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TENSOR PRODUCT MODEL TRANSFORMATION BASED CONTROL AND SYNCHRONIZATION OF A CLASS OF FRACTIONAL-ORDER CHAOTIC SYSTEMS

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ABSTRACT

Fractional-order chaotic systems are the complex systems that involve non-integer order derivatives. In this paper, tensor product (TP) model transformation-based controller design for control and synchronization of a class of the fractional-order chaotic systems is investigated. We propose a novel linear matrix inequality (LMI)-based stabilization condition for fractional-order TP models with a controller derived via a parallel distributed compensation (PDC) structure. In the controller design, the controlled system first is transformed into a convex state-space TP model using the TP model transformation. Based on the transformed TP model, the controller is determined by solving the proposed LMI condition. To the best of our knowledge, this is the first investigation of TP model transformation based design in fractional-order systems. Several illustrative examples are given to demonstrate the convenience of the proposed LMI condition and the effectiveness of the controller design.

Key Words: Tensor product (TP) model transformation, fractional-order systems, chaotic systems, chaos control, linear matrix inequality (LMI), parallel distributed compensation (PDC).

I. INTRODUCTION

Fractional calculus can be considered to be a generalization of integration and differentiation to arbitrary non-integer orders [1,2]. Although fractional calculus is a 300-year-old mathematical topic, its practical applications have been investigated only recently. Fractional-order systems are the dynamic systems that involve fractional derivatives. Many physics and engineering systems have been found that display fractional-order dynamics [1]. Fractional-order systems can also behave chaotically [2]. Some examples of the fractional-order chaotic systems include the fractional-order Lorenz system [3], the fractional-order Chen system [4], the fractional-order Lü system [5], and the fractional-order Liu system [6].

Control and synchronization of chaotic systems have been studied intensively during the last two decades. The chaos control attempts to suppress the chaotic behavior of systems while the chaos synchronization controls a chaotic system so that it follows another chaotic system. The pioneering method of chaos control was proposed by Ott *et al*. [7]. This method is now known as the OGY method. The pioneering work on chaos synchronization was done by Pecora and Carroll [8]. They reported that identical synchronization is possible in two chaotic systems.

Nowadays, whereas chaos control and synchronization of integer-order chaotic systems have been extensively studied [9–18], chaos control and synchronization of their fractional-order counterparts have been investigated only recently. It is still considered a challenging research topic. Some approaches for chaos control and synchronization of fractional-order chaotic systems have been proposed, such as linear control [19,20], active control [21,22], sliding-mode control [23,24], and adaptive control [25,26].

Tensor product (TP) model transformation is an effective numerical technique based on the recently developed high order singularity value decomposition (HOSVD) [27–29]. It transforms a linear parameter varying (LPV) system into a TP model form, which is described by a convex combination of linear time invariant (LTI) systems. Various types of convex hulls also can be derived. The transformation originally was introduced to reduce the complexity of fuzzy systems [30]. Nowadays, it has been extended to solve controller design problems [27,31]. If an exact transformation is not possible, the transformation can determine a TP model that is an approximation of the given system. The approximation property already has been investigated in [32,33]. Computationally relaxed TP transformation was also introduced in [34,35]. It

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can reduce considerably the computational load when dealing with higher dimension problems.

Tensor product model transformation based-controller design is a convenient numerical methodology for nonlinear systems [27–36]. It is assumed that the controlled system can be represented as an LPV system. The design is first to transform the controlled system defined over a bounded space into a state-space TP model form. Within a parallel distributed compensation (PDC) controller design framework, any linear controller design technique then can be used to determine each controller for the LTI system. Finally, the TP model-based controller is obtained readily by the convex combination of the linear controllers, where the convex combination is inherited from the TP model. Note that the controller design can be reduced promptly to solving a linear matrix inequality (LMI) problem [27]. It has been shown that the feasibility of the LMI and the resulting control performance also are influenced by the convex hull [37,38].

The TP model transformation based-controller design method has been applied successfully to various integer-order systems, including an aeroelastic system [39,40], the TORA system [41], a canard rotor/wing UAV [42], a quadrotor system [43], automatic transmission systems [44], an air-breathing hypersonic vehicle [45], and a force reflecting tele-grasping system [46]. Nevertheless, to the best of our knowledge, its application to fractional-order systems has not been investigated.

This paper presents TP model transformation-based control and synchronization of a class of fractional-order chaotic systems. A novel LMI-based stabilization condition for fractional-order TP models with a parallel distributed compensation (PDC) controller is proposed. The rest of the paper is organized as follows. In the next section, some preliminaries are provided. Main results are presented in Section III. Numerical simulations are given in Section IV. The paper is concluded in Section V.

