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Control Stability of TP Model Transformation Design via Probabilistic Error Bound of Plant Model

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***Abstract*—Recently, an emerging numerical controller design based on Tensor-Product model transformation has proven to work on a number of nonlinear systems by both simulations and practical implementations. In this paper, aiming at fully utilizing the power of approximation of TP model transformation, a new control stability analysis is proposed in a probabilistic framework. We utilize the matrix 2-norm to characterize the maximum deviation of the approximated system from the actual one. However, the theoretical boundary of the maximum deviation is hard to determine. Therefore, probabilistic error bound estimation is introduced with the aid of Chebyshev inequality to define a practical bound of deviation for the approximated system. This new analysis technique allows practical controller design for approximated TP system model by solving a set of Linear Matrix Inequalities (LMI). Examples using the benchmark Translational Oscillator with Rotational Actuator (TORA) system are presented to illustrate our formulation, showing that the designed controller can guarantee asymptotic stability with sufficiently high probability.**

***Index Terms*—Nonlinear Control, TP Model Transformation, LMI**

I. Introduction

Nonlinear control design has been developed for several decades and many theoretical design methods have been introduced, such as feedback linearization, backstepping, etc [1]. These methods have proven to be practical in many real plants. However, when the plants are very complex, the derivations using analytical design could be extremely complicated. To overcome this kind of complication, it is natural to seek for a computational approach for the controller design. Recently, Tensor-Product (TP) model based control design methodology has become a promising candidate for the computational approach, which allows automatic controller design using efficient algorithms even for complex systems [2], [3], [4], [5], [6], [7], [8], [9].

TP control system design can be applied to a class of nonlinear systems expressing in quasi-Linear Parameter-Varying (qLPV) state-space form [2]. The procedures involve two steps. First, the qLPV model is transformed into affine or convex polytopic representation with the aid of Singular Value Decomposition (SVD)-based algorithm first proposed in [10] and [11]. This TP model transformation can either be exact or approximated. Upon the TP model of the system, a LMI based optimization is executed for multi-objective control design. Both steps can be done without analytical derivation and thus are fully automated.

There are many successful examples such as the benchmark Translational Oscillator with Rotational Actuator (TORA) system [12] where exact TP model of system exists. However in practice, some classes of systems may not have an exact TP representation [13]. Or even in those cases with exact TP representation, if the order of the expression is too high, it is still undesirable to use it in designing a controller since the computational cost will explode exponentially in terms of the order [14]. Fortunately, the SVD-based transformation process enables approximation of the LPV model using a lower order TP representation. If the existing LMI design approach is applied, the stability and control specification are only guaranteed for the approximated model but not for the original plant. In the present paper, a set of LMI conditions ensuring the stability of original plant is derived by exploiting probabilistic error bound. The upper bound of deviation of the TP system model from the original plant is estimated by the well-known Chebyshev inequality [15]. As a result, the stability of closed loop system is guaranteed up to a certain required confidence in probability sense.

The structure of the paper is organized as follows. Section II introduces the numerical TP model transformation. Section III describes our LMI design for approximated TP system model. Section IV introduces the idea of probabilistic error bound estimation. Section V presents the simulation results that verify our proposed methods.

II. Numerical TP model transformation

In order to automatically design a controller for a given dynamical model using TP model, one has to begin with the numerical TP model transformation. The following description is based on [2] and [12]. Given a dynamical system in the following state-space representation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\big(\mathbf{p}(t)\big)\mathbf{x}(t) + \mathbf{B}\big(\mathbf{p}(t)\big)\mathbf{u}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$. The above representation is generally recognized as the LPV model with a time varying parameter $\mathbf{p}(t) \in \mathbb{R}^q$ in input $\mathbf{u}(t) \in \mathbb{R}^m$. A special case of (1) is a class of nonlinear systems where the parameter $\mathbf{p}(t)$ includes some elements of the state $\mathbf{x}(t)$. This is sometimes referred as the qLPV model.

Let $\mathbf{S}(\mathbf{p}(t)) = \big[\mathbf{A}\big(\mathbf{p}(t)\big) \quad \mathbf{B}\big(\mathbf{p}(t)\big)\big]$, then (1) can be written as:

$$\dot{\mathbf{x}} = \mathbf{S}(\mathbf{p}) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{2}$$

TP model transformation basically involves three major steps: discretization of parameter space, application of Higher Order Singular Value Decomposition (HOSVD) [16] and convex hull manipulation [17]. For discretization, first we have to define the operation range of the parameter $\mathbf{p}$. Let Ω be a bounded hyper rectangular parameter space with $\mathbf{p} \in \Omega : [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_q, b_q]$. Then for the $(i_1, i_2, \ldots, i_q)^{th}$ grid point there is a corresponding position vector $\mathbf{g}_{i_1,i_2,\ldots,i_q}$. We define a $(q+2)$ tensor $\mathcal{S}^D \in \mathbb{R}^{I_1\times I_2\times\ldots\times I_q\times n\times(n+m)}$ by

$$\mathcal{S}_{i_1,i_2,\ldots,i_q} = \mathbf{S}(\mathbf{g}_{i_1,i_2,\ldots,i_q}) \tag{3}$$

for $i_n = 1, \ldots, I_n$, $n = 1, \ldots, q$.

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In fact, the tensor consists of numerical values of $\mathbf{S}(\mathbf{p})$ sampled at every grid point which stores the information about the system matrix $\mathbf{S}$ in a numerical form. Thus a denser grid captures more system information, but at the same time increases the size of the tensor. However, since the system is known to be continuous with respect to the parameter space, and suppose the grid is dense enough, we would expect the tensor may have excess information about the system. Here HOSVD [16] is applied to extract the low-rank tensor product structure from the system tensor.

