\mathcal{G}$ is defined by $L = [l_{ij}] \in \Re^{n\times n}$, and $l_{ii} = \sum_{j=1}^{n} a_{ij}, l_{ij} = -a_{ij}$. A direct path from node i to node j is a sequence of successive edges in the form $\{(i, k), (k, l), \ldots, (m, j)\}$. If there is a special node such that there is a directed path from this node to every other node in $\mathcal{G}$, then $\mathcal{G}$ is said to have a spanning tree, and this special node is called the root node.

Lemma 1 [34]**.** *The directed graph $\mathcal{G}$ has a spanning tree if and only if Laplacian L of $\mathcal{G}$ has a simple zero eigenvalue (with eigenvector $\mathbf{1}_n$). In addition, all the other eigenvalues have positive real parts.*

2.2. Model description

Consider the network contains n agents indexed by $1, 2, \ldots, n$, where the ith agent is assumed to have the following linear coupling dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + Dw_i(t) \tag{1}$$

$$y_i(t) = Cx_i(t) \tag{2}$$

where $x_i(t) \in \Re^m$, $u_i(t) \in \Re^q$ and $y_i(t) \in \Re^o$ are the state, control input and measurement output, respectively. $w_i(t) \in \Re^p$ is the external disturbance input belonging to $L_2[0, \infty) \cup L_\infty[0, \infty)$. A, B, D, C are constant matrices with proper dimensions. Regarding the ith agent as the node i, the topology relationship among the N agents are described by a weighted directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ with $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{A} = [a_{ij}] \in \Re^{n\times n}$.

Assumption 1. (A, B) is controllable and $\text{rank}(B) = q$.

Assumption 2. (A, C) is detectable.

Definition 1 [30]**.** The scaled consensus of multi-agent system (1) is achieved if $\lim_{t\to\infty}[\frac{x_i(t)}{\alpha_i} - \frac{x_j(t)}{\alpha_j}] = 0, \forall i, j = 1, 2, \ldots, n.$

The scalars $\alpha_1, \ldots, \alpha_n$ are assumed to be non-zero and represent the scales of states $x_1(t), \ldots, x_n(t)$, respectively. Denote $\alpha_{ij} = \frac{\alpha_i}{\alpha_j}$ as a transformation, which can transfer the state $x_j(t)$ to possess the same scale as the state of $x_i(t)$. For the transcale coordination control of agents with different scales, the scaled consensus defined in Definition 1 means that the state of agent i agrees with the the states of other agents but of its own scale α_i, i.e., $\lim_{t\to\infty} x_i(t) = \alpha_i c$, where c is a common quantity for all agents.

Since the multi-agent system (1) is influenced by external disturbances, it is hard to guarantee the accurate scaled consensus defined in Definition 1. So, this paper will attempt to design distributed H_∞ SMC protocol such that the interference of external disturbances to the scaled consensus performance be attenuated.

Remark 1. This paper considers the multi-agent systems with general linear coupling dynamics, where the multi-agent systems with second-order or high-order integrator dynamics can be seen as the special case of system (1).

3. Main results

3.1. H_∞ *SMC based distributed state feedback control*

Define

$$z_i(t) = x_i(t) - \frac{1}{n}\sum_{j=1}^{n} \alpha_{ij} x_j(t) \tag{3}$$

as the controlled output, then we have

$$\frac{1}{\alpha_i} z_i(t) = \frac{1}{\alpha_i} x_i(t) - \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\alpha_j} x_j(t) \tag{4}$$

which means that $\frac{1}{\alpha_i} z_i(t)$ reflects the disagreement of $\frac{1}{\alpha_i} x_i(t)$ to the average state of $\frac{1}{\alpha_j} x_j(t), j = 1, 2, \ldots, N$. Denote $x(t) = [x_1(t)^T, \ldots, x_n^T(t)]^T \in \Re^{mn}, u(t) = [u_1(t)^T, \ldots, u_n^T(t)]^T \in \Re^{qn}, w(t) = [w_1(t)^T, \ldots, w_n^T(t)]^T \in \Re^{pn}, z(t) = [z_1(t)^T, \ldots, z_n^T(t)]^T \in \Re^{mn}$. Then, we can use Kronecker product to describe the matrix form equation of (1) and (3) as

$$\begin{aligned} \dot{x}(t) &= (I_n \otimes A)x(t) + (I_n \otimes B)u(t) + (I_n \otimes D)w(t) \\ z(t) &= (\alpha L_c \alpha^{-1} \otimes I_m)x(t) \end{aligned} \tag{5}$$

where $\alpha = \text{diag}\{\alpha_1, \ldots, \alpha_n\}$, $L_c = [l_{c_{ij}}] \in \Re^{n\times n}$, $l_{c_{ij}} = \frac{n-1}{n}, i = j$, and $l_{c_{ij}} = -\frac{1}{n}, i \neq j$.

The objectives of H_∞ SMC based distributed state feedback control are to design u_i for system (1) such that

(i) the state of system (1) will arrive at the sliding surface, and not leave there in the subsequent time;

(ii) the system (1) achieves scaled consensus with H_∞ disturbance attenuation index γ on the sliding surface, that is, for $w(t) = 0$, the overall closed-loop system achieves scaled consensus on the sliding surface; for $w(t) \neq 0$, the overall closed-loop system achieves $\|T_{zw}(s)\|_\infty < \gamma$ on the sliding surface, or equivalently, the sliding mode dynamics satisfy the dissipation inequality $\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t$ where $\gamma > 0$ is a given constant and $\|T_{zw}(s)\|_\infty$ is defined as $\|T_{zw}(s)\|_\infty = \sup_{0\neq w(t)\in L_2[0,\infty)} \frac{\|z(t)\|_2}{\|w(t)\|_2}$ [49].

3.1.1. State based integral sliding function design

The distributed integral-type sliding function for the ith agent is constructed as:

$$s_i(t) = B^T X x_i(t) - \int_0^t (B^T X A x_i(s) + B^T X B K \sum_{j \in N_i} a_{ij}(\alpha_{ij} x_j(s) - x_i(s)))\mathrm{d}s \tag{6}$$

where $X \in \Re^{m \times m}$ is a positive definite matrix and $K \in \Re^{q \times m}$ is a constant matrix such that $A - BK$ is Hurwitz, thus $B^T XB$ is nonsingular. Denote $s(t) = [s_1(t)^T, \ldots, s_n^T(t)]^T \in \Re^{qn}$, we have

$$s(t) = (I_n \otimes B^T X)x(t) - \int_0^t ((I_n \otimes B^T XA)x(s) - (\alpha L \alpha^{-1} \otimes B^T XBK)x(s))\mathrm{d}s \tag{7}$$

Substituting the solution $x(t)$ of system (5) into (7), we can obtain

$$s(t) = (I_n \otimes B^T X)x(0) + \int_0^t ((\alpha L \alpha^{-1} \otimes B^T XBK)x(s) + (I_n \otimes B^T XB)w(s) + (I_n \otimes B^T XB)u(s))\mathrm{d}s \tag{8}$$

If the system states are driven onto the sliding surface, we have $s(t) = 0$ and $\dot{s}(t) = 0$, and from $\dot{s}(t) = 0$, we get the equivalent control law as

$$u_{eq}(t) = -(\alpha L \alpha^{-1} \otimes K)x(t) - (I_n \otimes (B^T XB)^{-1} B^T XD)w(t) \tag{9}$$

Substituting $u_{eq}(t)$ into (5), we get the following sliding mode dynamics

$$\begin{aligned} \dot{x}(t) &= (I_n \otimes A - \alpha L \alpha^{-1} \otimes BK)x(t) + (I_n \otimes \bar{D})w(t) \\ z(t) &= (\alpha L_c \alpha^{-1} \otimes I_m)x(t) \end{aligned} \tag{10}$$

where $\bar{D} = (I_m - B(B^T XB)^{-1} B^T X)D$.