II. PRELIMINARIES

2.1 Fractional-order systems

Fractional-order systems are the dynamic systems that involve fractional derivatives. The frequently used definitions for fractional derivatives are the Riemann-Liouville, Grünwald-Letnikov, and Caputo definitions [1]. The Riemann-Liouville definition is given as:

$$\frac{d^q f(t)}{dt^q} = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-q)}\int_0^t \frac{f(\tau)}{(t-\tau)^{q+1-n}}\,d\tau\right], \tag{1}$$

where n is an integer satisfying $n-1 < q \le n$ and $\Gamma(.)$ is the Gamma function. The Grüwald-Letnikov definition can be written as:

$$\frac{d^q f(t)}{dt^q} = \lim_{h\to 0}\frac{1}{h^q}\sum_{j=0}^{\left[\frac{t}{h}\right]}(-1)^j\binom{q}{j} f(t-jh), \tag{2}$$

where [.] means the integer part. The Caputo definition is described by:

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}}\,d\tau, \tag{3}$$

where n is an integer satisfying $n-1 < q \le n$ and $\Gamma(.)$ is the Gamma function. These three definitions are equivalent under some conditions [1].

In this study, we adopt the Caputo derivative definition. An advantage of using the Caputo derivative definition is that the initial conditions for the fractional-order systems are in the same form as for the integer-order systems.

An autonomous fractional-order system with no input can be described as:

$$\begin{aligned}
\frac{d^{q_1}x_1}{dt^{q_1}} &= f_1(x_1(t), x_2(t), \ldots, x_n(t)),\\
\frac{d^{q_2}x_2}{dt^{q_2}} &= f_2(x_1(t), x_2(t), \ldots, x_n(t)),\\
&\vdots\\
\frac{d^{q_n}x_n}{dt^{q_n}} &= f_n(x_1(t), x_2(t), \ldots, x_n(t)),
\end{aligned} \tag{4}$$

where $x_1, x_2, \ldots, x_n$ are the state variables and $q_1, q_2, \ldots, q_n$ are the fractional orders. Note that the order of the system is $q_1 + q_2 + \ldots + q_n$. The system is called a commensurate-order system if $q_1 = q_2 = \ldots = q_n = q$. The vector representation of the commensurate-order system can be expressed as:

$$\frac{d^q\mathbf{x}}{dt^q} = \mathbf{f}(\mathbf{x}), \tag{5}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ is the state vector and q is the fractional commensurate order. A linear time-invariant (LTI) version of System (5) is written as:

$$\frac{d^q\mathbf{x}}{dt^q} = \mathbf{A}\mathbf{x}. \tag{6}$$

Lemma 1 [47]**.** The fractional-order LTI system (6) with $0 < q < 1$ is asymptotically stable if and only if there exist two symmetric positive-definite matrices $\mathbf{P}_{k1} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, and two skew-symmetric matrices $\mathbf{P}_{k2} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, such that:

$$\sum_{i=1}^{2}\sum_{j=1}^{2} Sym\{\boldsymbol{\Theta}_{ij}\otimes(\mathbf{A}\mathbf{P}_{ij})\}<0,$$
$$\begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ -\mathbf{P}_{12} & \mathbf{P}_{11}\end{bmatrix}>0,\quad \begin{bmatrix}\mathbf{P}_{21} & \mathbf{P}_{22}\\ -\mathbf{P}_{22} & \mathbf{P}_{21}\end{bmatrix}>0, \tag{7}$$

where

$$\boldsymbol{\Theta}_{11}=\begin{bmatrix} sin\left(\frac{\pi}{2}q\right) & -cos\left(\frac{\pi}{2}q\right)\\ cos\left(\frac{\pi}{2}q\right) & sin\left(\frac{\pi}{2}q\right)\end{bmatrix},$$

$$\boldsymbol{\Theta}_{12}=\begin{bmatrix} cos\left(\frac{\pi}{2}q\right) & sin\left(\frac{\pi}{2}q\right)\\ -sin\left(\frac{\pi}{2}q\right) & cos\left(\frac{\pi}{2}q\right)\end{bmatrix},$$

$$\boldsymbol{\Theta}_{21}=\begin{bmatrix} sin\left(\frac{\pi}{2}q\right) & cos\left(\frac{\pi}{2}q\right)\\ -cos\left(\frac{\pi}{2}q\right) & sin\left(\frac{\pi}{2}q\right)\end{bmatrix},$$

$$\boldsymbol{\Theta}_{22}=\begin{bmatrix} -cos\left(\frac{\pi}{2}q\right) & sin\left(\frac{\pi}{2}q\right)\\ -sin\left(\frac{\pi}{2}q\right) & -cos\left(\frac{\pi}{2}q\right)\end{bmatrix},$$

$$Sym\{\mathbf{Z}\}=\mathbf{Z}^T+\mathbf{Z},$$

and $\otimes$ is the Kronecker product.