HOSVD is a purely computational method to obtain a low-rank approximation of a tensor. Loosely speaking, the process is similar to doing ordinary Singular Value Decomposition (SVD) for each dimension of parameter $\mathbf{p}$. Hence, a decreasing sequence of singular values can be obtained for each dimension. There are two types of HOSVD. The first type is called Compact HOSVD (CHOSVD) where the zero singular values and their corresponding singular vectors are discarded. The second type is the Reduced HOSVD (RHOSVD) where some nonzero but small singular values and their corresponding singular vectors are discarded as well.

From a system point of view, the above TP model transformation process can be comprehended as follows. Through discretization, the system matrix $\mathbf{S}(\mathbf{p})$ is approximated by a finite element TP type polytopic model:

$$\mathbf{S}_{TP}(\mathbf{p}) = \sum_{i_1=1}^{I_1} \cdots \sum_{i_q=1}^{I_q} \prod_{n=1}^{q} w_{n,i_n}(p_n)\mathbf{S}_{i_1,\ldots,i_q} \tag{4}$$

where $\mathbf{S}_{i_1,\ldots,i_q} \in \mathbb{R}^{n\times(n+m)}$ and $w_{n,i_q}(p_n)$ are interpolating functions corresponding to n^{th} entry of $\mathbf{p}$ at the grid point i_n. For linear interpolation, $w_{n,i_n}(p_n)$ are overlapped triangular-shaped functions.

After applying CHOSVD, (4) can be extracted to the Exact TP (ETP) model:

$$\mathbf{S}_{ETP}(\mathbf{p}) = \sum_{i_1=1}^{R_1} \cdots \sum_{i_q=1}^{R_q} \prod_{n=1}^{q} \bar{w}_{n,i_n}(p_n)\bar{\mathbf{S}}_{i_1,\ldots,i_q} \stackrel{\text{def}}{=} \sum_{r=1}^{R} w_r(\mathbf{p})\mathbf{S}_r \tag{5}$$

where $\bar{\mathbf{S}}_{i_1,\ldots,i_q}$ are the extracted Linear Time-Invariant (LTI) matrix and $\bar{w}_{n,i_n}(p_n)$ are the extracted weighting functions. We further define

$$\mathbf{S}_r \stackrel{\text{def}}{=} [\mathbf{A}_r \quad \mathbf{B}_r] \stackrel{\text{def}}{=} \bar{\mathbf{S}}_{i_1,\ldots,i_q}, \tag{6}$$

$$w_r(\mathbf{p}) \stackrel{\text{def}}{=} \prod_{n=1}^{q} \bar{w}_{n,i_n}(p_n), \tag{7}$$

and $R = R_1R_2\cdots R_q$. Since this ETP model $\mathbf{S}_{ETP}$ is exactly the same as $\mathbf{S}_{TP}$, this is referred as the exact model. If RHOSVD is applied, then (4) will be approximated by the Reduced TP (RTP) model:

$$\mathbf{S}_{RTP}(\mathbf{p}) = \sum_{i_1=1}^{R'_1} \cdots \sum_{i_q=1}^{R'_q} \prod_{n=1}^{q} \bar{w}_{n,i_n}(p_n)\bar{\mathbf{S}}_{i_1,\ldots,i_q} \stackrel{\text{def}}{=} \sum_{r=1}^{R'} w_r(\mathbf{p})\mathbf{S}_r \tag{8}$$

where $R'_n < R_n$ for $n = 1,\ldots,q$ and $R' = R'_1R'_2\ldots R'_q$.

The final step of TP model transformation is the convex hull manipulation. Since we would like to design the controller by solving LMI, the system is required to be convex. Convex hull manipulation allows (5) and (8) to possess the convex property. In this paper, we simply focus on Close to NOrmal (CNO) type convex TP model whose weighting functions satisfy the following conditions for all $\mathbf{p} \in \Omega$:

$$\text{(Non-negativeness condition)} \quad \forall n,i,p_n : w_{n,i}(p_n) \in [0,1]$$

$$\text{(Sum Normalized condition)} \quad \forall n,p_n : \sum_{i=1}^{I_n} w_{n,i}(p_n) = 1$$

$$\text{(CNO condition)} \quad \forall n,i,p_n : max(w_{n,i}(p_n)) \approx 1$$

This CNO type convex TP model is found to be suitable for designing state feedback controller [18]. For properties of different types of convex hulls and computational details of convex hull manipulation, readers may refer to [11], [4], [17]. Note that the whole process of TP model transformation is numerical computation and can be executed automatically without human intervention.

III. LMI-BASED CONTROLLER DESIGN FOR REDUCED TP SYSTEM MODEL

Since our goal is to design a desirable controller for the dynamical system in the convex TP model, it is natural to define our controller to be also in the TP model representation [2]. One way to achieve this is by Parallel Distributed Compensation (PDC) framework [19]. As observed in (5), there is a sequence of LTI subsystems inside the summation. The concept of PDC is to design a linear state feedback controller for every subsystem. The resulting nonlinear controller

$$\mathbf{u} = \mathbf{K}(\mathbf{p})_{TP}\mathbf{x} \tag{9}$$

would have the same set of weighting functions as the convex TP model of the system. The nonlinear gain becomes

$$\begin{aligned}\mathbf{K}_{TP}(\mathbf{p}) &= -\sum_{i_1=1}^{R_1} \cdots \sum_{i_q=1}^{R_q} \prod_{n=1}^{q} \bar{w}_{n,i_n}(p_n)\mathbf{K}_{i_1,\ldots,i_q} \\ &\stackrel{\text{def}}{=} -\sum_{r=1}^{R} w_r(\mathbf{p})\mathbf{K}_r\end{aligned} \tag{10}$$

where $\mathbf{K}_{i_1,\ldots,i_q}$ are the LTI gain for the subsystems and $\bar{w}_{n,i_n}(p_n)$ are the same weighting functions in (5). We further define $\mathbf{K}_r \stackrel{\text{def}}{=} \mathbf{K}_{i_1,\ldots,i_q}$. [2] has shown that such framework can be used for controller design by solving LMI using efficient algorithms. However, existing literatures only discuss the case for ETP model of $\mathbf{S}_{ETP}$. If the same design methodology is used for RTP model $\mathbf{S}_{RTP}$, as the RTP model is just an approximation of the original model, the designed controller does not guarantee the same stability performance if it is applied to the original system.