Define the transformation

$$\begin{aligned} \tilde{x}(t) &= (\alpha^{-1} \otimes I_m)x(t) \\ \tilde{w}(t) &= (\alpha^{-1} \otimes I_p)w(t) \\ \tilde{z}(t) &= (\alpha^{-1} \otimes I_m)z(t) \end{aligned} \tag{11}$$

then from the above transformation, we have

$$\begin{aligned} \dot{\tilde{x}}(t) &= (I_n \otimes A - L \otimes BK)\tilde{x}(t) + (I_n \otimes \bar{D})\tilde{w}(t) \\ \tilde{z}(t) &= (L_c \otimes I_m)\tilde{x}(t) \end{aligned} \tag{12}$$

Denote $\bar{x}(t) = \tilde{x}(t) - \mathbf{1}_n \otimes \frac{\sum_{i=1}^n \int_0^t e^{A(t-s)} \bar{D}\tilde{w}_i(s)\mathrm{d}s}{n}$ and define the transformation $[\begin{smallmatrix} \delta(t) \\ \bar{\delta}(t) \end{smallmatrix}] = (U^T \otimes I_m)\bar{x}(t)$, where $U = [U_1, U_2] \in \Re^{n \times n}$ with $U_2 = \frac{\mathbf{1}_n}{\sqrt{n}}$ is an orthogonal matrix such that $U^T L_c U = [\begin{smallmatrix} I_{n-1} & 0 \\ * & 0 \end{smallmatrix}]$, $U^T LU = [\begin{smallmatrix} \bar{L} & 0 \\ U_2^T LU_1 & 0 \end{smallmatrix}]$ and $\bar{L} = U_1^T LU_1 \in \Re^{(n-1)\times(n-1)}$ [34]. Then, we can obtain

$$\begin{aligned} \begin{bmatrix} \dot{\delta}(t) \\ \dot{\bar{\delta}}(t) \end{bmatrix} &= (U^T \otimes I_m)\dot{\bar{x}}(t) = (U^T \otimes I_m)\left(\dot{\tilde{x}}(t) - \mathbf{1}_n \otimes \frac{\sum_{i=1}^n \int_0^t Ae^{A(t-s)}\bar{D}\tilde{w}_i(s)\mathrm{d}s + \sum_{i=1}^n \bar{D}\tilde{w}_i(t)}{n}\right) \\ &= (U^T \otimes I_m)((I_n \otimes A - L \otimes BK)\tilde{x}(t) + (I_n \otimes \bar{D})\tilde{w}(t)) - \left[\begin{matrix} 0 \\ \frac{\sum_{i=1}^n \int_0^t Ae^{A(t-s)}\bar{D}\tilde{w}_i(s)\mathrm{d}s + \sum_{i=1}^n \bar{D}\tilde{w}_i(t)}{\sqrt{n}} \end{matrix}\right]\Bigg) \\ &= (I_n \otimes A - U^T LU \otimes BK)\begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} + \begin{bmatrix} (U_1^T \otimes \bar{D})\tilde{w}(t) \\ 0 \end{bmatrix} \end{aligned} \tag{13}$$

$$\begin{aligned} \tilde{z}(t) =& (L_c \otimes I_m)\tilde{x}(t) = (L_c \otimes I_m)\bar{x}(t) = (L_c UU^T \otimes I_m)\bar{x}(t) \\ =& \left(\begin{bmatrix} U_1 & 0 \end{bmatrix} \otimes I_m\right)\begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} = (U_1 \otimes I_m)\delta(t) \end{aligned} \tag{14}$$

and (13) and (14) can be decomposed to the following two subsystems

$$\begin{cases} \dot{\delta}(t) = (I_{n-1} \otimes A - \bar{L} \otimes BK)\delta(t) + (U_1^T \otimes \bar{D})\tilde{w}(t) \\ \tilde{z}(t) = (U_1 \otimes I_m)\delta(t) \end{cases} \tag{15}$$

$$\dot{\bar{\delta}}(t) = (A - U_2^T LU_1 BK)\delta(t) \tag{16}$$

which means that the state $\bar{\delta}(t)$ is completely dependent on the state $\delta(t)$. From (14), we can conclude that if $\lim_{t\to\infty}\delta(t) = 0$, then $\lim_{t\to\infty}\tilde{z}(t) = 0$, that is, the scaled consensus is achieved. Moreover, if $\int_0^\infty \|\tilde{z}(t)\|^2 \mathrm{d}t \le \gamma^2 \int_0^\infty \|\tilde{w}(t)\|^2 \mathrm{d}t$, we have

$$\int_0^\infty \sum_{i=1}^n \frac{1}{\alpha_i^2} z_i(t)^T z_i(t)\mathrm{d}t \le \gamma^2 \int_0^\infty \sum_{i=1}^n \frac{1}{\alpha_i^2} w_i(t)^T w_i(t)\mathrm{d}t$$

which means that $\frac{1}{\alpha_{\max}}\int_0^\infty \|z(t)\|^2\mathrm{d}t \le \frac{\gamma^2}{\alpha_{\min}}\int_0^\infty \|w(t)\|^2\mathrm{d}t$ is achieved, where $\alpha_{\min} = \min\{\alpha_i^2\}, \alpha_{\max} = \max\{\alpha_i^2\}$. Thus, we can conclude that if the H_∞ norm of the closed-loop transfer function matrix $T_{\tilde{z}\tilde{w}}(s)$ of system (15) satisfies $\|T_{\tilde{z}\tilde{w}}(s)\|_\infty < \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}}\gamma$, then the H_∞ norm of the closed-loop transfer function matrix $T_{zw}(s)$ of sliding mode dynamics satisfies $\|T_{zw}(s)\|_\infty < \gamma$.

Theorem 1. *Assume that the directed graph $\mathcal{G}$ has a spanning tree, then for a given $\gamma > 0$, the scaled consensus with H_∞ disturbance attenuation index γ of multi-agent system (1) is achieved on the sliding surface $s(t) = 0$, if there exists matrix $X > 0$ satisfying*

$$\begin{bmatrix} \Omega + U_1^T U_1 \otimes I_m & (I_{n-1}\otimes X)(I_{n-1}\otimes B) & (I_{n-1}\otimes X)(U_1^T\otimes D) \\ * & -(I_{n-1}\otimes B^T)(I_{n-1}\otimes X)(I_{n-1}\otimes B) & 0 \\ * & * & (U_1\otimes D^T)(I_{n-1}\otimes X)(I_{n-1}\otimes D) - \bar{\gamma}^2(I_n\otimes I_p) \end{bmatrix} < 0 \quad (17)$$

where $\Omega = (I_{n-1}\otimes X)(I_{n-1}\otimes A - \bar{L}\otimes BK) + (I_{n-1}\otimes A - \bar{L}\otimes BK)^T(I_{n-1}\otimes X)$, $\bar{\gamma} = \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}}\gamma$.

Proof. If there exists matrix $X > 0$ satisfying LMI (17), we have $\Omega < 0$. Definite the Lyapunov function $V = \delta(t)^T(I_{n-1}\otimes X)\delta(t)$, then we have

$$\dot{V} = \delta(t)^T\Omega\delta(t) < 0 \quad (18)$$

for $\tilde{w}(t) = 0$, Thus, the system (15) is asymptotically stable for $\tilde{w}(t) = 0$, i.e., all the agents can reach scaled consensus.

Moreover, we will prove $\|T_{\tilde{z}\tilde{w}}(s)\|_\infty < \bar{\gamma}$ for all nonzero $\tilde{w}(t) \in L_2[0,\infty)$ for system (15). The zero initial condition of system (15) is assumed, and

$$\dot{V} = \delta(t)^T\Omega\delta(t) + 2\delta(t)^T(I_{n-1}\otimes X)(U_1^T\otimes D)\tilde{w}(t) - 2\delta(t)^T(I_{n-1}\otimes X)(U_1^T\otimes B(B^TXB)^{-1}B^TXD)\tilde{w}(t) \quad (19)$$

and

$$-2\delta(t)^T(I_{n-1}\otimes X)(U_1^T\otimes B(B^TXB)^{-1}B^TXD\tilde{w}(t) \le \delta(t)^T(I_{n-1}\otimes XB(B^TXB)^{-1}B^TX)\delta(t) + \tilde{w}(t)^T(U_1U_1^T\otimes D^TXD)\tilde{w}(t) \quad (20)$$

Definite

$$J_T = \int_0^T (\tilde{z}(t)^T\tilde{z}(t) - \bar{\gamma}^2\tilde{w}(t)^T\tilde{w}(t))\mathrm{d}t \quad (21)$$

for any $T > 0$. Based on the assumption of zero initial condition of system (15), we can obtain

$$J_T = \int_0^T (\tilde{z}(t)^T\tilde{z}(t) - \bar{\gamma}^2\tilde{w}(t)^T\tilde{w}(t) + \dot{V})\mathrm{d}t - V(\delta(T)) \le \int_0^T [\delta(t)^T\ \tilde{w}(t)^T]\Xi[\delta(t)^T\ \tilde{w}(t)^T]^T\mathrm{d}t - V(\delta(T)) \quad (22)$$

where

$$\Xi = \begin{bmatrix} \Omega + U_1^TU_1\otimes I_m + I_{n-1}\otimes XB(B^TXB)^{-1}B^TX & U_1^T\otimes XD \\ * & U_1U_1^T\otimes D^TXD - \bar{\gamma}^2 I_n\otimes I_p \end{bmatrix} \quad (23)$$

