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Robust inverse optimal control for discrete-time nonlinear system stabilization

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ABSTRACT

This paper presents an inverse optimal control approach in order to achieve stabilization of discrete-time nonlinear systems, avoiding the need to solve the associated Hamilton–Jacobi–Bellman equation, and minimizing a cost functional. Then, the proposed approach is extended to discrete-time disturbed nonlinear systems. The synthesized stabilizing optimal controller is based on a discrete-time control Lyapunov function. The applicability of the proposed approach is illustrated via simulations.

1. Introduction

Optimal nonlinear control is related to determining a control law for a given system, such that a performance criterion is minimized. This criterion is usually formulated as a cost functional, which is a function of state and control variables. The major drawback for optimal nonlinear control is the need to resolve the associated Hamilton–Jacobi–Bellman (HJB) equation [18,40]. This equation, as far as we are aware, has not been solved for general nonlinear systems; the solution exists only for the linear regulator problem, for which it is particularly well-suited [3]. Hence, in order to avoid the HJB equation solution, the inverse optimal control approach was proposed [27,40]. In the inverse approach, a stabilizing feedback control law is synthesized first, and then it is established that this control law optimizes a cost functional.

On the other hand, considering that systems usually present parameter uncertainties and are exposed to disturbances, it is desirable to obtain a robust optimal control scheme; unfortunately to do so, it is required to solve the Hamilton–Jacobi–Isaacs (HJI) partial differential equation [8].

The inverse approach was proposed initially by Kalman [11] for linear systems with quadratic cost functions. We refer the reader to [3,25,43] for the inverse optimal control addressed in the frequency domain for discrete-time linear systems based on the return difference function, and to [6,8,19,21,23,27] and references therein for the nonlinear continuous-time case. Despite the properties and applicabilities of inverse optimal controllers, to the best of our knowledge, there are few results on discrete-time nonlinear inverse optimal control; see [1,26], where the proposed control law depends on the knowledge of a positive (semi)definite function, which is difficult to obtain for general nonlinear systems. The inverse optimal control scheme for nonlinear systems is based on a control Lyapunov function (CLF) [40,8,19]. The existence of a CLF implies stabilizability and every CLF can be considered as a cost functional [5,8]. Systematic techniques for determining a CLF do not exist for general nonlinear systems; however, this approach has been applied successfully to classes of systems for which a CLF can be determined such as feedback linearizable, strict feedback and feed-forward systems [9,37]. Moreover, by using a CLF, it is not required that the system has to be stable for zero input. In this work, a specific structure for a parameter-dependent CLF is proposed, then this parameter is determined by using the particle swarm optimization (PSO) algorithm such that stability and optimality for the system are achieved.

In this paper, the authors present an inverse optimal control approach for stabilization of discrete-time nonlinear systems, which are affine in the control input, avoiding the need to solve the associated HJB equation, and for minimizing a cost functional. Furthermore, for realistic situations, a system includes disturbances, uncertain parameters and unmodeled dynamics that could cause an undesirable performance of a model-based controller;

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this fact motivates the need to derive a robust inverse optimal control scheme to deal with disturbed nonlinear systems and avoiding the associated HJI equation solution. Our main contributions are established as Theorem 1 for stabilization and as Theorem 2 for robust stabilization. Besides, there are two advantages to work in a discrete-time framework: (a) appropriate technology can be used to implement digital controllers rather than analog ones and (b) the synthesized controller is directly implemented in a digital processor.

Related works on the inverse optimal control scheme have been presented. In [31,34], an inverse optimal control scheme is proposed based on passivity concepts, which departs from a storage function to be used as a control Lyapunov function and where the output feedback becomes the stabilizing control law. In [33], trajectory tracking is treated as a stabilization problem if the system can be described as a block control structure, and in [7] the inverse optimal approach has been proposed for discrete-time stochastic nonlinear systems. This paper is an extended version of [30,32]; in addition to these works, in this contribution a parameter-dependent CLF is proposed, whose parameter is calculated through a PSO algorithm. Moreover, an example is presented to illustrate the robust stabilization scheme for discrete-time disturbed nonlinear systems.