In the following, RTP model of system (8) with CNO properties is considered. To simplify the notation, let

$$\mathbf{A}_{TP}(\mathbf{p}) \stackrel{\text{def}}{=} \sum_{r=1}^{R'} w_r(\mathbf{p})\mathbf{A}_r \tag{11}$$

$$\mathbf{B}_{TP}(\mathbf{p}) \stackrel{\text{def}}{=} \sum_{r=1}^{R'} w_r(\mathbf{p})\mathbf{B}_r \tag{12}$$

$$\mathbf{K}_{TP}(\mathbf{p}) \stackrel{\text{def}}{=} -\sum_{r=1}^{R'} w_r(\mathbf{p})\mathbf{K}_r \tag{13}$$

and

$$\Delta\mathbf{A}(\mathbf{p}) \stackrel{\text{def}}{=} \mathbf{A}(\mathbf{p}) - \mathbf{A}_{TP}(\mathbf{p}) \tag{14}$$

$$\Delta\mathbf{B}(\mathbf{p}) \stackrel{\text{def}}{=} \mathbf{B}(\mathbf{p}) - \mathbf{B}_{TP}(\mathbf{p}) \tag{15}$$

where R' is the same as that in (8). Given the original dynamical system of the form (1), the ETP model of system can be defined as

$$\dot{\mathbf{x}}_{TP} = \sum_{r=1}^{R'} w_r(\mathbf{p})\left(\mathbf{A}_i\mathbf{x} + \mathbf{B}_i\mathbf{u}\right) = \mathbf{A}_{TP}(\mathbf{p})\mathbf{x} + \mathbf{B}_{TP}(\mathbf{p})\mathbf{u} \tag{16}$$

Using PDC as in (10), the TP controller is defined as:

$$\mathbf{u}_{TP}(\mathbf{p}) \overset{\text{def}}{=} -\sum_{r=1}^{R'} w_r(\mathbf{p})\mathbf{K}_r\mathbf{x} = \mathbf{K}_{TP}(\mathbf{p})\mathbf{x} \tag{17}$$

Lyapunov direct method is employed to derive the LMI for designing a desirable controller that can stabilize the original system (1). Let the Lyapunov candidate for system (1) and controller (17) be

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x} \tag{18}$$

Then the time derivative of V is

$$\begin{aligned}\dot{V} =& \mathbf{x}^T\Big(\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\mathbf{P} \\ &+ \mathbf{P}\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big) + \Delta\mathbf{A}(\mathbf{p})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}(\mathbf{p}) \\ &+ \big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\mathbf{P} + \mathbf{P}\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)\Big)\mathbf{x}\end{aligned} \tag{19}$$

Since

$$\begin{aligned}&(\Delta\mathbf{A}(\mathbf{p})\mathbf{x} - \mathbf{P}\mathbf{x})^T(\Delta\mathbf{A}(\mathbf{p})\mathbf{x} - \mathbf{P}\mathbf{x}) \geq 0 \\ \Leftrightarrow\ &\mathbf{x}^T\Big(\Delta\mathbf{A}(\mathbf{p})^T\mathbf{P} + \mathbf{P}\Delta\mathbf{A}(\mathbf{p})\Big)\mathbf{x} \leq \mathbf{x}^T\Big(\Delta\mathbf{A}(\mathbf{p})^T\Delta\mathbf{A}(\mathbf{p}) + \mathbf{P}^T\mathbf{P}\Big)\mathbf{x}\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\Big(\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)\mathbf{x} - \mathbf{P}\mathbf{x}\Big)^T\Big(\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)\mathbf{x} - \mathbf{P}\mathbf{x}\Big) \geq 0 \\ \Leftrightarrow\ &\mathbf{x}^T\Big(\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\mathbf{P} + \mathbf{P}\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)\Big)\mathbf{x} \\ &\leq \mathbf{x}^T\Big(\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big) + \mathbf{P}^T\mathbf{P}\Big)\mathbf{x}\end{aligned} \tag{21}$$

Then

$$\begin{aligned}\dot{V} \leq& \mathbf{x}^T\Big(\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\mathbf{P} \\ &+ \mathbf{P}\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big) + \Delta\mathbf{A}(\mathbf{p})^T\Delta\mathbf{A}(\mathbf{p}) \\ &+ \big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\big(\Delta\mathbf{B}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big) + 2\mathbf{P}^T\mathbf{P}\Big)\mathbf{x}\end{aligned} \tag{22}$$

It is known that $\Delta\mathbf{A}(\mathbf{p})$ and $\Delta\mathbf{B}(\mathbf{p})$ are finite for all $\mathbf{p}$ and thus their norms are bounded. To define the upper bound, we introduce the matrix 2-norm which is defined as $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{A}\mathbf{x}\|_2$. Let $\|\Delta\mathbf{A}(\mathbf{p})\|_2 \leq \epsilon_A$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2 \leq \epsilon_B$ for all $\mathbf{p} \in \Omega$. The detailed of estimating the norm upper bounds using probabilistic approach is discussed in Section IV.