Thus, by Schurs complement, if there exists matrix $X > 0$ satisfying LMI (17), we have $J_T < 0$. Hence, for $T\to\infty$, we can obtain $\int_0^\infty \|\tilde{z}(t)\|^2\mathrm{d}t \le \bar{\gamma}^2\int_0^\infty \|\tilde{w}(t)\|^2\mathrm{d}t$ for all nonzero $\tilde{w}(t) \in L_2[0,\infty)$ from (21), that is, $\|T_{zw}(s)\|_\infty < \gamma$ for all nonzero $w(t) \in L_2[0,\infty)$. □

Remark 2. If $\mathcal{G}$ has a spanning tree, then the eigenvalues of matrix $\bar{L}$ have positive real parts, which means that the eigenvalues of matrix $I_{n-1}+\bar{L}$ have positive real parts. Because of $A - BK$ is Hurwitz, we can obtain the matrix $(I_{n-1}+\bar{L})\otimes(A-BK)$ is Hurwitz by the property of Kronecker product. Thus, $\mathcal{G}$ has a spanning tree can guarantee that there exists a positive matrix $\mathcal{X}$ such that $((I_{n-1}+\bar{L})\otimes(A-BK))^T\mathcal{X} + \mathcal{X}((I_{n-1}+\bar{L})\otimes(A-BK)) < 0$. Thus, if $\mathcal{X} = I_{n-1}\otimes X$, $\Omega < 0$ can be guaranteed. If $\mathcal{X}$ dose not satisfy the form of $\mathcal{X} = I_{n-1}\otimes X$, then we can give a matrix $X > 0$ (ex. we can choose $X = I_p$) and a index $\gamma > 0$, if $\mathcal{X}$ satisfies the following LMI

$$\begin{bmatrix} \tilde{\Omega} + U_1^TU_1\otimes I_m & \mathcal{X}(I_{n-1}\otimes\bar{D}) \\ * & -\bar{\gamma}^2(I_n\otimes I_p) \end{bmatrix} < 0$$

where $\tilde{\Omega} = \mathcal{X}(I_{n-1}\otimes A - \bar{L}\otimes BK) + (I_{n-1}\otimes A - \bar{L}\otimes BK)^T\mathcal{X}$, then the scaled consensus with H_∞ disturbance attenuation index γ of multi-agent system (1) is achieved on the sliding surface $s(t) = 0$.

Remark 3. The zero initial condition of system (15) can be guaranteed by choosing $x(0) = \beta\alpha\mathbf{1}_n$, where β is an arbitrary constant.

3.1.2. State based distributed adaptive SMC law design

Assumption 3. $\|w_i(t)\| \leq \chi_i$, where $\chi_i > 0$ is an unknown constant.

Remark 4. Assumption 3 is reasonable for general linear or nonlinear systems with disturbances, and the similarity assumptions are also given in [4,30], however, the precise upper bounds of disturbances are required.

Since Assumption 3 is satisfied, then there exists an unknown constant $\varphi_i > 0$ such that $\|(I_n \otimes (B^T XB)^{-1} B^T XD) w_i(t)\| \leq \varphi_i$.

Theorem 2. *Consider the multi-agent system (1) with Assumptions 1 and 3. The sliding functions are given in (6), where X is solved from LMI (17). Then the states of all agents will be driven onto $s(t) = 0$, if the desired distributed adaptive SMC law is designed as*

$$u_i(t) = K \sum_{j \in N_i} a_{ij}(\alpha_{ij} x_j(t) - x_i(t)) - \hat{\rho}_i \mathrm{sgn}(s_i(t)) \tag{24}$$

with the update law

$$\dot{\hat{\varphi}}_i = \kappa_i \|s_i(t)\| \tag{25}$$

where $\hat{\rho}_i = \hat{\varphi}_i + \varepsilon_i$, κ_i and ε_i are the designed positive parameters.

Proof. From (6) and (24), we can obtain

$$\dot{s}(t) = (I_n \otimes B^T XD) w(t) - (I_n \otimes B^T XB) \hat{\rho} \mathrm{sgn}(s(t)) \tag{26}$$

where $\mathrm{sgn}(s(t)) = [\mathrm{sgn}(s_1(t))^T, \ldots, \mathrm{sgn}(s_n(t))^T]^T$, $\hat{\rho} = \mathrm{diag}\{\hat{\rho}_1, \ldots, \hat{\rho}_n\} \otimes I_q$. Considering the Lyapunov function $V_1 = \frac{1}{2} s^T(t)(I_n \otimes B^T XB)^{-1} s(t) + \frac{1}{2}\sum_{i=1}^{n} \kappa_i^{-1} \tilde{\varphi}_i^2$, where $\tilde{\varphi}_i = \varphi_i - \hat{\varphi}_i$. Using (25), we have

$$\begin{aligned}
\dot{V}_1 &= s^T(t)(I_n \otimes B^T XB)^{-1} \dot{s}(t) - \sum_{i=1}^{n} \kappa_i^{-1} \tilde{\varphi}_i \dot{\hat{\varphi}}_i \\
&\leq \sum_{i=1}^{n} \|s_i(t)\| \|(I_n \otimes (B^T XB)^{-1} B^T XD) w_i(t)\| - \sum_{i=1}^{n} \hat{\rho}_i \|s_i(t)\|_1 - \sum_{i=1}^{n} \kappa_i^{-1} \tilde{\varphi}_i \dot{\hat{\varphi}}_i \\
&\leq -\sum_{i=1}^{n} \varepsilon_i \|s_i(t)\| = -\varpi(t) \leq 0
\end{aligned} \tag{27}$$

where $\varpi(t) = \sum_{i=1}^{n} \varepsilon_i \|s_i(t)\|$ and $\|s_i(t)\|_1 \geq \|s_i(t)\|$ is applied. Integrating (27) from zero to t yields $V_1(0) \geq V_1(t) + \int_0^t \varpi(s) ds \geq \int_0^t \varpi(s) ds$. When $t \to \infty$, the above integral is always less than or equal to $V_1(0)$. Since $V_1(0)$ is positive and finite, thus, from the Barbalat's Lemma given in [50], we can obtain $\lim_{t\to\infty} \varpi(t) = \lim_{t\to\infty} \sum_{i=1}^{n} \varepsilon_i \|s_i(t)\| = 0$, which means that the state trajectories of all agents will asymptotic arrive at the sliding surface, that is, $\lim_{t\to\infty} s_i(t) = 0$. □

Remark 5. If $B = D$, the external disturbance $w_i(t)$ satisfies the matching condition, which means that the sliding mode dynamics will not be influenced by external disturbance and the scaled consensus can be achieved when $\Omega < 0$.

3.2. H_∞ SMC based distributed output feedback control

Comparing with the state feedback based distributed protocol given in (24) and Li et al. [33,34,36], the output feedback based distributed protocol would be more important in practical [7,35,47,48], because of the relative information of all states constraints is sometimes hard to directly measure in practice, and only the relative information of all outputs can be available. Hence, the output feedback based scaled consensus protocols are also discussed in this paper.

Now, we proposed the following distributed observer for the ith agent as

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + BF\left(\sum_{j \in N_i} a_{ij}(\alpha_{ij} y_j(t) - y_i(t)) - \sum_{j \in N_i} a_{ij}(\alpha_{ij} C\hat{x}_j(t) - C\hat{x}_i(t))\right) \tag{28}$$

where $F \in \Re^{q \times o}$ is a constant matrix such that $A + BFC$ is Hurwitz.

Remark 6. Because (A, C) is detectable, there exists a matrix M such that $A + MC$ is Hurwitz. From Assumption 1, we can choose $F = (B^T B)^{-1} B^T M$ such that $BF = M$, which means $A + BFC$ is Hurwitz.