The analysis and control synthesis for discrete-time dynamical systems is an interesting topic. These systems appear in disciplines such as economy, chemical engineering, biology and medicine [17], or they come from discretization of continuous-time systems [22,42]. Additionally, different approaches have been dedicated to the analysis of the discrete-time system properties [16,26] and to the determination of adequate discrete-time controllers [4,12,42]. On the other hand, there is no guarantee that a continuous-time synthesized control scheme preserves its properties when implemented in real-time; even worse, it is known that continuous-time schemes could become unstable after sampling [29].

The contents of the paper are as follows. Section 2 establishes the inverse optimal control and its solution by means of a quadratic CLF. In Section 3, the robust inverse optimal control scheme is developed. The applicability of the proposed control methodologies is presented in Section 4. Finally, Section 5 concludes the paper.

2. Inverse optimal control stabilization

Let us consider the discrete-time affine nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad x_0 = x(0) \tag{1}$$

with the associated cost functional

$$V(x_k) = \sum_{n=k}^{\infty} l(x_n) + u_n^T R u_n \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time $k \in \mathcal{N}$. $\mathcal{N}$ denotes the set of nonnegative integers. $u \in \mathbb{R}^m$ is the input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings; $f(0)=0$ and $g(x_k) \neq 0$ for all $x_k \neq 0$. It is required that $V : \mathbb{R}^n \to \mathbb{R}_+$ has a boundary condition $V(0)=0$, so that $V(x_k)$ can be used as a Lyapunov function; $l : \mathbb{R}^n \to \mathbb{R}_+$ is a positive semidefinite[1] function and $R : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite[2] weighting matrix.

Cost functional (2) is considered a performance measure [15]. The entries of R could be functions of the system state in order to vary the weighting on control effort according to the state value [15]. This paper considers affine nonlinear systems, which represents a great variety of systems, most of which are approximate discretizations of continuous-time systems [28,39]. For mathematical preliminaries on stability analysis and optimal control in the discrete-time framework, we refer the reader to [39] and references therein.

Since it is not easy to solve the optimal nonlinear control for system (1) and (2) due the need to solve the associated HJB equation, it is proposed to solve the inverse optimal control [40]. For the inverse approach, a stabilizing feedback control law is first developed, and then it is established that this control law optimizes a cost functional. Along the lines of [40], the discrete-time inverse optimal control law for nonlinear systems is defined as follows.

Definition 1. The control law [39]

$$u_k^* = -\frac{1}{2} R^{-1} g^T(x_k) \frac{\partial V(z)}{\partial z}\bigg|_{z=f(x_k)+g(x_k)u_k^*} \tag{3}$$

is inverse optimal if

(i) it achieves (global) exponential stability of the equilibrium point $x_k = 0$ for system (1);
(ii) it minimizes a cost functional defined as (2), for which $l(x_k) = -\overline{V}$ with
$$\overline{V} := V(x_{k+1}) - V(x_k) + u_k^{*T} R(x_k) u_k^* \leq 0. \tag{4}$$

As established in Definition 1, the inverse optimal control is based on the knowledge of $V(x_k)$; therefore, in order to guarantee (i) and (ii), we propose a CLF defined as

$$V(x_k) = \tfrac{1}{2} x_k^T P x_k, \quad P = P^T > 0 \tag{5}$$

for control law (3) in order to ensure stability for system (1), which will be achieved by determining appropriately matrix P. Moreover, it will be established that the inverse control law (3) with (5) optimizes a cost functional as (2).

Consequently, by considering $V(x_k)$ as in (5), control law (3) takes the following form:

$$\begin{aligned} u_k^* &= -\frac{1}{2} R^{-1} g^T(x_k) \frac{\partial V(z)}{\partial z}\bigg|_{z=f(x_k)+g(x_k)u_k^*} \\ &= -\frac{1}{2} R^{-1} g^T(x_k) P z\Big|_{z=f(x_k)+g(x_k)u_k^*} \\ &= -\frac{1}{2} R^{-1} g^T(x_k) P \big(f(x_k) + g(x_k)u_k^*\big). \end{aligned}$$

Then

$$\Big(I + \tfrac{1}{2} R^{-1} g^T(x_k) P g(x_k)\Big) u_k^* = -\tfrac{1}{2} R^{-1} g^T(x_k) P f(x_k). \tag{6}$$

Multiplying (6) by R, we obtain

$$\big(R + \tfrac{1}{2} g^T(x_k) P g(x_k)\big) u_k^* = -\tfrac{1}{2} g^T(x_k) P f(x_k) \tag{7}$$

which results in the following state feedback control law:

$$\alpha(x_k) := u_k^* = -\tfrac{1}{2}(R + P_2(x_k))^{-1} P_1(x_k) \tag{8}$$

where $P_1(x_k) = g^T(x_k) P f(x_k)$ and $P_2(x_k) = \frac{1}{2} g^T(x_k) P g(x_k)$. Note that $P_2(x_k)$ is positive definite and symmetric matrix, which ensures that the inverse matrix in (8) exists.