2.2 Tensor product model transformation

The tensor product (TP) model transformation is an effective numerical methodology to transform linear parameter varying (LPV) systems into convex TP model representations [27,28]. In this paper, we consider the following LPV system:

$$\begin{aligned}\frac{d^q\mathbf{x}}{dt^q}&=\mathbf{A}(\mathbf{p}(t))\mathbf{x}+\mathbf{B}(\mathbf{p}(t))\mathbf{u}\\ &=\mathbf{S}(\mathbf{p}(t))\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix},\end{aligned} \tag{8}$$

where

$$\mathbf{S}(\mathbf{p}(t))=(\mathbf{A}(\mathbf{p}(t))\quad \mathbf{B}(\mathbf{p}(t))) \tag{9}$$

is the system matrix, $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector, q is the fractional commensurate order, and $\mathbf{p}(t)$ is a time varying vector in a bounded space. The system is called quasi LPV (qLPV) if $\mathbf{p}(t)$ includes some elements of $\mathbf{x}$. Note that, when $q = 1$, the system is a conventional integer-order LPV system.

The TP model transformation converts the system matrix (9) into a convex combination of R constant linear time invariant (LTI) system matrices

$$\mathbf{S}_r=(\mathbf{A}_r\quad \mathbf{B}_r),\quad r=1,2,3,\dots,R \tag{10}$$

as

$$\mathbf{S}(\mathbf{p}(t))=\sum_{r=1}^{R}\omega_r(\mathbf{p}(t))\mathbf{S}_r, \tag{11}$$

where $\omega_r(\cdot)$ are the weighting functions with $\left\{\forall\mathbf{p}(t):\sum_{r=1}^{R}\omega_r(\mathbf{p}(t))=1,\forall r,\mathbf{p}(t):\omega_r(\mathbf{p}(t))\geqslant 0\right\}$. Thus, System (8) can be written as:

$$\frac{d^q\mathbf{x}}{dt^q}=\sum_{r=1}^{R}\omega_r(\mathbf{p}(t))(\mathbf{A}_r\mathbf{x}+\mathbf{B}_r\mathbf{u}). \tag{12}$$

The system can also be reformulated in terms of tensor algebra as [27,28]:

$$\frac{d^q\mathbf{x}}{dt^q}=\left(S\mathop{\boxtimes}_{n=1}^{N}\mathbf{w}_n(p_n(t))\right)\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}, \tag{13}$$

where S is the tensor constructed from the vertex system matrices, $\mathbf{w}_n(\cdot)$ is the row vector containing the weighting functions, and $p_n(t)$ are the elements of $\mathbf{p}(t)$. Here, $\boxtimes$ is used instead of $\otimes$ to express that the core tensor S has higher structure. $\boxtimes$ has been used in most recent papers [35,37,38]. Also note that, if an exact transformation is not possible, the equality signs in (11), (12), and (13) should be replaced by approximately equal signs.

The TP model transformation consists of several steps, which can be summarized as follows. First, the system matrix $\mathbf{S}(\mathbf{p}(t))$ is sampled over a defined hyper rectangular grid of the transformation space of $\mathbf{p}(t)$. Then, the sampled matrices are stored to form a tensor. After that, the higher order singular value decomposition (HOSVD) is executed on the tensor to find the minimal number of vertex systems. Then, the convex hull manipulation is executed and the weighting functions are generated. Note that the convex hull manipulation step ensures that the resulting weighting functions are convex. This step is crucial for a parallel distributed compensation (PDC) controller design framework since the design framework requires the convexity of the TP model. There are various types of the convex hull derived by the TP model transformation. In this paper, the CNO (Close to NOrmal) type is used. The CNO type convex hull constrains the largest values of all weighting functions to be 1 or close to 1. The reader is referred to [27,28] for more details of the TP model transformation.

Note that the TP model transformation can be executed by the TP tool [29].

2.3 Fractional-order chaotic systems

Chaotic systems are the dynamic systems that are highly sensitive to initial conditions. Chaos is defined as the existence of at least one positive Lyapunov exponent. A class of fractional-order chaotic systems considered in this paper is described as [20]:

$$\begin{aligned}\frac{d^q x}{dt^q} &= y\cdot f(x,y,z) + z\cdot\Phi(x,y,z) - \alpha x,\\ \frac{d^q y}{dt^q} &= x\cdot g(x,y,z) - \beta y,\\ \frac{d^q z}{dt^q} &= y\cdot h(x,y,z) - x\cdot\Phi(x,y,z) - \gamma z,\end{aligned} \tag{14}$$

where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, and $\Phi(\cdot)$ are smooth functions. Many fractional-order chaotic systems belong to this class, such as the fractional-order Lorenz system [3], the fractional-order Chen system [4], the fractional-order Lü system [5], and the fractional-order Liu system [6]. This class of the systems can be expressed in the LPV model form as:

$$\frac{d^q \mathbf{x}}{dt^q} = \begin{bmatrix} -\alpha & f(x,y,z) & \Phi(x,y,z) \\ g(x,y,z) & -\beta & 0 \\ -\Phi(x,y,z) & h(x,y,z) & -\gamma \end{bmatrix}\mathbf{x}, \tag{15}$$

where $\mathbf{x} = [x, y, z]^T$. It is worth noting that the chaotic systems are dissipative. This means that all of the system trajectories are bounded. An example of a chaotic attractor is shown in Fig. 1.