With the above upper bounds, (22) can be simplified into

$$\begin{aligned}\dot{V} \leq& \mathbf{x}^T\Big(\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big)^T\mathbf{P} \\ &+ \mathbf{P}\big(\mathbf{A}_{TP}(\mathbf{p}) + \mathbf{B}_{TP}(\mathbf{p})\mathbf{K}_{TP}(\mathbf{p})\big) + \epsilon_A^2\mathbf{I} \\ &+ \epsilon_B^2\mathbf{K}_{TP}^T\mathbf{K}_{TP} + 2\mathbf{P}^T\mathbf{P}\Big)\mathbf{x}\end{aligned} \tag{23}$$

It can be proved that $\mathbf{x}^T\left(\mathbf{K}_{TP}(\mathbf{p})^T\mathbf{K}_{TP}(\mathbf{p})\right)\mathbf{x}$ has the following upper bound:

$$\mathbf{x}^T\Big(\mathbf{K}_{TP}(\mathbf{p})^T\mathbf{K}_{TP}(\mathbf{p})\Big)\mathbf{x} \leq \sum_{i=1}^{R'} w_i(\mathbf{p})\mathbf{x}^T\mathbf{K}_i^T\mathbf{K}_i\mathbf{x} \tag{24}$$

Then we can get a *sufficient condition* for asymptotically stability of the control system in Theorem 1.

Theorem 1. *Assume that the TP controller (17) is applied to the original LPV system (1), the closed loop system is asymptotically stable for $\boldsymbol{p} \in \Omega$ if there exists a positive definite matrix $\boldsymbol{P}$ such that the following inequalities are satisfied:*

$$\begin{aligned}&(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j) \\ &\quad + \epsilon_A^2\boldsymbol{I} + \epsilon_B^2\boldsymbol{K}_j^T\boldsymbol{K}_j + 2\boldsymbol{P}^2 < 0\end{aligned} \tag{25}$$

for $i, j = 1, \ldots, R'$.

Proof: By substituting (11) into (23), we get

$$\begin{aligned}\dot{V} \leq& \sum_{i=1}^{R'}\sum_{j=1}^{R'} w_i(\mathbf{p})w_j(\mathbf{p}) \\ &\mathbf{x}^T\Big((\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j) \\ &+ \epsilon_A^2\mathbf{I} + \epsilon_B^2\mathbf{K}_j^T\mathbf{K}_j + 2\mathbf{P}^2\Big)\mathbf{x}\end{aligned} \tag{26}$$

If the inequalities (25) are satisfied, $\dot{V} < 0$ and hence the closed loop system is asymptotically stable as the weighting functions $w_r(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Omega$. □

Notice that the inequalities (25) are difficult to solve analytically. Fortunately, they can be reformulated into LMI problems in which the solution is equivalent to solving (25). It is well known that LMI problems can be solved efficiently through some convex optimization algorithms such as the interior point method. One way to formulate the LMI problem for our controller design is to introduce new variables $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{Y}_j = \mathbf{K}_j\mathbf{X}$. Then (25) becomes

$$\mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{Y}_j^T\mathbf{B}_i^T - \mathbf{B}_i\mathbf{Y}_j + \epsilon_A^2\mathbf{X}^2 + \epsilon_B^2\mathbf{Y}_j^T\mathbf{Y}_j + 2\mathbf{I} < 0 \tag{27}$$

Using the idea of Schur complements, (25) is equivalent to the following LMI:

$$\begin{bmatrix} \mathbf{X}\mathbf{A}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{Y}_j^T\mathbf{B}_i^T - \mathbf{B}_i\mathbf{Y}_j + 2\mathbf{I} & \epsilon_B\tilde{\mathbf{Y}}_j^T & \epsilon_A\mathbf{X} \\ \epsilon_B\tilde{\mathbf{Y}}_j & -\mathbf{I} & \mathbf{0} \\ \epsilon_A\mathbf{X} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0 \tag{28}$$

where $\tilde{\mathbf{Y}}_j \overset{\text{def}}{=} [\mathbf{Y}_j \quad \mathbf{0}]^T \in \mathbb{R}^{n\times n}$. By solving for $\mathbf{X}$ and $\mathbf{Y}_j$ $j = 1, \ldots, R'$, the LTI gains $\mathbf{K}_j$ can be obtained by $\mathbf{K}_j = \mathbf{Y}_j\mathbf{X}^{-1}$.

The above LMI can be the foundation of our TP controller design. However, the information of weighting functions is not utilized. Exploiting the fact that the controller has the same set of weighting functions as the system, it is possible to reduce the number of LMI using Theorem 2.

Theorem 2. *Assume that the TP controller (17) is applied to the original LPV system (1), the closed loop system is asymptotically stable for $\boldsymbol{p} \in \Omega$ if there exists a positive definite matrix $\boldsymbol{X}$ such that the following LMI are satisfied:*