Once we have proposed a CLF for solving the inverse optimal control in accordance with Definition 1, the first contribution of the paper is established as the following theorem.

Theorem 1. *Consider the affine discrete-time nonlinear system (1). If there exists a matrix $P = P^T > 0$ such that the following inequality holds*:

$$V_f(x_k) - \tfrac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \leq -\zeta_Q \|x_k\|^2 \tag{9}$$

[1] A function $l(z)$ is positive semidefinite (or nonnegative definite) function if for all vectors z, $l(z) \geq 0$. In other words, there are some vectors z for which $l(z)=0$, and for all others z, $l(z) > 0$ [15].

[2] A real symmetric matrix R is positive definite if $z^T R z > 0$ for all $z \neq 0$ [15].

where $V_f(x_k) = V(f(x_k)) - V(x_k)$, *with* $V(f(x_k)) = \frac{1}{2} f^T(x_k) P f(x_k)$ *and* $\zeta_Q > 0$; $P_1(x_k)$ *and* $P_2(x_k)$ *as defined in* (8); *then, the equilibrium point* $x_k = 0$ *of system* (1) *is globally exponentially stabilized by the control law* (8).

Moreover, with (5) *as a CLF, this control law is inverse optimal in the sense that it minimizes the cost functional given by*

$$V(x_k) = \sum_{k=0}^{\infty} (l(x_k) + u_k^T R u_k) \tag{10}$$

with

$$l(x_k) = -\overline{V}|_{u_k^* = \alpha(x_k)} \tag{11}$$

and optimal value function $V^*(x_k) = V(x_0)$.

Proof. First, we analyze stability. Global stability for the equilibrium point x_k=0 of system (1) with (8) as input is achieved if inequality (4) is satisfied. To do so, let rewrite (4) as follows:

$$\begin{aligned}\overline{V} &= V(x_{k+1}) - V(x_k) + \alpha^T(x_k) R \alpha(x_k) \\ &= \frac{f^T(x_k) P f(x_k) + 2 f^T(x_k) P g(x_k) \alpha(x_k)}{2} \\ &\quad + \frac{\alpha^T(x_k) g^T(x_k) P g(x_k) \alpha(x_k) - x_k^T P x_k}{2} \\ &\quad + \alpha^T(x_k) R(x_k) \alpha(x_k) \\ &= V_f(x_k) - \frac{1}{2} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \\ &\quad + \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \\ &= V_f(x_k) - \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k).\end{aligned} \tag{12}$$

Selecting P in (12), with $P_1(x_k)$ and $P_2(x_k)$ as defined in (8), such that $\overline{V} \le 0$, stability of x_k=0 is then guaranteed. Furthermore, by means of P, we can achieve a desired negativity amount [9] for the closed-loop function $\overline{V}$ in (12). This negativity amount can be bounded using a positive definite matrix Q as follows:

$$\begin{aligned}\overline{V} &= V_f(x_k) - \tfrac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \le -x_k^T Q x_k \\ &\le -\lambda_{min}(Q) \|x_k\|^2 = -\zeta_Q \|x_k\|^2, \quad \zeta_Q = \lambda_{min}(Q)\end{aligned} \tag{13}$$

where $\|\cdot\|$ stands for the Euclidean norm and $\zeta_Q > 0$ denotes the minimum eigenvalue of matrix Q ($\lambda_{min}(Q)$). Thus, from (13) follows condition (9).

Considering (12) and (13) we obtain $\overline{V} = V(x_{k+1}) - V(x_k) + \alpha^T(x_k) R \alpha(x_k) \le -\zeta_Q \|x_k\|^2 \Rightarrow \Delta V = V(x_{k+1}) - V(x_k) \le -\zeta_Q \|x_k\|^2$.