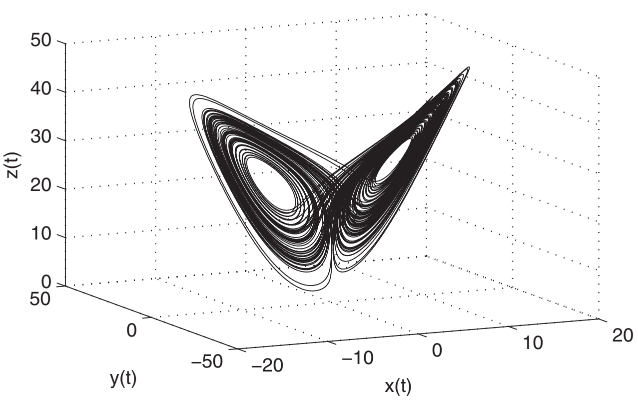

Fig. 1. A chaotic attractor of the fractional-order Lorenz system with $q = 0.993$.

III. MAIN RESULTS

In this section, based on a parallel distributed compensation (PDC) technique [48], we first propose a stabilization condition for the fractional-order TP model system (12). The condition then is simplified to achieve an LMI-based condition, which is solved readily.

The PDC controller is defined as:

$$\mathbf{u} = -\left(\sum_{r=1}^{R}\omega_r(\mathbf{p}(t))\mathbf{K}_r\right)\mathbf{x}, \tag{16}$$

which can also be reformulated in terms of tensor algebra as [27,28]:

$$\mathbf{u} = -\left(K \mathop{\boxtimes}_{n=1}^{N} \mathbf{w}_n(p_n(t))\right)\mathbf{x}, \tag{17}$$

where K is a feedback tensor. Note that the TP model and the controller share the same weighting functions.

Theorem 1. The fractional-order TP model system (12) with the PDC controller (16) and $0 < q < 1$ is asymptotically stable if there exist two symmetric positive-definite matrices $\mathbf{P}_{k1} \in \mathbb{R}^{n\times n}$, $k = 1, 2$ and two skew-symmetric matrices $\mathbf{P}_{k2} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, such that:

$$\begin{aligned}&\mathbf{F}_r = \sum_{i=1}^{2}\sum_{j=1}^{2} Sym\{\boldsymbol{\Theta}_{ij}\otimes(\mathbf{A}_r\mathbf{P}_{ij} - \mathbf{B}_r\mathbf{K}_r\mathbf{P}_{ij})\} < 0,\\ &\quad r = 1, 2, \ldots, R,\\ &\mathbf{G}_{rs} = \sum_{i=1}^{2}\sum_{j=1}^{2} Sym\{\boldsymbol{\Theta}_{ij}\otimes(\mathbf{A}_r\mathbf{P}_{ij} - \mathbf{B}_r\mathbf{K}_s\mathbf{P}_{ij} + \mathbf{A}_s\mathbf{P}_{ij} - \mathbf{B}_s\mathbf{K}_r\mathbf{P}_{ij})\}\\ &\quad < 0, \quad r < s,\\ &\begin{bmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ -\mathbf{P}_{12} & \mathbf{P}_{11}\end{bmatrix} > 0, \quad \begin{bmatrix}\mathbf{P}_{21} & \mathbf{P}_{22}\\ -\mathbf{P}_{22} & \mathbf{P}_{21}\end{bmatrix} > 0.\end{aligned} \tag{18}$$

Proof. By substituting (16) into (12), one obtains the following closed loop system:

$$\begin{aligned}\frac{d^q \mathbf{x}}{dt^q} &= \sum_{r=1}^{R}\sum_{s=1}^{R}\omega_r(\mathbf{p}(t))\omega_s(\mathbf{p}(t))(\mathbf{A}_r - \mathbf{B}_r\mathbf{K}_s)\mathbf{x}\\ &= \mathbf{A}_c\mathbf{x},\end{aligned} \tag{19}$$

where

$$\begin{aligned}\mathbf{A}_c &= \sum_{r=1}^{R}\omega_r^2(\mathbf{p}(t))(\mathbf{A}_r - \mathbf{B}_r\mathbf{K}_r)\\ &\quad + \sum_{r=1}^{R}\sum_{s=r+1}^{R}\omega_r(\mathbf{p}(t))\omega_s(\mathbf{p}(t))(\mathbf{A}_r - \mathbf{B}_r\mathbf{K}_s + \mathbf{A}_s - \mathbf{B}_s\mathbf{K}_r).\end{aligned} \tag{20}$$

Suppose there exist $\mathbf{P}_{k1}$ and $\mathbf{P}_{k2}$ satisfying (18). Since $\left\{\forall\mathbf{p}(t): \sum_{r=1}^{R}\omega_r(\mathbf{p}(t)) = 1, \forall r, \mathbf{p}(t): \omega_r(\mathbf{p}(t)) \geqslant 0\right\}$,

$$\sum_{r=1}^{R}\omega_r^2(\mathbf{p}(t))\mathbf{F}_r+\sum_{r=1}^{R}\sum_{s=r+1}^{R}\omega_r(\mathbf{p}(t))\omega_s(\mathbf{p}(t))\mathbf{G}_{rs}<0. \quad (21)$$

After rearranging the above inequality, one obtains:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\mathbf{\Theta}_{ij}\otimes(\mathbf{A}_c\mathbf{P}_{ij})\}<0. \quad (22)$$

Thus, it follows from Lemma 1 that (19) is asymptotically stable. Therefore, the fractional-order TP model system (12) with the PDC controller (16) is asymptotically stable and the proof is complete.