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{Y}_i^T\boldsymbol{B}_i^T - \boldsymbol{B}_i\boldsymbol{Y}_i + 2\boldsymbol{I} & \epsilon_B\tilde{\boldsymbol{Y}}_i^T & \epsilon_A\boldsymbol{X} \\ \epsilon_B\tilde{\boldsymbol{Y}}_i & -\boldsymbol{I} & \boldsymbol{0} \\ \epsilon_A\boldsymbol{X} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0 \tag{29}$$

for $i = 1, \ldots, R'$ and

$$\begin{bmatrix} \boldsymbol{G}_{ij} + \boldsymbol{G}_{ji} + 4\boldsymbol{I} & \epsilon_B\tilde{\boldsymbol{Y}}_i^T & \epsilon_B\tilde{\boldsymbol{Y}}_j^T & \epsilon_A\boldsymbol{X} \\ \epsilon_B\tilde{\boldsymbol{Y}}_i & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \epsilon_B\tilde{\boldsymbol{Y}}_j & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \epsilon_A\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\frac{1}{2}\boldsymbol{I} \end{bmatrix} < 0 \tag{30}$$

for $i < j < R'$, where $\boldsymbol{G}_{ij} \overset{def}{=} \boldsymbol{X}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{Y}_j^T\boldsymbol{B}_i^T - \boldsymbol{B}_i\boldsymbol{Y}_j$ and $\tilde{\boldsymbol{Y}}_j \overset{def}{=} [\boldsymbol{Y}_j \quad \boldsymbol{0}]^T \in \mathbb{R}^{n\times n}$.

Proof: Start from (26),

$$\dot{V} \leq \sum_{i=1}^{R'} w_i^2(\mathbf{p})\mathbf{x}^T \Big((\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i) + \epsilon_A^2\mathbf{I} + \epsilon_B^2\mathbf{K}_i^T\mathbf{K}_i + 2\mathbf{P}^2\Big)\mathbf{x} + \sum_{i=1}^{R'}\sum_{i<j} w_i(\mathbf{p})w_j(\mathbf{p})\mathbf{x}^T \Big((\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j) (\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i) + 2\epsilon_A^2\mathbf{I} + \epsilon_B^2(\mathbf{K}_i^T\mathbf{K}_i + \mathbf{K}_j^T\mathbf{K}_j) + 4\mathbf{P}^2\Big)\mathbf{x} \tag{31}$$

If $\mathbf{Y}_j$ and $\mathbf{X}$ satisfying (29) and (30) are found, by letting $\mathbf{P} = \mathbf{X}^{-1}$ and $\mathbf{K}_j = \mathbf{Y}_j\mathbf{X}^{-1}$, (29) and (30) are equivalent to:

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i) + \epsilon_A^2\mathbf{I} + \epsilon_B^2\mathbf{K}_i^T\mathbf{K}_i + 2\mathbf{P}^2 < 0 \tag{32}$$

and

$$(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_j) + (\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_j - \mathbf{B}_j\mathbf{K}_i) + 2\epsilon_A^2\mathbf{I} + \epsilon_B^2(\mathbf{K}_i^T\mathbf{K}_i + \mathbf{K}_j^T\mathbf{K}_j) + 4\mathbf{P}^2 < 0 \tag{33}$$

Thus, $\dot{V} < 0$ and Theorem 2 is valid. □

IV. Probabilistic error bound estimation

In this section, the idea and procedures of probabilistic error bound estimation is introduced. In order to correctly and efficiently apply Theorem 1, we have to get an appropriate estimation of the upper bounds of $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2$ for $\mathbf{p} \in \Omega$. If the estimates are much larger than the actual maximum, Theorem 1 will become more conservative. Unfortunately, the theoretical least upper bounds are difficult to obtain analytically since two types of approximation errors may effectuate the deviation in the extracted RTP model. The first one is the error as a result of grid point sampling and linear interpolation while the other is the approximation error due to discarding some singular values and singular vectors in HOSVD.

Instead of theoretically calculating the least upper bound, we seek a numerical way to estimate an *effective upper bound* which may not be the least upper bound but sufficient for us to design a stable controller. Since the norms are not globally concave, it is impossible to get the global maximum via iterative algorithms. We propose a statistical way to get an effective upper bound. Because the original plant and RTP model are known, it is possible to sample a large number of the norms $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2$ by picking a large number of $\mathbf{p}$ within the range of Ω uniformly at random. For simplicity, consider the norm $\|\Delta\mathbf{X}(\mathbf{p})\|_2$ where $\mathbf{X}$ can either be $\mathbf{A}$ or $\mathbf{B}$. Therefore, the norm can be viewed as a continuous random variable with a finite maximum.

The above problem can be simplified as follows. Let X be a continuous random variable with a probability distribution function is $f(x)$ for $x \in [0, \theta]$ where θ is a finite real number, and 0 elsewhere. Our objective is to estimate θ through sampling.

The basic idea is to find the distribution of the largest order statistics estimator (i.e. the maximum of samples). First, we sample N sets of X. Each set consists of M random numbers, i.e. $\{X_1, \ldots, X_M\}$. Let Y be the largest order statistics of the set, i.e. $Y = max\{X_1, \ldots, X_M\}$, which is also a continuous random variable.

Assume $f(x)$ is known, then we have the following theorem:

Table 1. Procedures for probabilistic error bound estimation
Step 1: Uniformly generate M sample of points $\mathbf{p}$. Find the largest $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2$ among those sample points and record them.
Step 2: Repeat step 1 for N times. Now there should be N maxima of $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2$. Get the averages μ_A, μ_B and standard deviation σ_A, σ_B of the N maxima.
Step 3: For 95% confidence level, set $\epsilon_A = \mu_A + 4.36\sigma_A$ and $\epsilon_B = \mu_B + 4.36\sigma_B$ respectively, while for 99% confidence, set $\epsilon_A = \mu_A + 9.95\sigma_A$ and $\epsilon_B = \mu_B + 9.95\sigma_B$ repectively.