Moreover, as $V(x_k)$ is a radially unbounded function, then the solution x_k=0 of the closed-loop system (1) with (8) as input is globally exponentially stable.

When function $-l(x_k)$ is set to be the right-hand side of (13), then

$$l(x_k) = -\overline{V}|_{u_k^* = \alpha(x_k)} = -V_f(x_k) + \tfrac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k). \tag{14}$$

Hence $V(x_k)$ as proposed in (5), is a solution of the discrete-time HJB equation [39].

In order to obtain the optimal value for the meaningful cost functional (10), we substitute $l(x_k)$ given in (14) into (10); then

$$\begin{aligned}V(x_k) &= \sum_{k=0}^{\infty} (l(x_k) + u_k^T R u_k) \\ &= \sum_{k=0}^{\infty} (-\overline{V} + u_k^T R u_k) \\ &= -\sum_{k=0}^{\infty} \left[V_f(x_k) - \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \right] + \sum_{k=0}^{\infty} u_k^T R u_k.\end{aligned} \tag{15}$$

Factorizing (15), and adding the identity matrix $I_m = (R + P_2(x_k))(R + P_2(x_k))^{-1}$, $I_m \in \mathbb{R}^{m \times m}$, we obtain

$$\begin{aligned}V(x_k) = -\sum_{k=0}^{\infty} \Big[& V_f(x_k) - \frac{1}{2} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \\ & + \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} \\ & \times P_2(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \\ & + \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} R \times (R + P_2(x_k))^{-1} P_1(x_k) \Big] + \sum_{k=0}^{\infty} u_k^T R u_k.\end{aligned} \tag{16}$$

Being $\alpha(x_k) = -\frac{1}{2}(R + P_2(x_k))^{-1} P_1(x_k)$, then (16) becomes

$$\begin{aligned}V(x_k) &= -\sum_{k=0}^{\infty} [V_f(x_k) + P_1^T(x_k) \alpha(x_k) \\ &\quad + \alpha^T(x_k) P_2(x_k) \alpha(x_k)] + \sum_{k=0}^{\infty} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)] \\ &= -\sum_{k=0}^{\infty} [V(x_{k+1}) - V(x_k)] \\ &\quad + \sum_{k=0}^{\infty} [u_k^T \times R u_k - \alpha^T(x_k) R \alpha(x_k)]\end{aligned} \tag{17}$$

which can be written as

$$\begin{aligned}V(x_k) &= -\sum_{k=1}^{\infty} [V(x_{k+1}) - V(x_k)] - V(x_1) \\ &\quad + V(x_0) + \sum_{k=0}^{\infty} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)] \\ &= -\sum_{k=2}^{\infty} [V(x_{k+1}) - V(x_k)] - V(x_2) \\ &\quad + V(x_1) - V(x_1) + V(x_0) \\ &\quad + \sum_{k=0}^{\infty} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)].\end{aligned} \tag{18}$$

For notation convenience in (18), the upper limit ∞ will be treated as $N \to \infty$, and thus

$$\begin{aligned}V(x_k) &= -V(x_N) + V(x_{N-1}) - V(x_{N-1}) \\ &\quad + V(x_0) + \lim_{N \to \infty} \sum_{k=0}^{N} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)] = -V(x_N) + V(x_0) \\ &\quad + \lim_{N \to \infty} \sum_{k=0}^{N} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)].\end{aligned}$$

Letting $N \to \infty$ and noting that $V(x_N) \to 0$ for all x_0, then

$$V(x_k) = V(x_0) + \sum_{k=0}^{\infty} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)]. \tag{19}$$

Thus, the minimum value of (19) is reached with $u_k = \alpha(x_k)$, and the control law (8) minimizes the cost functional (10). The optimal value function of (10) is $V^*(x_k) = V(x_0)$ for all x_0. □

In order to fulfill inequality (9), it is required the knowledge of matrix P. In [32], a fixed positive definite matrix P is selected by trial and error. In this paper, the PSO algorithm is used to establish this matrix. The PSO algorithm was developed in [13] as an optimization algorithm based on social simulation models to determine the minimum of nonlinear functions by iteratively improving a candidate solution with regard to a given measure of performance [13,36]. The PSO algorithm is used for the controller (8) by defining the components of a particle as the entries of matrix P. Details on the applicability of the PSO algorithm for computing matrix P in the discrete-time inverse optimal controller is described in [38]. Other techniques can be used for determining P, for instance, in [35] a speed-gradient algorithm is used to recursively calculate matrix P such that stability and optimality conditions are fulfilled. Additionally, in [2,41] the sum-of-squares (SOS) programming is presented as a tool for the computation of Lyapunov functions.