Solving the stabilization condition (18) is a tedious task since it is a nonlinear matrix inequality (NMI) problem. For ease of computation, we simplify the condition (18) by setting $\mathbf{P}_{12}=\mathbf{P}_{22}=\mathbf{0}$, $\mathbf{P}_{11}=\mathbf{P}_{21}=\mathbf{P}$, and $\mathbf{X}_r=\mathbf{K}_r\mathbf{P}$, resulting in the following corollary.

Corollary 1. The fractional-order TP model system (12) with the PDC controller (16) and $0<q<1$ is asymptotically stable if there exists a symmetric positive-definite matrix $\mathbf{P}\in\mathbb{R}^{n\times n}$ such that:

$$\begin{aligned}&\sum_{i=1}^{2}Sym\{\mathbf{\Theta}_{i1}\otimes(\mathbf{A}_r\mathbf{P}-\mathbf{B}_r\mathbf{X}_r)\}<0,\quad r=1,2,\ldots,R,\\&\sum_{i=1}^{2}Sym\{\mathbf{\Theta}_{i1}\otimes(\mathbf{A}_r\mathbf{P}-\mathbf{B}_r\mathbf{X}_s+\mathbf{A}_s\mathbf{P}-\mathbf{B}_s\mathbf{X}_r)\}<0,\quad r<s,\\&\mathbf{P}>0.\end{aligned} \quad (23)$$

Moreover, the PDC feedback gains are provided by:

$$\mathbf{K}_r=\mathbf{X}_r\mathbf{P}^{-1},\quad r=1,2,\ldots,R.$$

The stabilization condition given in the above corollary is an LMI in $\mathbf{P}$ and $\mathbf{X}_r$, $r=1,2,\ldots,R$, and it can be solved by various LMI solvers, such as the LMI Robust Control Toolbox of MATLAB. Note that the condition is a sufficiency condition. For a particular type of convex hull or TP model, if there is no feasible solution, one should try with a different type. It was shown in [37,38] that the convex hull strongly influences the feasibility of the LMI solution, as well as the control performance.

IV. NUMERICAL SIMULATIONS

The Adams-type predictor-corrector method [49,50] with the time step of 0.001 sec is used in all simulations.

4.1 Chaos control

We consider the fractional-order Lorenz system and the fractional-order Liu system as illustrative examples. The objective of the control is to suppress the chaotic behavior of the systems.

The fractional-order Lorenz system is described as [3]:

$$\begin{aligned}\frac{d^qx}{dt^q}&=a(y-x),\\\frac{d^qy}{dt^q}&=bx-y-xz,\\\frac{d^qz}{dt^q}&=-cz+xy,\end{aligned} \quad (24)$$

where $a>0$, $b>0$, and $c>0$ are the system parameters and $0<q<1$ is the fractional commensurate order. When $a=10$, $b=28$, $c=8/3$, and $q=0.993$, the system has a chaotic attractor, as shown in Fig. 1. The system (24) belongs to the class of chaotic systems (14) by setting $\alpha=a$, $f(x,y,z)=a$, $\beta=1$, $g(x,y,z)=b-z$, $\gamma=c$, $h(x,y,z)=x$ and $\Phi(x,y,z)=0$. To control chaos in the system, we add the control input $u(t)$ to the second state equation. The system can be written in the LPV model form (8) as:

$$\begin{aligned}\frac{d^q\mathbf{x}}{dt^q}&=\begin{bmatrix}-a&a&0\\b-z&-1&0\\0&x&-c\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\1\\0\end{bmatrix}u\\&=\mathbf{A}(\mathbf{p}(t))\mathbf{x}+\mathbf{B}u.\end{aligned} \quad (25)$$

Note that the time varying vector $\mathbf{p}(t)$ contains $x(t)$ and $z(t)$. Since the system is dissipative, all state variables are bounded. The bounds estimated through simulations were found to be $-20<x<30$, $-25<y<30$, and $0<z<50$, respectively. Thus, the space of $\mathbf{p}(t)$ is selected as $[-20,30]\times[0,50]$.