Denote $e_i(t) = x_i(t) - \hat{x}_i(t)$ as the estimation error, then, from (1), (2) and (28), we have

$$\dot{e}_i(t) = Dw_i(t) - BFC \sum_{j \in N_i} a_{ij}(\alpha_{ij} e_j(t) - e_i(t)) \tag{29}$$

Definite

$$\bar{z}_i(t) = \begin{bmatrix} \hat{x}_i(t) - \frac{1}{n}\sum_{j=1}^{n} \alpha_{ij}\hat{x}_j(t) \\ e_i(t) - \frac{1}{n}\sum_{j=1}^{n} \alpha_{ij}e_j(t) \end{bmatrix} \tag{30}$$

as the controlled output. Denote $\hat{x}(t) = [\hat{x}_1(t)^T, \dots, \hat{x}_n^T(t)]^T \in \Re^{mn}, y(t) = [y_1(t)^T, \dots, y_n^T(t)]^T \in \Re^{on}, e(t) = [e_1(t)^T, \dots, e_n^T(t)]^T \in \Re^{mn}$. Then, we can use Kronecker product to describe the matrix form equations of (28) and (30) as

$$\dot{\hat{x}}(t) = (I_n \otimes A)\hat{x}(t) + (I_n \otimes B)u(t) - (\alpha L_c \alpha^{-1} \otimes BF)(y(t) - (I_n \otimes C)\hat{x}(t)) \tag{31}$$

$$\dot{e}(t) = (I_n \otimes A)e(t) + (\alpha L \alpha^{-1} \otimes BFC)e(t) + (I_n \otimes D)w(t) \tag{32}$$

$$\bar{z}(t) = \begin{bmatrix} (\alpha L_c \alpha^{-1} \otimes I_m)\hat{x}(t) \\ (\alpha L_c \alpha^{-1} \otimes I_m)e(t) \end{bmatrix} \tag{33}$$

Thus, if $\lim_{t\to\infty} \bar{z}(t) = 0$ is guaranteed, then the scaled consensus defined in Definition 1 can be satisfied.

The objectives of H_∞ SMC based distributed output feedback control are to design u_i for system (1) such that

(i) the state of observer system (28) will arrive at the sliding surface, and not leave there in the subsequent time;

(ii) the system (1) achieves scaled consensus with H_∞ disturbance attenuation index γ on the sliding surface, that is, for $w(t) = 0$, the overall closed-loop system achieves scaled consensus on the sliding surface; for $w(t) \neq 0$, the overall closed-loop system on the sliding surface the satisfies the dissipation inequality $\int_0^\infty \|\bar{z}(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t$ where $\gamma > 0$ is a given constant.

3.2.1. State estimation based integral sliding function design

The state estimation based distributed integral-type sliding function for the ith agent is constructed as:

$$\bar{s}_i(t) = B^T X\hat{x}_i(t) - \int_0^t \left(B^T XA\hat{x}_i(s) + B^T XBK\sum_{j\in N_i} a_{ij}(\alpha_{ij}\hat{x}_j(s) - \hat{x}_i(s)) \right)\mathrm{d}s \tag{34}$$

where $X \in \Re^{m\times m}$ is defined as in (6). Denote $\bar{s}(t) = [\bar{s}_1(t)^T, \dots, \bar{s}_n^T(t)]^T \in \Re^{qn}$, we have

$$\bar{s}(t) = (I_n \otimes B^T X)\hat{x}(t) - \int_0^t ((I_n \otimes B^T XA)\hat{x}(s) - (\alpha L \alpha^{-1} \otimes B^T XBK)\hat{x}(s))\mathrm{d}s \tag{35}$$

Substituting (31) into $\dot{\bar{s}}(t)$ yields

$$\dot{\bar{s}}(t) = (I_n \otimes B^T XB)u(t) - (\alpha L \alpha^{-1} \otimes B^T XBFC)e(t) + (\alpha L \alpha^{-1} \otimes B^T XBK)\hat{x}(t) \tag{36}$$

From $\dot{\bar{s}}(t) = 0$, we can obtain the equivalent control law as

$$u_{eq}(t) = -(\alpha L \alpha^{-1} \otimes K)\hat{x}(t) + (I_n \otimes FC)e(t) \tag{37}$$

Substituting $u_{eq}(t)$ into (31) yields the sliding mode dynamics

$$\dot{\hat{x}}(t) = (I_n \otimes A - \alpha L \alpha^{-1} \otimes BK)\hat{x}(t) \tag{38}$$

Define the transformation

$$\begin{aligned} \tilde{\hat{x}}(t) &= (\alpha^{-1} \otimes I_m)\hat{x}(t) \\ \tilde{e}(t) &= (\alpha^{-1} \otimes I_m)e(t) \\ \tilde{\bar{z}}(t) &= (\alpha^{-1} \otimes I_{2m})\bar{z}(t) \end{aligned} \tag{39}$$

Then, we can obtain the following overall closed-loop system on the sliding surface

$$\dot{\tilde{\hat{x}}}(t) = (I_n \otimes A - L \otimes BK)\tilde{\hat{x}}(t) \tag{40}$$

$$\dot{\tilde{e}}(t) = (I_n \otimes A)\tilde{e}(t) + (L \otimes BFC)\tilde{e}(t) + (I_n \otimes D)\tilde{w}(t) \tag{41}$$

$$\tilde{\bar{z}}(t) = \begin{bmatrix} (L_c \otimes I_m)\tilde{\hat{x}}(t) \\ (L_c \otimes I_m)\tilde{e}(t) \end{bmatrix} \tag{42}$$

Denote $\bar{e}(t) = \tilde{e}(t) - \mathbf{1}_n \otimes \frac{\sum_{i=1}^{n} \int_0^t e^{A(t-s)} D\tilde{w}_i(s)\mathrm{d}s}{n}$ and define the transformations $[\begin{smallmatrix} \delta_1(t) \\ \bar{\delta}_1(t) \end{smallmatrix}] = (U^T \otimes I_m)\tilde{\hat{x}}(t)$ and $[\begin{smallmatrix} \delta_2(t) \\ \bar{\delta}_2(t) \end{smallmatrix}] = (U^T \otimes I_m)\bar{e}(t)$, where U is defined in Section III.A. Then, as in (13)–(16), we can obtain the following two subsystems

$$\begin{cases} \dot{\delta}_1(t) = (I_{n-1} \otimes A - \bar{L} \otimes BK)\delta_1(t) \\ \dot{\delta}_2(t) = (I_{n-1} \otimes A + \bar{L} \otimes BFC)\delta_2(t) + (U_1^T \otimes D)\tilde{w}(t) \\ \tilde{\bar{z}}(t) = \begin{bmatrix} (U_1 \otimes I_m)\delta_1(t) \\ (U_1 \otimes I_m)\delta_2(t) \end{bmatrix} \end{cases} \tag{43}$$

$$\begin{cases} \dot{\bar{\delta}}_1(t) = A\delta_1(t) - U_2^T LU_1 B_K \delta_1(t) \\ \dot{\bar{\delta}}_2(t) = A\delta_2(t) + U_2^T LU_1 BFC\delta_2(t) \end{cases} \tag{44}$$

From (43), we can conclude that if $\lim_{t\to\infty} \delta_1(t) = 0$ and $\lim_{t\to\infty} \delta_2(t) = 0$, then $\lim_{t\to\infty} \tilde{\bar{z}}(t) = 0$, that is, the scaled consensus is achieved. Moreover, if $\int_0^\infty \|\tilde{\bar{z}}(t)\|^2 dt \le \gamma^2 \int_0^\infty \|\tilde{w}(t)\|^2 dt$, we have $\int_0^\infty \sum_{i=1}^n \frac{1}{\alpha_i} \bar{z}_i(t)^T \bar{z}_i(t) dt \le \gamma^2 \int_0^\infty \sum_{i=1}^n \frac{1}{\alpha_i} w_i(t)^T w_i(t) dt$, which means that $\frac{1}{\alpha_{\max}} \int_0^\infty \|\bar{z}(t)\|^2 dt \le \frac{\gamma^2}{\alpha_{\min}} \int_0^\infty \|w(t)\|^2 dt$ is achieved. Thus, we can conclude that if the H_∞ norm of the overall closed-loop transfer function matrix $T_{\tilde{\bar{z}}\tilde{w}}(s)$ of system (43) satisfies $\|T_{\tilde{\bar{z}}\tilde{w}}(s)\|_\infty < \gamma$, then the H_∞ norm of the overall closed-loop transfer function matrix $T_{\bar{z}w}(s)$ of systems (32) and (33) and (38) satisfies $\|T_{\bar{z}v}(s)\|_\infty < \sqrt{\frac{\alpha_{\max}}{\alpha_{\min}}}\gamma$.