3. Robust inverse optimal control stabilization

In this section, we establish a robust inverse optimal controller to achieve stabilization of discrete-time disturbed nonlinear systems described as

$$x_{k+1} = f(x_k) + g(x_k)u_k + d_k, \quad x_0 = x(0) \tag{20}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time $k \in \mathcal{N}$, $u \in \mathbb{R}^m$ is the control input, $d_k \in \mathbb{R}^n$ is the disturbance term. The disturbance term d_k could result from modeling errors, uncertainties and disturbances, which exist in any realistic problem [14]. Let us consider that the disturbance $d_k \in \mathbb{R}^n$ is bounded by

$$\|d_k\| \le \ell'_k + \alpha_4(\|x_k\|) \tag{21}$$

with $\ell'_k \le \ell$; ℓ is a positive constant and $\alpha_4(\|x_k\|)$ is a $\mathcal{K}_\infty$–function, and suppose that $\alpha_4(\|x_k\|)$ in (21) is of the same order as the $\mathcal{K}_\infty$–function $\alpha_3(\|x_k\|)$, i.e.

$$\alpha_4(\|x_k\|) = \delta\alpha_3(\|x_k\|), \quad \delta > 0. \tag{22}$$

For discrete-time disturbed nonlinear systems, along the lines of [8], we propose the discrete-time Isaacs equation described as

$$V^*(x_k) = \min_{u_k}\{l(x_k) + u_k^T R u_k + V^*(x_k, u_k, d_k)\} \tag{23}$$

where $V^*(x_k, u_k, d_k) = V(x_{k+1})$; then the HJI equation becomes

$$\begin{aligned} 0 &= \inf_u \sup_{d \in \mathcal{D}}\{l(x_k) + u_k^T R u_k + V(x_{k+1}) - V(x_k)\} \\ &= \inf_u \sup_{d \in \mathcal{D}}\{l(x_k) + u_k^T R u_k + V(x_k, u_k, d_k) - V(x_k)\} \end{aligned} \tag{24}$$

where $\mathcal{D}$ is the set of locally bounded functions, and function $V(x_k)$ is unknown. However, determining a solution of the HJI equation (24) for $V(x_k)$ is the main drawback of the robust optimal control; this solution may not exist or may be pretty difficult to solve [8]. Hence, we propose to develop the robust inverse optimal control, avoiding the need to solve the HJI partial differential equation [8].

We establish the discrete-time *robust* inverse optimal control as follows.

Definition 2. The control law (3) is robust inverse optimal (globally) stabilizing if:

(i) It achieves (global) input-to-state stability (ISS) for system (20);
(ii) $V(x_k)$ is (radially unbounded) positive definite such that the inequality

$$\overline{V}_d(x_k, d_k) := V(x_{k+1}) - V(x_k) + u_k^T R u_k \le -\sigma(x_k) + \ell_d\|d_k\| \tag{25}$$

is satisfied, where $\sigma(x_k)$ is a positive definite function and ℓ_d is a positive constant. The value of the function $\sigma(x_k)$ represents a desired amount of negativity [9] of the closed-loop difference $\overline{V}_d(x_k, d_k)$.

For the robust inverse optimal control solution, let us consider the state feedback control law (3), with (5) as a candidate CLF, where P is assumed to be positive definite. Taking one step ahead for (5), then control law (3) results in (8).

Hence, the robust inverse optimal control is solved as follows.