By executing the TP model transformation of the system given in (25), using the TP Tool [29] with 50×50 sampling grid points, the rank of the sampled tensor was found to be 2 on both dimensions, which implies that four vertex systems can exactly represent the system. The weighting functions are shown in Figs 2 and 3. Solving the LMI condition yields the following four linear feedback gains:

$$\mathbf{K}_{1,1}=(-2.8022\quad 7.1602\quad -0.1521),$$

$$\mathbf{K}_{2,1}=(-2.8022\quad 7.1602\quad 0.1521),$$

$$\mathbf{K}_{1,2}=(16.5025\quad 15.4600\quad -0.2990),$$

$$\mathbf{K}_{2,2}=(16.5025\quad 15.4600\quad 0.2990).$$

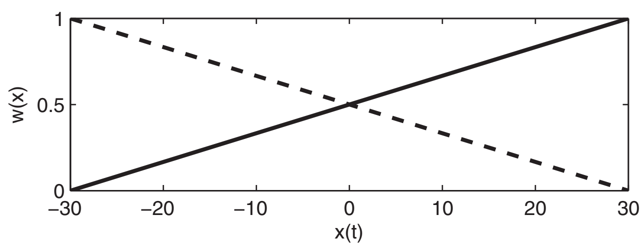

Fig. 2. Weighting functions on the dimension x.

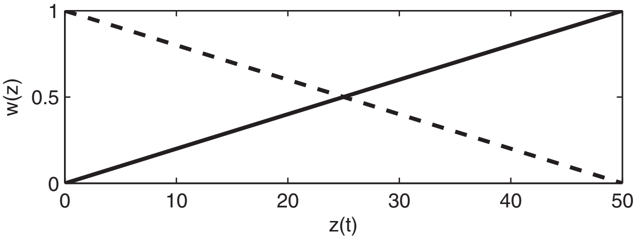

Fig. 3. Weighting functions on the dimension z.

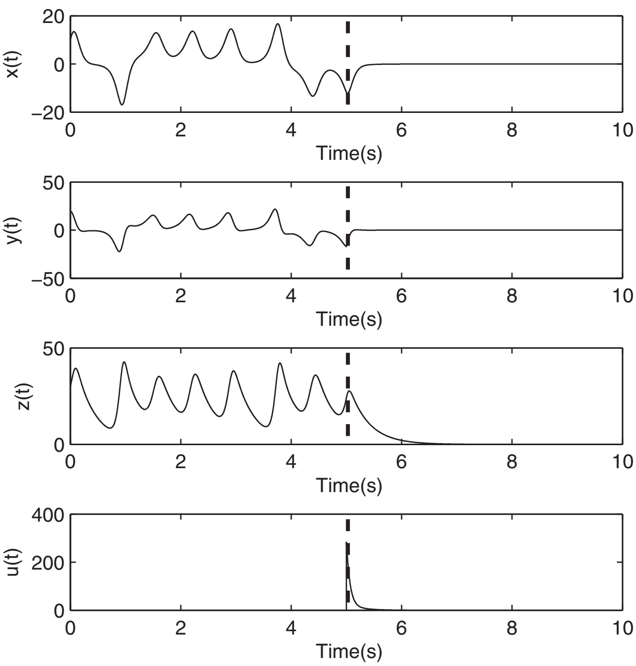

Fig. 4. State trajectories of the fractional-order Lorenz system. The controller was activated at time = 5 sec (indicated by the dashed line).

The control results are shown in Fig. 4. It shows that the chaotic behavior was suppressed within 2 seconds after the controller was activated. The convergence was very smooth and there was no overshoot observed. We conclude that asymptotic stability was achieved, as guaranteed by Corollary 1.

Note that Corollary 1 concludes nothing about control performance, except asymptotic stability. Thus, if some specific control performance is desired, it might be possible to tune the control performance empirically via convex hull manipulation, as done in [37,38]. We consider the relation between the control performances and the convex hull as a further development of this paper.

As another example, consider the fractional-order Liu system [6]:

$$\begin{aligned}\frac{d^q x}{dt^q} &= -ax - ey^2,\\ \frac{d^q y}{dt^q} &= by - kxz,\\ \frac{d^q z}{dt^q} &= -cz + mxy,\end{aligned} \tag{26}$$

where $a = e = 1$, $b = 2.5$, $c = 5$, $k = m = 4$, and $q = 0.98$ yield chaotic trajectory. By setting $\alpha = a$, $f(x, y, z) = -ey$, $\beta = -b$, $g(x, y, z) = -kz$, $\gamma = c$, $h(x, y, z) = mx$, and $\Phi(x, y, z) = 0$, the system belongs to (14). After adding the control input $u(t)$, the system can be written in the LPV model form as:

$$\frac{d^q \mathbf{x}}{dt^q} = \begin{bmatrix} -a & -ey & 0 \\ -kz & b & 0 \\ 0 & mx & -c \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u. \tag{27}$$

In this case, the time varying vector $\mathbf{p}(t)$ contains $x(t)$, $y(t)$, and $z(t)$. The bounds of $x(t)$, $y(t)$, and $z(t)$ estimated through simulations were found to be $-8 < x < 5$, $-8 < y < 7$, and $-6 < z < 8$, respectively. Thus, the space of $\mathbf{p}(t)$ is selected as $[-8, 5] \times [-8, 7] \times [-6, 8]$.