Theorem 3. *Assume that the directed graph $\mathcal{G}$ has a spanning tree, then for a given $\gamma > 0$, the scaled consensus with H_∞ disturbance attenuation index γ of multi-agent system (1) is achieved on the sliding surface $\bar{s}(t) = 0$, if there exists matrix $X > 0$ satisfying*

$$\begin{bmatrix} \bar{\Omega} + U_1^T U_1 \otimes I_{2m} & \begin{bmatrix} 0 \\ (I_{n-1} \otimes X)(U_1^T \otimes D) \end{bmatrix} \\ * & -\bar{\gamma}^2 (I_n \otimes I_p) \end{bmatrix} < 0 \tag{45}$$

where

$$\bar{\Omega} = \begin{bmatrix} I_{n-1} \otimes X & 0 \\ 0 & I_{n-1} \otimes X \end{bmatrix} \begin{bmatrix} I_{n-1} \otimes A - \bar{L} \otimes BK & 0 \\ 0 & I_{n-1} \otimes A + \bar{L} \otimes BFC \end{bmatrix} + \begin{bmatrix} I_{n-1} \otimes A - \bar{L} \otimes BK & 0 \\ 0 & I_{n-1} \otimes A + \bar{L} \otimes BFC \end{bmatrix}^T \begin{bmatrix} I_{n-1} \otimes X & 0 \\ 0 & I_{n-1} \otimes X \end{bmatrix},$$

and $\bar{\gamma}$ is defined in Theorem 1.

Proof. The proof of Theorem 3 is similar to that of Theorem 1, thus is omitted for brevity. □

3.2.2. State estimation based distributed SMC law design

Theorem 4. *Consider the multi-agent system (1) with Assumptions 1–3. The sliding functions are given in (34), where X is solved from LMI (45). Then the states of all observers (28) will be driven onto $\bar{s}(t) = 0$ in finite time, if the desired distributed SMC law is designed as*

$$u_i(t) = K \sum_{j\in N_i} a_{ij}(\alpha_{ij}\hat{x}_j(t) - \hat{x}_i(t)) - \bar{\rho}_i \mathrm{sgn}(s_i(t)) \tag{46}$$

where $\bar{\rho}_i = \|F \sum_{j\in N_i} a_{ij}(\alpha_{ij} y_j(t) - y_i(t))\| + \|F \sum_{j\in N_i} a_{ij}(\alpha_{ij} C\hat{x}_j(t) - C\hat{x}_i(t))\| + \varepsilon_i$ and $\varepsilon_i > 0$ is the designed parameter.

Proof. From (6) and (24), we can obtain

$$\dot{\bar{s}}(t) = -(\alpha L\alpha^{-1} \otimes B^T XBF)y(t) + (\alpha L\alpha^{-1} \otimes B^T XBFC)\hat{x}(t) - (I_n \otimes B^T XB)\bar{\rho}\,\mathrm{sgn}(\bar{s}(t)) \tag{47}$$

where $\bar{\rho} = \mathrm{diag}\{\bar{\rho}_1, \dots, \bar{\rho}_n\} \otimes I_q$. Definite the Lyapunov function $V_2 = \frac{1}{2}\bar{s}^T(t)(I_n \otimes B^T XB)^{-1}\bar{s}(t)$, we can obtain

$$\begin{aligned} \dot{V}_2 &= \bar{s}^T(t)(I_n \otimes B^T XB)^{-1}\dot{\bar{s}}(t) \\ &\le \sum_{i=1}^n \|\bar{s}_i(t)\| (\|F \sum_{j\in N_i} a_{ij}(\alpha_{ij} y_j(t) - y_i(t))\| + \|F \sum_{j\in N_i} a_{ij}(\alpha_{ij} C\hat{x}_j(t) - C\hat{x}_i(t))\|) - \sum_{i=1}^n \bar{\rho}_i \|\bar{s}_i(t)\|_1 \\ &\le -\sum_{i=1}^n \varepsilon_i \|\bar{s}_i(t)\| \le -\varepsilon_{\min} \sqrt{\frac{2}{\lambda_{\max}((I_n \otimes B^T XB)^{-1})}} V_2^{\frac{1}{2}} \end{aligned} \tag{48}$$

where $\varepsilon_{\min} = \min\{\varepsilon_i\}$. Thus, the state trajectories of all observers (28) will arrive at the sliding surface $\bar{s}_i(t) = 0$ in finite time $t \le \frac{V_2(0)^{\frac{1}{2}}}{\frac{\varepsilon_{\min}\sqrt{\frac{2}{\lambda_{\max}((I_n \otimes B^T XB)^{-1})}}}{2}}$. □

Remark 7. Compared with [28–30], the output feedback based scaled consensus protocol is first designed, which can simple the constraints on the state feedback based scaled consensus protocols designed in [28–30].

Remark 8. For $\alpha_i = 1, i = 1, 2, \ldots, n,$ the robust scaled consensus discussed in this paper reduces to the robust consensus discussed in [33–36], where the robust consensus problem is transformed into H_∞ control problem. Compared with the state feedback based H_∞ consensus control of disturbed multi-agent systems discussed in [33,34,36], the distributed H_∞ SMC proposed in Section 3 has stronger robustness. Because of if the external disturbances satisfy the matching condition, the control law (24) can completely eliminate the influence of disturbances for consensus, and if the external disturbances don't satisfy the matching condition, we only need to design K and X in the control law (24) and sliding function (6) to attenuate the influence of unmatched disturbances in the sliding mode dynamics for consensus.

Remark 9. Compared with the sliding mode based consensus control of multi-agent systems with matched externals in [8,38–40], the proposed distributed H_∞ SMC can deal with the consensus problem for multi-agent systems with unmatched disturbances. Moreover, the output feedback design under integral-type sliding mode is not considered in [38,40].

Remark 10. The system (1) reduces to the general linear systems with disturbances if the network has only one agent, and the methods proposed in this paper can cover the partly results in [42–45,47]. In fact, the H_∞ sliding mode control method in distributed framework is more significant in practice because of the distributed coordinated control is important for many fields [1–3].

3.3. Extension the distributed H_∞ SMC to multi-agent systems under switching topology

In Sections 3.1 and 3.2, we consider the distributed H_∞ SMC for multi-agent systems under fixed topology, however, the directed graph may change over time. Now, denote $S = \{\mathcal{G}_1, \ldots, \mathcal{G}_o\}$ as the set which includes all possible directed communication topologies among the agents. Assume that there exists an infinite sequence of uniformly bounded non-overlapping time intervals $[t_l, t_{l+1}), l = 0, 1, \ldots$ with $t_0 = 0, t_{l+1} - t_l > 0,$ across which the communication graph is time-invariant. The time sequence $t_1, t_2, \ldots$ is called the switching sequence, which means the communication graph changes at $t_1, t_2, \ldots$. Defined the switching signal as $\sigma(t) : [0, \infty) \to \{1, \ldots, o\},$ and it is obvious that $\mathcal{G}_{\sigma(t)} \in S$ for all $t \geq 0$.

Assumption 4. Each switching graph $\mathcal{G}_{\sigma(t)}$ has a spanning tree.

3.3.1. Distributed state feedback control

Now, the desired distributed adaptive SMC law under switching topology is proposed as

$$u_i(t) = K\sum_{j\in N_i} a_{ij}(t)(\alpha_{ij}(t)x_j(t) - x_i(t)) - \hat{\rho}_i \mathrm{sgn}(s_i(t)) \tag{49}$$

with the update law

$$\dot{\hat{\varphi}}_i = \kappa_i\|s_i(t)\| \tag{50}$$

where $\hat{\rho}_i$ and κ_i are defined as in Theorem 2, $\alpha_{ij}(t) = \frac{\alpha_i(t)}{\alpha_j(t)}$. The distributed integral-type sliding function for the ith agent under switching topology is defined as:

$$s_i(t) = B^T X x_i(t) - \int_0^t \left(B^T X A x_i(s) + B^T X B K \sum_{j\in N_i} a_{ij}(s)(\alpha_{ij}(s)x_j(s) - x_i(s)) \right) \mathrm{d}s \tag{51}$$

where X and K are defined as in (6).

From the description of switching topology in the initial paragraph of Section 3.3, we have $a_{ij}(t) = a_{ij,l}$ for $t \in [t_l, t_{l+1})$. Likewise, we make the following assumption on time-varying scales $\alpha_i(t), i \in \mathcal{V}$ as

Assumption 5. $\alpha_i(t) = \alpha_{i,l}$ for $t \in [t_l, t_{l+1}),$ where $\alpha_{i,\,l}$ is a non-zero constant.