Theorem 2. *Consider the disturbed affine discrete-time nonlinear system (20). If there exists a matrix $P = P^T > 0$ such that the following inequality holds*

$$V_f(x_k) - \tfrac{1}{4}P_1^T(x_k)\left(R + P_2(x_k)\right)^{-1}P_1(x_k) \le -\zeta\alpha_3(\|x_k\|), \ \forall \|x_k\| \ge \rho(\|d_k\|) \tag{26}$$

with δ in (22) satisfying

$$\delta < \frac{\eta}{\ell_d} \tag{27}$$

where $V_f(x_k) = \frac{1}{2}[V(f(x_k)) - V(x_k)]$, and $P_1(x_k) = g^T(x_k)Pf(x_k)$ and $P_2(x_k) = \frac{1}{2}g^T(x_k)Pg(x_k)$; ζ, $\ell_d > 0$, $\eta = (1-\theta)\zeta > 0$, $0 < \theta < 1$, and with ρ a $\mathcal{K}_\infty$–function. Then, the solution of the closed-loop system (20) with (8) is ISS [10].

Moreover, with (5) as CLF, control law (8) is inverse optimal in the sense that it minimizes the cost functional given as

$$V(x_k) = \sup_{d \in \mathcal{D}}\left\{\lim_{\tau \to \infty}\left[V(x_\tau) + \sum_{k=0}^{\tau}(l_d(x_k) + u_k^T \times Ru_k + \ell_d\|d_k\|)\right]\right\} \tag{28}$$

where $\mathcal{D}$ is the set of locally bounded functions and

$$l_d(x_k) := -V_d(x_k, d_k) \tag{29}$$

with $V_d(x_k, d_k)$ a negative definite function.

Proof. The Lyapunov difference for the disturbed system is defined as

$$\Delta V_d(x_k, d_k) = V(x_{k+1}) - V(x_k). \tag{30}$$

Then, adding and subtracting $V_{nom}(x_{k+1})$ in (30) result in:

$$\Delta V_d(x_k, d_k) = \underbrace{V(x_{k+1}) - V_{nom}(x_{k+1})}_{\Lambda V(x_k, d_k)} + \underbrace{V_{nom}(x_{k+1}) - V(x_k)}_{\Delta V(x_k)} \tag{31}$$

where $V_{nom}(x_{k+1})$ is defined as the Lyapunov function for the nominal system (1).

From (4) with $\sigma(x_k) = \zeta\alpha_3(\|x_k\|)$, $\zeta > 0$ and the control law (8), we obtain

$$\Delta V(x_k) = V_f(x_k) - \tfrac{1}{4}P_1^T(x_k)(R + P_2(x_k))^{-1} \times P_1(x_k) \le -\zeta\alpha_3(\|x_k\|)$$

for (31), which is ensured by determining an appropriate matrix $P = P^T > 0$. On the other hand, since $V(x_k)$ is a C^1 function in x_k for all k, then ΛV becomes

$$\begin{aligned} |\Lambda V(x_k, d_k)| &= |V(x_{k+1}) - V_{nom}(x_{k+1})| \\ &\le \ell_d\|f(x_k) + g(x_k)u_k(x_k) + d_k \\ &\quad - f(x_k) - g(x_k)u_k(x_k)\| = \ell_d\|d_k\| \\ &\le \ell_d\ell + \ell_d\alpha_4(\|x_k\|) \end{aligned}$$

where ℓ_d and ℓ are positive constants. Hence, the Lyapunov difference $\Delta V_d(x_k, d_k)$ for the disturbed system becomes

$$\begin{aligned} \Delta V_d(x_k, d_k) &= \Lambda V(x_k, d_k) + \Delta V(x_k) \\ &\le |\Lambda V(x_k, d_k)| + \Delta V(x_k) \\ &\le \ell_d\alpha_4(\|x_k\|) + \ell_d\ell + V_f(x_k) \\ &\quad - \frac{1}{4}P_1^T(x_k)(R + P_2(x_k))^{-1}P_1(x_k) \\ &\le -\zeta\alpha_3(\|x_k\|) + \ell_d\alpha_4(\|x_k\|) + \ell_d\ell \\ &= -\zeta\alpha_3(\|x_k\|) + \ell_d\alpha_4(\|x_k\|) + \ell_d\ell \\ &= -(1-\theta)\zeta\alpha_3(\|x_k\|) - \theta\zeta\alpha_3(\|x_k\|)\ell_d\alpha_4(\|x_k\|) + \ell_d\ell \\ &= -(1-\theta)\zeta\alpha_3(\|x_k\|) + \ell_d\alpha_4(\|x_k\|), \quad \forall \|x_k\| \ge \alpha_3^{-1}\left(\frac{\ell_d\ell}{\theta\zeta}\right) \end{aligned}$$