Similar to the previous case, by executing the TP model transformation with $50 \times 50 \times 50$ sampling grid points, the rank of the sampled tensor was found to be 2 on all dimensions. Thus, eight vertex systems can exactly represent the system. Then, by solving the LMI, we obtain the following eight linear feedback gains:

$$\mathbf{K}_{1,1,1} = (27.7917 \quad 11.8248 \quad -0.1882),$$

$$\mathbf{K}_{2,1,1} = (27.7804 \quad 10.9593 \quad 0.0144),$$

$$\mathbf{K}_{1,2,1} = (14.3771 \quad 9.2609 \quad -0.1600),$$

$$\mathbf{K}_{2,2,1} = (13.8960 \quad 8.1542 \quad -0.0083),$$

$$\mathbf{K}_{1,1,2} = (-20.7369 \quad 10.3548 \quad -0.2043),$$

$$\mathbf{K}_{2,1,2} = (-20.2075 \quad 9.4116 \quad -0.0322),$$

$$\mathbf{K}_{1,2,2} = (-34.3924 \quad 12.8824 \quad -0.2464),$$

$$\mathbf{K}_{2,2,2} = (-34.3510 \quad 12.1207 \quad -0.0248).$$

The control results are shown in Fig. 5. The states asymptotically converged to zeros after the controller was activated. The convergence was very smooth, and there was no overshoot observed.

4.2 Chaos synchronization

The fractional-order Lorenz system (24) is employed as an illustrative example. The two systems in synchronization

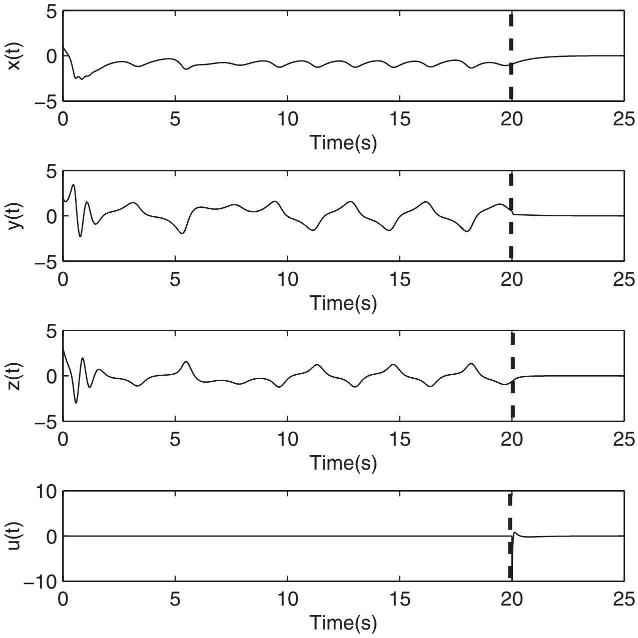

Fig. 5. State trajectories of the fractional-order Liu system. The controller was activated at time = 20 s (indicated by the dashed line).

are called the master system and the slave system, respectively. The objective of the synchronization is to control the behavior of the slave system to follow the behavior of the master system.

From (24), we define the master and slave systems as:

$$\begin{aligned}\frac{d^q x_m}{dt^q} &= a(y_m - x_m),\\ \frac{d^q y_m}{dt^q} &= bx_m - y_m - x_m z_m,\\ \frac{d^q z_m}{dt^q} &= -cz_m + x_m y_m,\end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{d^q x_s}{dt^q} &= a(y_s - x_s),\\ \frac{d^q y_s}{dt^q} &= bx_s - y_s - x_s z_s + u,\\ \frac{d^q z_s}{dt^q} &= -cz_s + x_s y_s,\end{aligned} \tag{29}$$

where the lower scripts m and s stand for the master and slave, respectively, and u is the controller that is designed such that the two systems are synchronized.

Let us define the synchronous errors as $e_1 = x_s - x_m$, $e_2 = y_s - y_m$, and $e_3 = z_s - z_m$. Using (28), and (29), we obtain the following error system:

$$\begin{aligned}\frac{d^q e_1}{dt^q} &= a(e_2 - e_1),\\ \frac{d^q e_2}{dt^q} &= (b - z_m)e_1 - e_2 - x_s e_3 + u,\\ \frac{d^q e_3}{dt^q} &= y_m e_1 + x_s e_2 - ce_3,\end{aligned} \tag{30}$$

which can be written in the LPV model form as:

$$\frac{d^q \mathbf{x}}{dt^q} = \begin{bmatrix} -a & a & 0 \\ b - z_m & -1 & -x_s \\ y_m & x_s & -c \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}u, \tag{31}$$

where $\mathbf{x} = [e_1, e_2, e_3]^T$. The time varying vector $\mathbf{p}(t)$ contains $x_s(t)$, $y_m(t)$, and $z_m(t)$. The space of $\mathbf{p}(t)$ is selected as $[-20, 30] \times [-25, 30] \times [0, 50]$.