Conclusion 1. Assume that the directed switching graph $\mathcal{G}_{\sigma(t)}$ satisfies Assumptions 4–5, then for a given $\gamma > 0$, the scaled consensus with H_∞ disturbance attenuation index γ of multi-agent system (1) is achieved on the sliding surface $s(t) = 0,$ if there exists matrix $X > 0$ satisfying

$$\begin{bmatrix} \Omega + U_1^T U_1 \otimes I_m & (I_{n-1}\otimes X)(I_{n-1}\otimes B) & (I_{n-1}\otimes X)(U_1^T \otimes D) \\ * & -(I_{n-1}\otimes B^T)(I_{n-1}\otimes X)(I_{n-1}\otimes B) & 0 \\ * & * & (U_1\otimes D^T)(I_{n-1}\otimes X)(I_{n-1}\otimes D) - \bar{\gamma}^2(I_n\otimes I_p) \end{bmatrix} < 0 \tag{52}$$

where $\Omega = (I_{n-1}\otimes X)(I_{n-1}\otimes A - \bar{L}_{\sigma(t)}\otimes BK) + (I_{n-1}\otimes A - \bar{L}_{\sigma(t)}\otimes BK)^T(I_{n-1}\otimes X), \bar{\gamma} = \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}}\gamma,$ and $\alpha_{\min} = \min\{\alpha_{i,l}\}, \alpha_{\max} = \max\{\alpha_{i,l}\}$.

Proof. As the proof of Theorem 1, by constructing the common Lyapunov function $V = \delta(t)^T(I_{n-1}\otimes X)\delta(t),$ we can obtain the similar results as in Theorem 1, so the proof is omitted for brevity. □

Conclusion 2. Consider the multi-agent system (1) under switching topology with Assumptions 1–5. The sliding functions are given in (51), where X is solved from LMI (52). Then the states of all agents will be driven onto $s(t)=0$, if the desired distributed adaptive SMC law is designed as in (49) with the update law (50)

Proof. The proof of Conclusion 2 is similar to that of Theorem 2, thus is omitted for brevity. □

3.3.2. Distributed output feedback control

Now, we proposed the following distributed observer for the ith agent under switching topology as

$$\dot{\hat{x}}_i(t)=A\hat{x}_i(t)+Bu_i(t)+BF\left(\sum_{j\in N_i}a_{ij}(t)(\alpha_{ij}(t)y_j(t)-y_i(t))-\sum_{j\in N_i}a_{ij}(t)(\alpha_{ij}(t)C\hat{x}_j(t)-C\hat{x}_i(t))\right) \tag{53}$$

where F is defined as in (28). The state estimation based distributed integral-type sliding function for the ith agent under switching topology is constructed as:

$$\bar{s}_i(t)=B^TX\hat{x}_i(t)-\int_0^t\left(B^TXA\hat{x}_i(s)+B^TXBK\sum_{j\in N_i}a_{ij}(s)(\alpha_{ij}(s)\hat{x}_j(s)-\hat{x}_i(s))\right)ds \tag{54}$$

where X and K are defined as in (34).

Conclusion 3. Assume that the directed switching graph $\mathcal{G}_{\sigma(t)}$ satisfies Assumptions 4–5, then for a given $\gamma>0$, the scaled consensus with H_∞ disturbance attenuation index γ of multi-agent system (1) is achieved on the sliding surface $\bar{s}(t)=0$, if there exists matrix $X>0$ satisfying

$$\begin{bmatrix}\bar{\Omega}+U_1^TU_1\otimes I_{2m} & \begin{bmatrix}0\\(I_{n-1}\otimes X)(U_1^T\otimes D)\end{bmatrix}\\ * & -\bar{\gamma}^2(I_n\otimes I_p)\end{bmatrix}<0 \tag{55}$$

where

$$\begin{aligned}\bar{\Omega}=&\begin{bmatrix}I_{n-1}\otimes X & 0\\0 & I_{n-1}\otimes X\end{bmatrix}\begin{bmatrix}I_{n-1}\otimes A-\bar{L}_{\sigma(t)}\otimes BK & 0\\0 & I_{n-1}\otimes A+\bar{L}_{\sigma(t)}\otimes BFC\end{bmatrix}\\&+\begin{bmatrix}I_{n-1}\otimes A-\bar{L}_{\sigma(t)}\otimes BK & 0\\0 & I_{n-1}\otimes A+\bar{L}_{\sigma(t)}\otimes BFC\end{bmatrix}^T\begin{bmatrix}I_{n-1}\otimes X & 0\\0 & I_{n-1}\otimes X\end{bmatrix},\end{aligned}$$

and $\bar{\gamma}$ is defined in Conclusion 1.

Proof. The proof of Conclusion 3 is similar to that of Conclusion 1, thus is omitted for brevity. □

Conclusion 4. Consider the multi-agent system (1) with Assumptions 1–5. The sliding functions are given in (54), where X is solved from LMI (55). Then the states of all observers (53) will be driven onto $\bar{s}(t)=0$ in finite time, if the desired distributed SMC law is designed as

$$u_i(t)=K\sum_{j\in N_i}a_{ij}(t)(\alpha_{ij}(t)\hat{x}_j(t)-\hat{x}_i(t))-\bar{\rho}_i\mathrm{sgn}(s_i(t)) \tag{56}$$

where $\bar{\rho}_i=\|F\sum_{j\in N_i}a_{ij}(t)(\alpha_{ij}(t)y_j(t)-y_i(t))\|+\|F\sum_{j\in N_i}a_{ij}(t)(\alpha_{ij}(t)C\hat{x}_j(t)-C\hat{x}_i(t))\|+\varepsilon_i$ and $\varepsilon_i>0$ is defined as in Theorem 4.

Proof. The proof of Conclusion 4 is similar to that of Theorem 4, thus is omitted for brevity. □

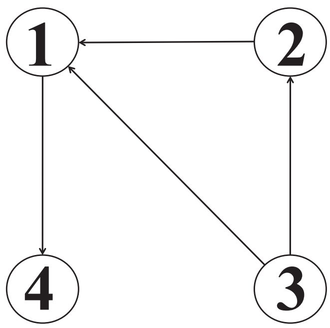

Fig. 1. The interaction topology among agents.

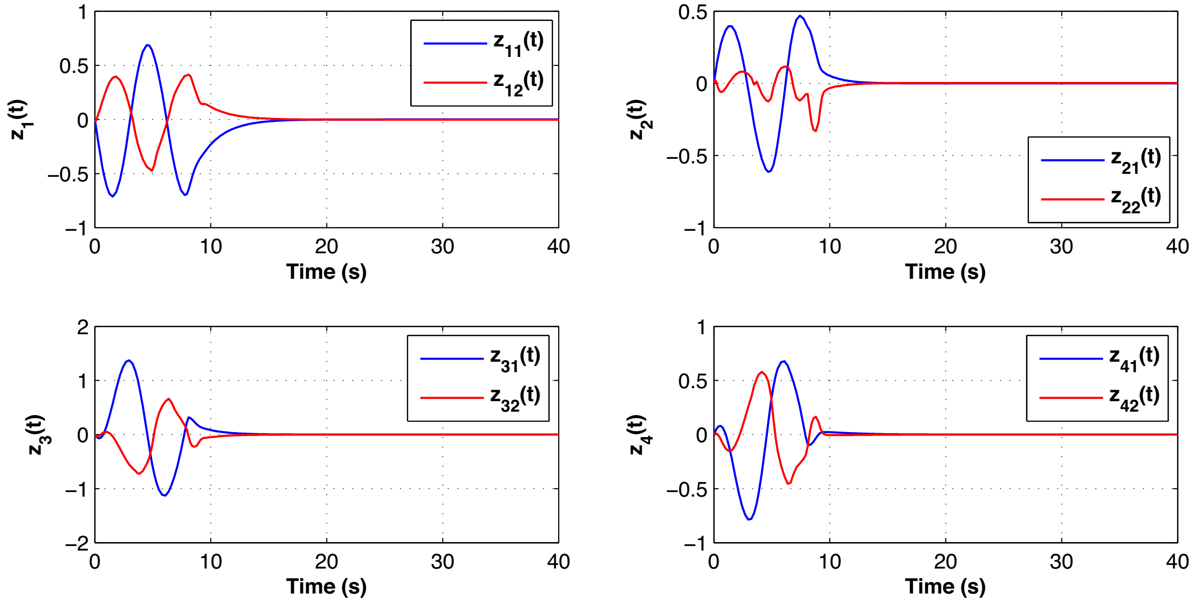

Fig. 2. The time responses of $z_i(t) = [z_{i1}(t), z_{i2}(t)]^T, i = 1, 2, 3, 4$.