where $0 < \theta < 1$. In particular, using condition (22) in the previous expression, we obtain

$$\begin{aligned} \Delta V_d(x_k, d_k) &\le -(1-\theta)\zeta\alpha_3(\|x_k\|) + \ell_d\alpha_4(\|x_k\|) \\ &= -\eta\alpha_3(\|x_k\|) + \ell_d\delta\alpha_3(\|x_k\|) \\ &= -(\eta - \ell_d\delta)\alpha_3(\|x_k\|) \end{aligned} \tag{32}$$

which is negative definite if (27) is satisfied, where $\eta = (1-\theta)\zeta > 0$. Then, considering (5) as a radially unbounded CLF, the closed-loop system (20) and (8) is ISS, which implies bounded-input bounded-state (BIBS) stability and $\mathcal{K}$–asymptotic gain [10].

The solution of the closed-loop system (20) with (8) is ultimately bounded with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$, which results in $\gamma = \alpha_3^{-1} (\ell_d \ell / \theta \zeta) \sqrt{\lambda_{max}(P)/\lambda_{min}(P)}$, where $\lambda_{min}(P)$ denotes the minimum eigenvalue of matrix P and $\lambda_{max}(P)$ the maximum eigenvalue of matrix P.

In order to establish inverse optimality, considering that the control (8) achieves ISS for the system (20), and substituting in (28) function $l_d(x_k)$ as defined in (29) which is only valid for $\forall \| x_k \| \geq \alpha_3^{-1}(\ell_d \ell / \theta \zeta)$, it follows that

$$\begin{aligned} V(x_k) &= \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} (l_d(x_k) + u_k^T \times R u_k + \ell_d \| d_k \|) \right] \right\} \\ &= \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} (-\Lambda V(x_k, d_k) - \Delta V(x_k) + u_k^T R u_k + \ell_d \| d_k \|) \right] \right\} \\ &= \lim_{\tau \to \infty} \left[V(x_\tau) - \sum_{k=0}^{\tau} \left(V_f(x_k) - \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} P_1(x_k) \right) \right. \\ &\quad \left. + \sum_{k=0}^{\tau} u_k^T R u_k + \sup_{d \in \mathcal{D}} \left\{ \sum_{k=0}^{\tau} (\ell_d \| d_k \| - \ell_d \ell - \ell_d \delta \alpha_3(\| x_k \|)) \right\} \right]. \end{aligned} \tag{33}$$

Adding the term

$\frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} R (R + P_2(x_k))^{-1} P_1(x_k)$

to the first right hand side term of (33) and subtracting it from the second right hand side term of (33), yields

$$\begin{aligned} V(x_k) &= \lim_{\tau \to \infty} \left[V(x_\tau) - \sum_{k=0}^{\tau} (V(x_{k+1}) - V(x_k)) \right. \\ &\quad + \sum_{k=0}^{\tau} \left(u_k^T R u_k - \frac{1}{4} P_1^T(x_k)(R + P_2(x_k))^{-1} R (R + P_2(x_k))^{-1} \times, P_1(x_k) \right) \\ &\quad \left. + \sup_{d \in \mathcal{D}} \left\{ \sum_{k=0}^{\tau} (\ell_d \| d_k \| - \ell_d \ell - \ell_d \delta \alpha_3(\| x_k \|)) \right\} \right] \\ &= \lim_{\tau \to \infty} \left[V(x_\tau) - \sum_{k=0}^{\tau} (V(x_{k+1}) - V(x_k)) + \sum_{k=0}^{\tau} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)] \right. \\ &\quad \left. + \sup_{d \in \mathcal{D}} \left\{ \sum_{k=0}^{\tau} (\ell_d \| d_k \| - \ell_d \ell - \ell_d \delta \alpha_3(\| x_k \|)) \right\} \right] \\ &= \lim_{\tau \to \infty} \left[V(x_\tau) - V(x_\tau) + V(x_0) + \sum_{k=0}^{\tau} [u_k^T R u_k - \alpha^T(x_k) R \alpha(x_k)] \right. \\ &\quad \left. + \sum_{k=0}^{\tau} \left(\sup_{d \in \mathcal{D}} \{ \ell_d \| d_k \| \} - \ell_d \ell - \ell_d \delta \alpha_3(\| x_k \|) \right) \right] \end{aligned} \tag{34}$$