Theorem 3. *The probability density function $g(y)$ of random variable Y defined above is given by*

$$g(y) = \begin{cases} M\left(\int_0^y f(x)dx\right)^{M-1} f(y) & \text{if } 0 < y < \theta \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

Proof: Since Y is less than x only when all X in the set is less than x, i.e.

$$Pr\{Y \leq x\} = (Pr\{X \leq x\})^M \tag{35}$$

which can also be expressed as

$$\int_0^y g(x)dx = \left(\int_0^y f(x)dx\right)^M \tag{36}$$

By the fundamental theorem of calculus, (36) implies (34). □

As $M \to \infty$, the term $\left(\int_0^y f(x)dx\right)^{M-1}$ is 0 except for $y = \theta$. And when $y = \theta$, $\left(\int_0^\theta f(x)dx\right)^{M-1} = 1$ and hence $g(\theta) = Mf(\theta) \to \infty$ as $M \to \infty$ if $f(\theta) \neq 0$. This implies that the probability density function of Y will converge to the shifted Dirac delta function peaked at θ.

In our formulation, we obtain N samples of the random variable Y. If $N \to \infty$, the sampling distribution will converge to $g(y)$. However, $f(x)$ is unknown and N is finite in our problem. Therefore, we can only estimate the mean μ and standard deviation σ of the distribution $g(y)$ using the sample average and standard deviation with the N samples of Y.

By applying the one-sided Chebyshev inequality [15], it is possible to obtain the effective upper bound. The one-sided Chebyshev inequality is applicable for any probability distribution and is given as:

$$Pr\{Y - \mu \geq k\sigma\} \leq \frac{1}{1+k^2} \tag{37}$$

where μ and σ are average and standard deviation of the random variable Y respectively. The effective upper bound can be set as $k\sigma$ under the required confidence. For instance, if we want 95% confidence, we can set $k = 4.36$, while for 99% confidence, we can set $k = 9.95$. In the present paper, 99% confidence is considered in the examples. The procedures of probabilistic error bound estimation are summarized in Table 1.

V. Simulations

In this section, the TORA system appeared in [12] is demonstrated to verify our LMI-based controller design for reduced TP system model. All simulations are done in MATLAB. TP Tool is used for TP model transformation while the YALMIP is applied as the LMI solver.

A. Example: TORA

The dynamical equations of TORA system are given by [20]:

$$(M+m)\ddot{q} + kq = m - me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \quad (38)$$

$$(I+me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N \quad (39)$$

With the normalization $\xi = \sqrt{\frac{M+m}{I+me^2}}q$, $\tau = \sqrt{\frac{k}{M+m}}t$, $u = \frac{M+m}{k(I+me^2)}N$ and $\rho = \frac{me}{\sqrt{(I+me^2)(M+m)}}$, the equations of motion of TORA can be written in the LPV form (1) with the state $\mathbf{x} = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^T = \begin{bmatrix}\xi & \dot{\xi} & \theta & \dot{\theta}\end{bmatrix}^T$,

$$\mathbf{A}(\mathbf{p}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\rho^2\cos^2(x_3)} & 0 & 0 & \frac{\rho x_4 \sin(x_3)}{1-\rho^2\cos^2(x_3)} \\ 0 & 0 & 0 & 1 \\ \frac{\rho\cos(x_3)}{1-\rho^2\cos^2(x_3)} & 0 & 0 & \frac{\rho^2 x_4\sin(x_3)\cos(x_3)}{1-\rho^2\cos^2(x_3)} \end{bmatrix} \quad (40)$$

and

$$\mathbf{B}(\mathbf{p}) = \begin{bmatrix} 0 \\ -\frac{\rho\cos(x_3)}{1-\rho^2\cos^2(x_3)} \\ 0 \\ \frac{1}{1-\rho^2\cos^2(x_3)} \end{bmatrix} \quad (41)$$

where $\mathbf{p} \overset{\text{def}}{=} [x_3 \quad x_4]^T \in \Omega$.

In our simulation, we define the parameter space $\Omega = [-a, a] \times [-0.5, 0.5]$ where $a = \frac{45}{180}\pi$ and $\rho = 0.2$. The density of the discretization grid is 137×137. By using TP Tool, result shows that the above TORA system can be exactly transformed into the convex TP model with 10 LTI subsystems:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{5}\sum_{j=1}^{2} \bar{w}_{1,i}\big(x_3(t)\big)\bar{w}_{2,j}\big(x_4(t)\big)\big(\bar{\mathbf{A}}_{i,j}\mathbf{x}(t) + \bar{\mathbf{B}}_{i,j}u(t)\big) \quad (42)$$

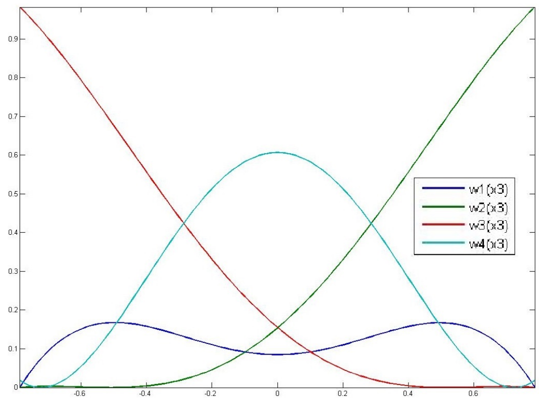
The LMI-based control design of this exact system has already been discussed in [12]. The problem we are interested is whether the reduced TP model can also be used for design. Here we consider the case where only 3 singular values of parameter x_3 are kept, i.e. discarding the 2 smallest nonzero singular values and their corresponding singular vectors. For parameter x_4, all singular values are kept. After the convex hull manipulation to CNO, the TORA can be approximated by a convex TP model with $4 \times 2 = 8$ LTI subsystems:

$$\mathbf{A}_{1,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0629 & 0 & 0 & -1.4498 \times 10^{-5} \\ 0 & 0 & 0 & 1 \\ 0.2893 & 0 & 0 & (-1)^{j+1}2.3908 \times 10^{-6} \end{bmatrix}$$