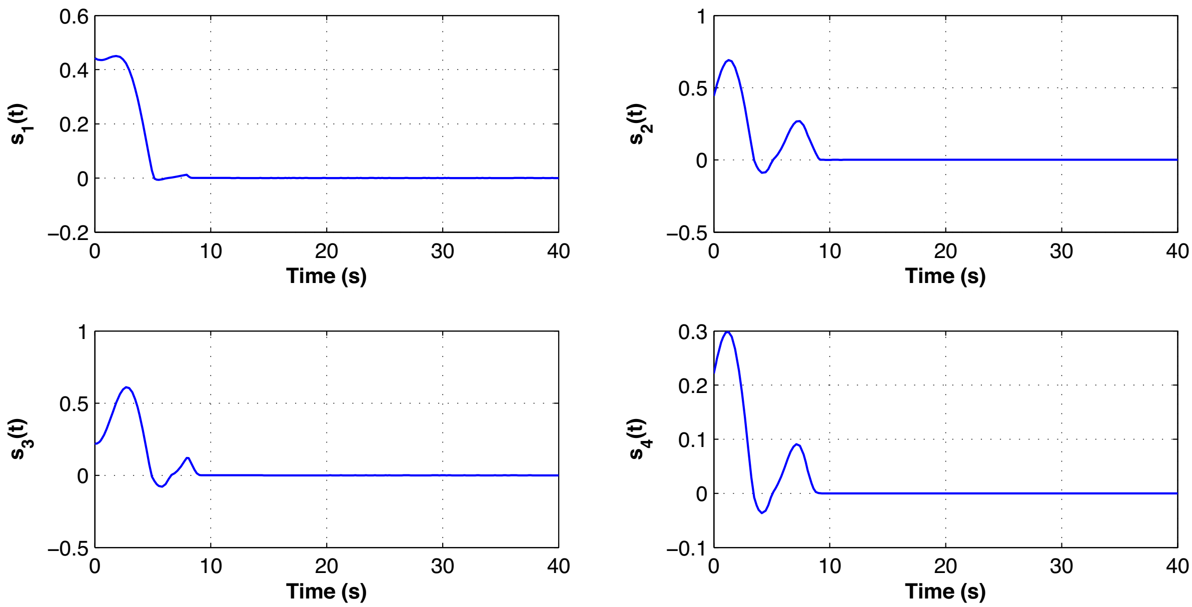

Fig. 3. The time responses of $s_i(t), i = 1, 2, 3, 4$.

4. Numerical results

To show the effectiveness of the presented distributed H_∞ SMC methods to robust scaled consensus problem, the following networked multi-agent system with four nodes is considered, where

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_i(t) \\ y_i(t) &= \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} x_i(t), i = 1, 2, 3, 4 \end{aligned} \tag{57}$$

The disturbance $w_i, i = 1, 2, 3, 4$ is chosen as

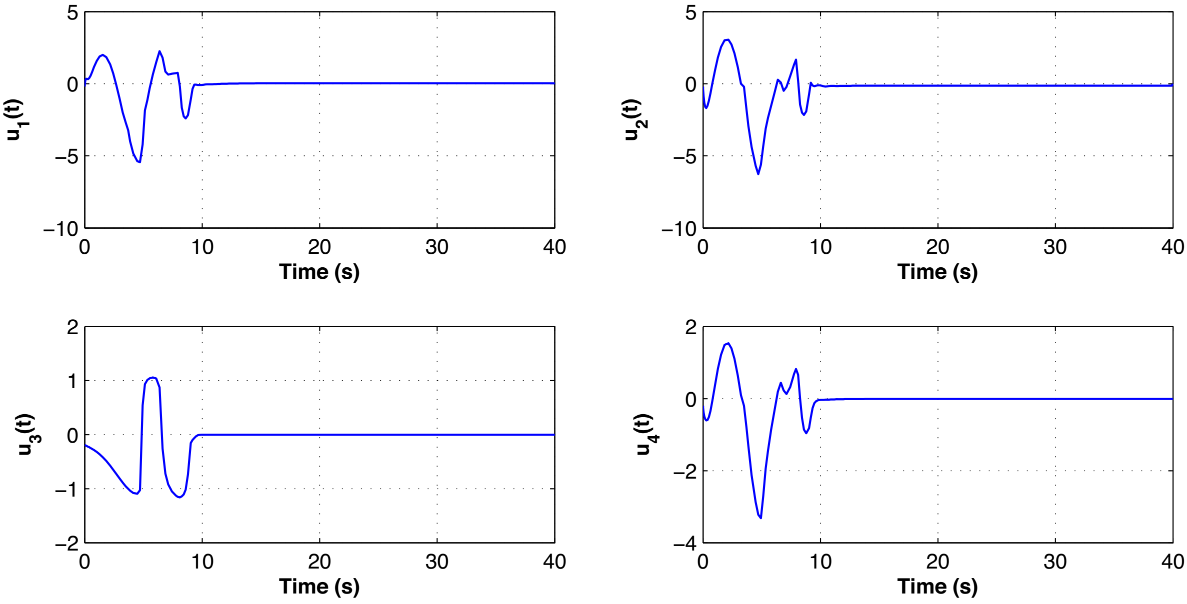

Fig. 4. The time responses of $u_i(t), i = 1, 2, 3, 4$.

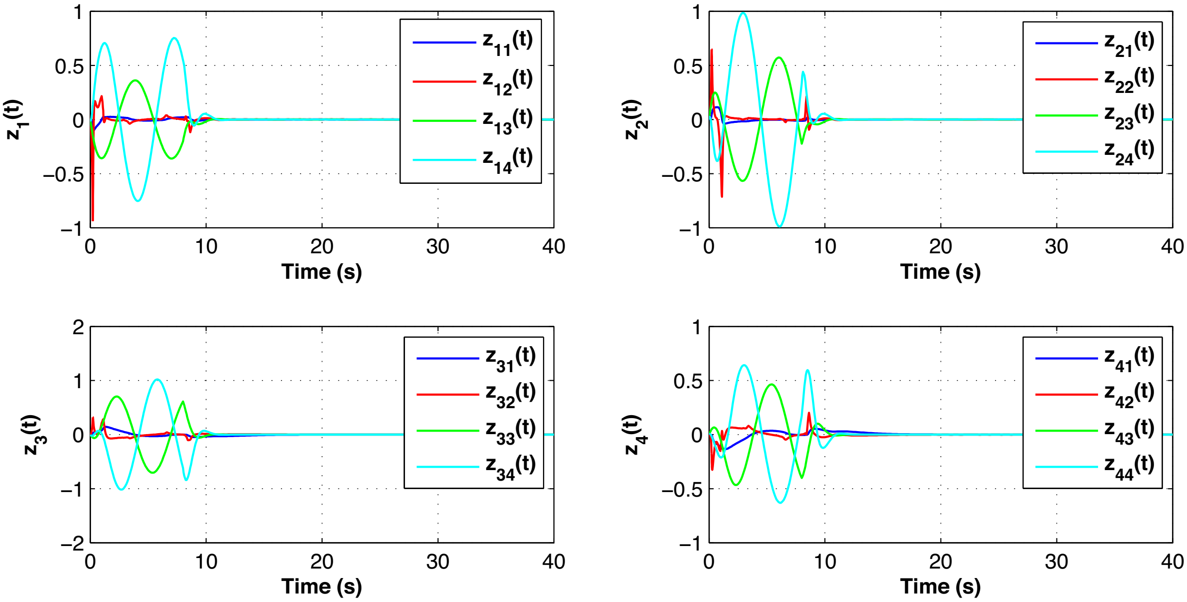

Fig. 5. The time responses of $\bar{z}_i(t) = [\bar{z}_{i1}(t), \bar{z}_{i2}(t), \bar{z}_{i3}(t), \bar{z}_{i4}(t)]^T, i = 1, 2, 3, 4$.

$$w_1(t) = \begin{cases} 0.4\sin(t), t < 8 \\ 0, t \geq 8 \end{cases}, \quad w_2(t) = \begin{cases} 1.5\cos(t), t < 8 \\ 0, t \geq 8 \end{cases}$$

$$w_3(t) = \begin{cases} 1.3\sin(t), t < 8 \\ 0, t \geq 8 \end{cases}, \quad w_4(t) = \begin{cases} 0.6\cos(t), t < 8 \\ 0, t \geq 8 \end{cases}$$

The directed graph $\mathcal{G}$ of four agents is given in Fig. 1, where the corresponding Laplacian matrix is given as $L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. The scaled scalars are chosen as $\alpha_1 = \alpha_2 = 2, \alpha_3 = \alpha_4 = 1$. The initial conditions of four agents are chosen as $x_1(0) = [2, 2]^T, x_2(0) = [2, 2]^T, x_3(0) = [1, 1]^T, x_4(0) = [1, 1]^T$.