By executing the TP model transformation of the error system, using the TP Tool [29] with $50 \times 50 \times 50$ sampling grid points, the rank of the sampled tensor was found to be 2 on all dimensions, which implies that eight vertex systems can represent the system exactly. Solving the LMI of the error system yields the following eight linear feedback gains:

$$\mathbf{K}_{1,1,1} = (53.2223 \quad 4.3423 \quad 0.5238),$$

$$\mathbf{K}_{2,1,1} = (83.1466 \quad 4.2800 \quad -4.3589),$$

$$\mathbf{K}_{1,2,1} = (77.6591 \quad 2.3615 \quad 7.5949),$$

$$\mathbf{K}_{2,2,1} = (63.5462 \quad 5.2833 \quad 2.4936),$$

$$\mathbf{K}_{1,1,2} = (14.1825 \quad 2.7146 \quad 1.2901),$$

$$\mathbf{K}_{2,1,2} = (44.0159 \quad 3.0285 \quad -2.7203),$$

$$\mathbf{K}_{1,2,2} = (43.8500 \quad 1.8663 \quad 4.6435),$$

$$\mathbf{K}_{2,2,2} = (24.7092 \quad 2.9929 \quad 0.8241).$$

The state responses of the master and slave systems and the synchronization errors are shown in Figs 6 and 7, respectively. The results show that the controller was able to drive the states of the slave system to asymptotically synchronize the states of the master system as desired. The synchronization was achieved within 1 second after the controller was activated. The convergence of the errors was very smooth, and there was no overshoot observed.

V. CONCLUSIONS

In this paper, TP model transformation based-controller design for control and synchronization of fractional-order

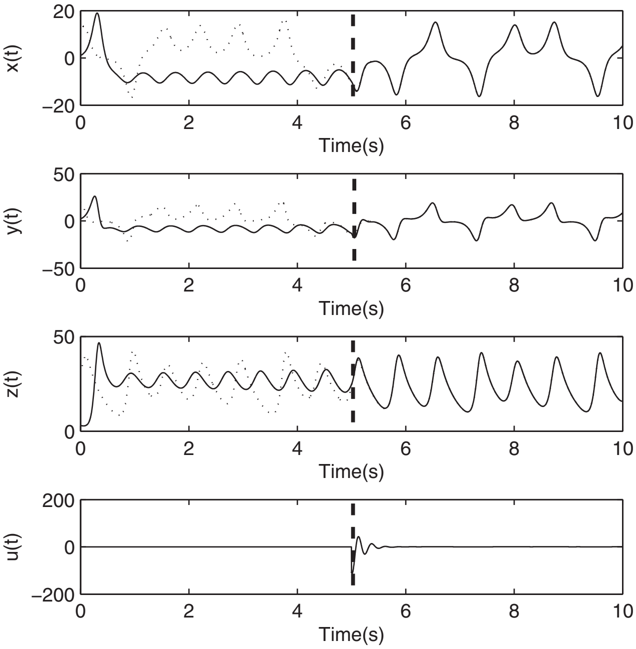

Fig. 6. State responses of the fractional-order master and slave Lorenz systems (solid line = slave system, dotted line = master system). The controller was activated at time = 5 s (indicated by the dashed line).

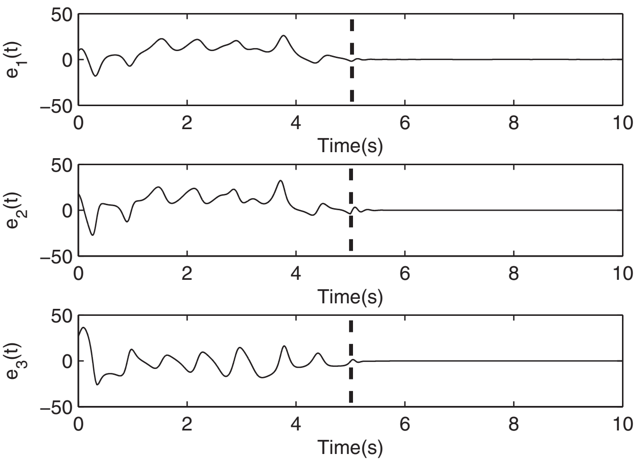

Fig. 7. Synchronization errors of the fractional-order Lorenz systems. The controller was activated at time = 5 sec (indicated by the dashed line).

chaotic systems was proposed. We presented a novel LMI condition for fractional-order TP models with a controller derived via a parallel distributed compensation structure. The method starts with transformation of a controlled system into a convex TP model form. After that, the controller is determined directly by solving the LMI condition based on the transformed TP model. Numerical results of the fractional-order Lorenz and Liu systems illustrated that the method is effective. In these numerical studies, the CNO type convex hull was used and exact transformations were achieved. Although the focus of this paper is on fractional-order chaotic systems, the method also can be applied to other fractional-order dynamical systems as well, assuming that the TP model transformation is possible. A future direction of this paper is to investigate how the resulting control performance and robustness will be influenced by the convex hull, especially when an exact transformation cannot be obtained.

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