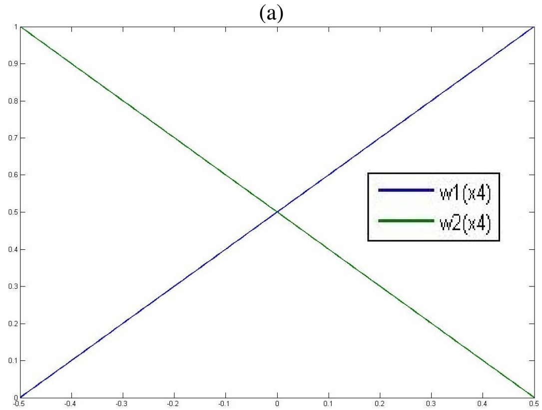
$$\mathbf{A}_{2,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0509 & 0 & 0 & (-1)^{j+1}1.8270 \times 10^{-5} \\ 0 & 0 & 0 & 1 \\ 0.2350 & 0 & 0 & -3.0128 \times 10^{-6} \end{bmatrix}$$

$$\mathbf{A}_{3,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0198 & 0 & 0 & (-1)^{j}0.0733 \\ 0 & 0 & 0 & 1 \\ 0.1426 & 0 & 0 & (-1)^{j+1}0.0121 \end{bmatrix}$$

$$\mathbf{A}_{4,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0198 & 0 & 0 & (-1)^{j+1}0.0733 \\ 0 & 0 & 0 & 1 \\ 0.1425 & 0 & 0 & (-1)^{j}0.0121 \end{bmatrix}$$



(a)



(b)

Fig. 1: Weighting functions (a) $w_{1,j}(x_3)$ and (b) $w_{2,j}(x_4)$

$$\mathbf{B}_{1,1} = \mathbf{B}_{1,2} = \begin{bmatrix} 0 \\ -0.2893 \\ 0 \\ 1.0629 \end{bmatrix} \quad \mathbf{B}_{2,1} = \mathbf{B}_{2,2} = \begin{bmatrix} 0 \\ -0.2350 \\ 0 \\ 1.0509 \end{bmatrix}$$

$$\mathbf{B}_{3,1} = \mathbf{B}_{3,2} = \begin{bmatrix} 0 \\ -0.1426 \\ 0 \\ 1.0198 \end{bmatrix} \quad \mathbf{B}_{4,1} = \mathbf{B}_{4,2} = \begin{bmatrix} 0 \\ -0.1425 \\ 0 \\ 1.0198 \end{bmatrix}$$

where $j = 1, 2$, and the weighting functions are presented in Fig. 1.

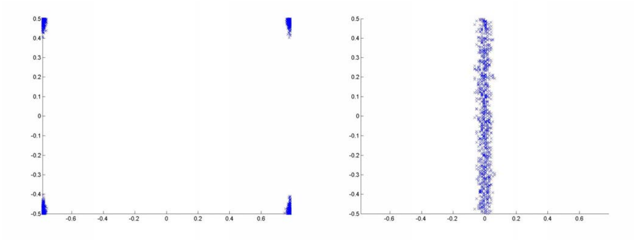
To apply Theorem 1 and 2, ϵ_A and ϵ_B have to be estimated probabilistic method according to the procedures described in Table 1. By using $M = 1000$ and $N = 1000$, it is found that μ_A and σ_A are 0.0012 and 2.1738×10^{-4} while μ_B and σ_B are 3.9873×10^{-6} and 4.2958×10^{-8} respectively. For 99% confidence, the ϵ_A and ϵ_B are set to be 0.0033 and 4.4148×10^{-6} respectively.

Figure 2 shows the locations of the 1000 sample maxima for $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and $\|\Delta\mathbf{B}(\mathbf{p})\|_2$ in Ω respectively. More than one local maximum exist in the figure so it is hard to search for the global maximum using iterative algorithms.

When the set of LMI in (28) is considered, there are 65 LMI in our case. Solving with YALMIP, it is found that

$$\mathbf{P} = \begin{bmatrix} 0.7683 & -0.2349 & -0.2913 & -0.7984 \\ -0.2349 & 0.1401 & 0.1364 & 0.2518 \\ -0.2913 & 0.1364 & 01622 & 0.3104 \\ -0.7984 & 0.2518 & 0.3104 & 0.8575 \end{bmatrix} > 0$$

and $\mathbf{K}_j = [-6.2388 \quad 1.2815 \quad 2.6581 \quad 6.8429]$ for $j = 1, \ldots, 8$



(a) (b)

Fig. 2: Locations of 1000 sample maxima of (a) $\|\Delta\mathbf{A}(\mathbf{p})\|_2$ and (b) $\|\Delta\mathbf{B}(\mathbf{p})\|_2$ in Ω

satisfy (28). Hence by Theorem 1, the closed loop system is asymptotically stable with 99% confidence in terms of probabilistic error bound estimation.

If the set of LMI in (37) is considered instead, there are altogether 37 LMI. It is found that

$$\mathbf{P} = \begin{bmatrix} 0.8804 & -0.2102 & -0.2723 & -0.9101 \\ -0.2102 & 0.1115 & 0.1040 & 0.2234 \\ -0.2723 & 0.1040 & 0.1264 & 0.2875 \\ -0.9101 & 0.2234 & 0.2875 & 0.9684 \end{bmatrix} > 0$$

and

$$\begin{array}{l} \mathbf{K}_1 = \mathbf{K}_2 = [-6.7729 \quad 0.9128 \quad 2.3094 \quad 7.3583] \\ \mathbf{K}_3 = \mathbf{K}_4 = [-6.6618 \quad 0.8841 \quad 2.2733 \quad 7.2363] \\ \mathbf{K}_5 = \mathbf{K}_8 = [-6.5976 \quad 0.8625 \quad 2.2512 \quad 7.1689] \\ \mathbf{K}_6 = \mathbf{K}_7 = [-6.7464 \quad 0.9053 \quad 2.3017 \quad 7.3237] \end{array}$$

satisfy (37). Hence by Theorem 2, the closed loop system is asymptotically stable with 99% confidence in terms of probabilistic error bound estimation.