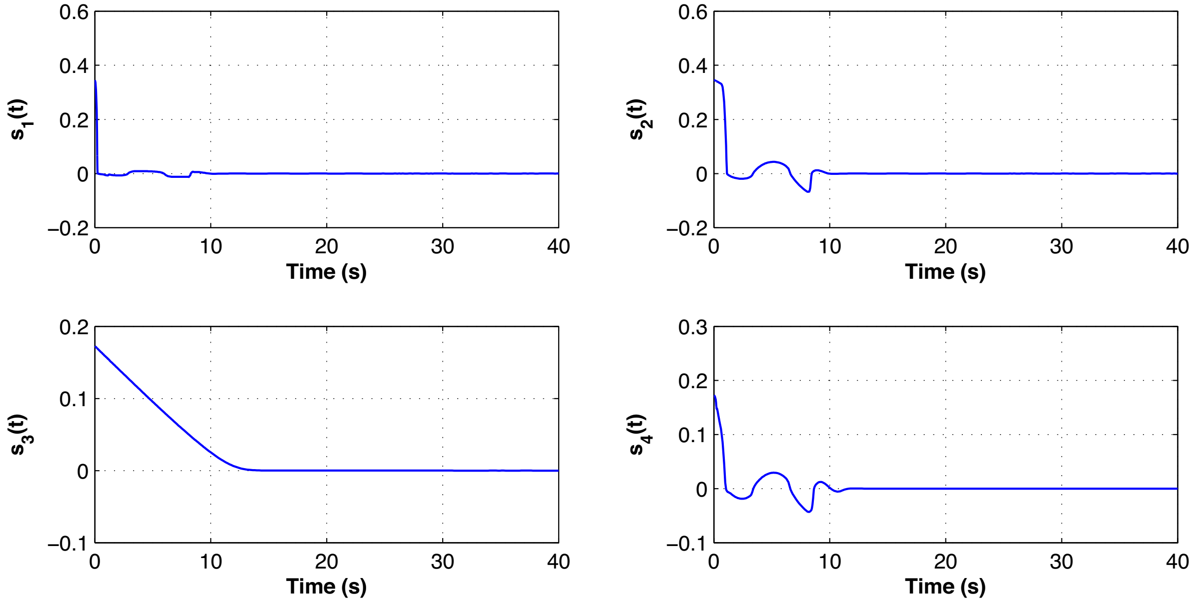

Fig. 6. The time responses of $\bar{s}_i(t), i = 1, 2, 3, 4$.

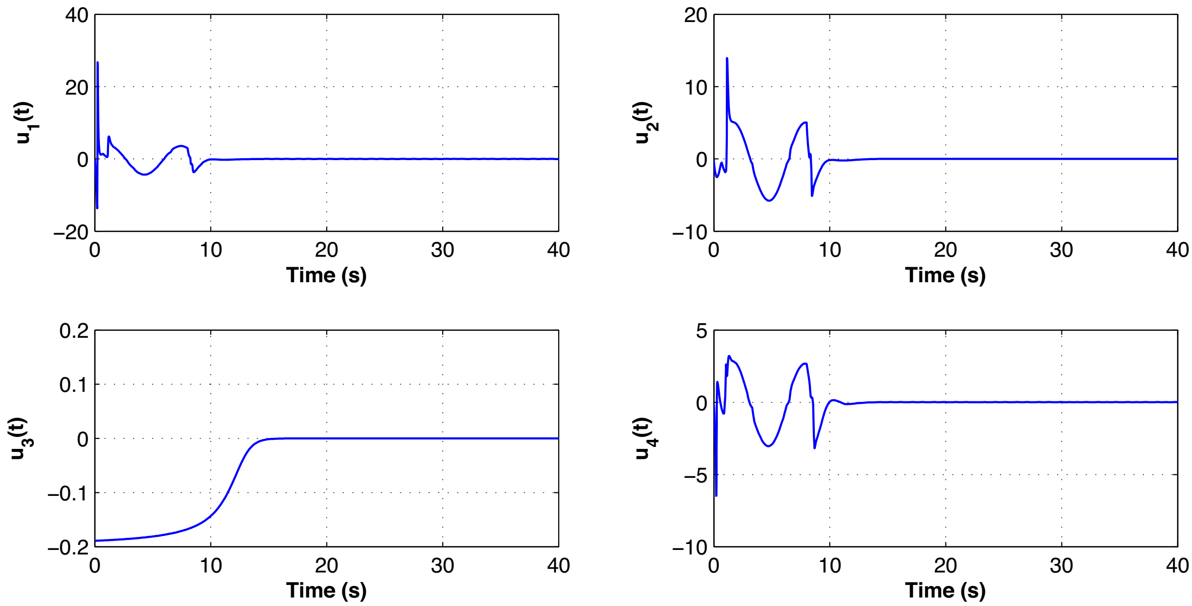

Fig. 7. The time responses of $u_i(t), i = 1, 2, 3, 4$.

We first consider the state feedback case. The controlled output $z_i(t), i = 1, 2, 3, 4$ is defined as $z_i(t) = x_i(t) - \frac{1}{4}\sum_{j=1}^{4}\alpha_{ij}x_j(t)$. The disturbance attention index is chosen as $\gamma = 1$, thus we have $\bar{\gamma} = 0.5$. Choosing $K = \begin{bmatrix} 3 & 6 \end{bmatrix}$, then we know that $A - BK$ is Hurwitz. According to the scaled consensus with γ-disturbance attenuation condition on the sliding surface in Theorem 1, by solving LMI (17), we obtain the matrix $X = \begin{bmatrix} 0.1979 & 0.0774 \\ 0.0774 & 0.1439 \end{bmatrix}$. Moreover, we choose κ_i, ε_i as $\kappa_i = 0.5$ and $\varepsilon_i = 0.1, i = 1, 2, 3, 4,$ respectively. To reduce the chattering, we use a continuous function $\frac{s_i(t)}{s_i(t)+\epsilon_i}$ to replace the discontinuous function $\mathrm{sgn}(s_i(t))(i = 1, 2, 3, 4)$ in simulations, where ϵ_i is a small positive constant. In this example, we choose $\epsilon_i = 0.01$. The time responses of controlled outputs $z_i(t), i = 1, 2, 3, 4$ under the distributed adaptive SMC law (24) are shown in Fig. 2, respectively. It can be seen that the scaled consensus is achieved at about 15 s. The time responses of sliding modes $s_i(t)$ and control inputs $u_i(t), i = 1, 2, 3, 4$ are shown in Figs. 3 and 4, respectively.

Then, we consider the output feedback case. The controlled output $\bar{z}_i(t), i = 1, 2, 3, 4$ is defined as $\bar{z}_i(t) = \begin{bmatrix} \hat{x}_i(t) - \frac{1}{n}\sum_{j=1}^{4} \alpha_{ij}\hat{x}_j(t) \\ e_i(t) - \frac{1}{n}\sum_{j=1}^{4} \alpha_{ij}e_j(t) \end{bmatrix}$. The disturbance attention index is also chosen as $\gamma = 1$, thus we have $\bar{\gamma} = 0.5$. Choosing $F = -40$, then we know that $A + BFC$ is Hurwitz. According to the scaled consensus with γ-disturbance attenuation condition on the sliding surface in Theorem 1, by solving LMI (45), we obtain the matrix $X = \begin{bmatrix} 0.4672 & 0.0894 \\ 0.0894 & 0.0834 \end{bmatrix}$. Moreover, we choose $\varepsilon_i = 0.1, i = 1, 2, 3, 4$. The time responses of controlled outputs $\bar{z}_i(t), i = 1, 2, 3, 4$ under the distributed SMC law (46) are shown in Fig. 5, respectively. It can be seen that the scaled consensus is achieved at about 15 s. The time responses of sliding modes $\bar{s}_i(t)$ and control inputs $u_i(t), i = 1, 2, 3, 4$ are shown in Figs. 6 and 7, respectively.

It can be seen from Figs. 2–7 that the proposed distributed state feedback and output feedback control laws are both robust against external disturbances.

5. Conclusion

This paper considers the scaled consensus control problem of networked multi-agent systems with linear coupling dynamics and external disturbances. The traditional H_∞ SMC design is extended to the distributed H_∞ SMC design for solving the robust scaled consensus under directed fixed topology and switching topology respectively, where the distributed state feedback design and distributed output feedback design are both established. By using the Lyapunov stability theory and LMI technique, the integral sliding function is established and the distributed SMC law is constructed, respectively, which can guarantee the states of all agents can be driven onto sliding surface and achieving scaled consensus with H_∞ disturbance attenuation index in there.

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