To check the efficacy of our controller, we randomly generated 200 initial conditions satisfying the condition of $\mathbf{p} \in \Omega$. It is found that the control system converges to the origin asymptotically in all the 200 cases. Hence, our probabilistic error bound estimation is believed to be sufficiently accurate for this example.

VI. Conclusions

The control stability of TP model transformation design is introduced with the aid of the probabilistic error bound estimation. This approach is found to be successful in practice when the difference between the reduced TP model and the original plant is small, i.e. the approximation is sufficient close to the original plant. As seen from the simulations, keeping fewer singular values would lead to fewer LMI and hence lower the computational cost which is more desirable. There is always a tradeoff between computational cost and complexity of TP model. We have provided a method to balance the two: while using less complex TP structure of controller, we ensure that it can achieve asymptotic stability with sufficiently high confidence. The concept of probability is introduced as a numerical approach to estimate the error bounds which are extremely difficult to obtain theoretically. Notice that the LMI based design process can be done fully numerically and automatically, which is one of the features of TP model transformation. Future researches include finding less conservative LMI conditions and extending the present results to observer-based control design.

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References

[1] H.K. Khalil and J.W. Grizzle, *Nonlinear systems*, vol. 3, Prentice hall, 1992.
[2] P. Baranyi, "TP model transformation as a way to LMI-based controller design," *IEEE Transactions on Industrial Electronics*, vol. 51, no. 2, pp. 387–400, 2004.
[3] P. Baranyi, "Tensor product model based control of 2-D aeroelastic system," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 2, pp. 391–400, 2006.
[4] P. Baranyi, "Output feedback control of 2-D aeroelastic system," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 3, pp. 762–767, 2006.
[5] P. Baranyi, Y. Yam, and P. Varlaki, *TP Model Transformation in Polytopic Model-Based Control*, Taylor & Francis, 2013.
[6] P. Galambos, P. Baranyi, and G. Arz, "TP model transformation based control design for force reflecting tele-grasping under time delay," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, 2013.
[7] R.-E. Precup, C.-A. Dragos, S. Preitl, M.-B. Radac, and E.M. Petriu, "Novel tensor product models for automatic transmission system control," *IEEE Systems Journal*, vol. 6, no. 3, pp. 488–498, 2012.
[8] J. Matusko, V. Lesic, F. Kolonic, and S Iles, "Tensor product based control of the single pendulum gantry process with stable neural network based friction compensation," in *IEEE/ASME International Conference on Advanced Intelligent Mechatronics*. IEEE, 2011, pp. 1010–1015.
[9] P. Baranyi, D. Tikk, Y. Yam, and R.J. Patton, "From differential equations to PDC controller design via numerical transformation," *Computers in Industry*, vol. 51, no. 3, pp. 281–297, 2003.
[10] Y. Yam, "Fuzzy approximation via grid point sampling and singular value decomposition," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 27, no. 6, pp. 933–951, 1997.
[11] Y. Yam, P. Baranyi, and C.T. Yang, "Reduction of fuzzy rule base via singular value decomposition," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 120–132, 1999.
[12] Z. Petres, "Polytopic decomposition of linear parameter-varying models by tensor-product model transformation," *Budapest: Budapest University of Technology and Economics*, 2006.
[13] D. Tikk, P. Baranyi, and R.J. Patton, "Approximation properties of TP model forms and its consequences to TPDC design framework," *Asian Journal of Control*, vol. 9, no. 3, pp. 221–231, 2007.
[14] S. Nagy, Z. Petres, P. Baranyi, and H. Hashimoto, "Computational relaxed TP model transformation: restricting the computation to subspaces of the dynamic model," *Asian Journal of Control*, vol. 11, no. 5, pp. 461–475, 2009.
[15] J.G. Saw, M.C.K. Yang, and T.C. Mo, "Chebyshev inequality with estimated mean and variance," *The American Statistician*, vol. 38, no. 2, pp. 130–132, 1984.
[16] L. DeLathauwer, B. DeMoor, and J. Vandewalle, "A multilinear singular value decomposition," *SIAM journal on Matrix Analysis and Applications*, vol. 21, no. 4, pp. 1253–1278, 2000.
[17] A. Matszangosz, S. Nagy, and P. Baranyi, "Numerical convex hull manipulation under tensor product transformation based control design," in *IEEE International Conference on Computational Cybernetics*, 2008, pp. 169–172.
[18] P. Gróf, P. Baranyi, and P. Korondi, "Convex hull manipulation based control performance optimisation," *WSEAS Transactions on Systems and Control*, vol. 5, no. 8, pp. 691–700, 2010.
[19] H.O. Wang, K. Tanaka, and M. Griffin, "Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model," in *Proceedings of 1995 IEEE International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and The Second International Fuzzy Engineering Symposium*. IEEE, 1995, vol. 2, pp. 531–538.
[20] C.J. Wan, D.S. Bernstein, and V.T. Coppola, "Global stabilization of the oscillating eccentric rotor," *Nonlinear dynamics*, vol. 10, no. 1, pp. 49–62, 1996.