If $\sup_{d \in \mathcal{D}} \{ \ell_d \| d_k \| \}$ is taken as the worst case by considering the equality for (22), then

$$\sup_{d \in \mathcal{D}} \{ \ell_d \| d_k \| \} = \ell_d \sup_{d \in \mathcal{D}} \{ \| d_k \| \} = \ell_d \ell + \ell_d \delta \alpha_3(\| x_k \|). \tag{35}$$

Therefore

$$\sum_{k=0}^{\tau} \left(\sup_{d \in \mathcal{D}} \{ \ell_d \| d_k \| \} - \ell_d \ell - \ell_d \delta \alpha_3(\| x_k \|) \right) = 0. \tag{36}$$

Thus, the minimum value of (34) is reached with $u_k = \alpha(x_k)$, and the control law (8) minimizes the cost functional (28). □

Remark 1. The term $V(x_\tau)$ in (28) avoids to impose the assumption that $x_k \to 0$ as $k \to \infty$ when nonvanishing disturbances are affecting the system.

4. Examples

4.1. Stabilization

The applicability of the control scheme presented in Section 2 is illustrated by synthesizing a control law to achieve stabilization of a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (1), with

$$f(x_k) = \begin{bmatrix} x_{1,k}\, x_{2,k} - 0.8 x_{2,k} \\ x_{1,k}^2 + 1.8 x_{2,k} \end{bmatrix} \tag{37}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ -2 + \cos(x_{2,k}) \end{bmatrix}. \tag{38}$$

According to (8), the stabilizing optimal control law is formulated as

$$\begin{aligned} \alpha(x_k) &= -\tfrac{1}{2}(R + \tfrac{1}{2} g^T(x_k) P g(x_k))^{-1} \\ &\quad \times g^T(x_k) P f(x_k) \end{aligned}$$

where matrix P is determined using the PSO algorithm as described in [38], which results in

$$P = \begin{bmatrix} 0.06210 & -0.0292 \\ -0.0292 & 394.892 \end{bmatrix}$$

and R is selected as the constant term $R = 1$. The state penalty term $l(x_k)$ in (10) is calculated according to (11).

The phase portrait for this unstable open-loop ($u_k = 0$) system with initial conditions $x_0 = [2\ -2]^T$ is displayed in Fig. 1. Fig. 2 presents the time evolution of x_k for this system with initial conditions $x_0 = [2\ -2]^T$ under the action of the proposed control law. This figure also includes the applied inverse optimal control law, which achieves stability.

4.2. Robust stabilization

The proposed robust inverse optimal control presented in Section 3 is illustrated by stabilizing the inverted pendulum on a cart at the upright position [24] (see Fig. 3), which is difficult to control due to the fact that it is an underactuated system, $\vec{F}$ being the only control input.

The continuous-time dynamics of the inverted pendulum is given as [24]

$$\begin{aligned} \dot{x} &= v_x \\ \dot{v}_x &= \frac{ml\omega^2 \sin\theta - mg \sin\theta \cos\theta + \vec{F}}{M + m \sin^2\theta} \\ \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{-ml\omega^2 \sin\theta \cos\theta + (M+m) g \sin\theta}{Ml + ml \sin^2\theta} - \frac{\vec{F} \cos\theta}{Ml + ml \sin^2\theta} \end{aligned} \tag{39}$$

where x is the car position, v_x is the car velocity, θ is the pendulum angle, ω is the angular velocity, M is the mass of the car, m is the point mass attached at the end of the pendulum, l is the length of the pendulum, g is the gravity constant and $\vec{F}$ is force applied to the cart.

After discretizing by the Euler approximation,[3] the discrete-time model for the inverted pendulum on a cart is rewritten as

$$\begin{aligned} x_{k+1} &= x_k + T v_{x,k} \\ v_{x,k+1} &= v_{x,k} + \frac{T}{M + m \sin^2\theta_k} \vec{F}_k \end{aligned}$$

[3] For the ordinary differential equation $dz/dt = f(z)$, the Euler discretization is defined as $(z_{k+1} - z_k)/T = f(z_k)$, such that $z_{k+1} = z_k + Tf(z_k)$, where T is the sampling time [20,22].

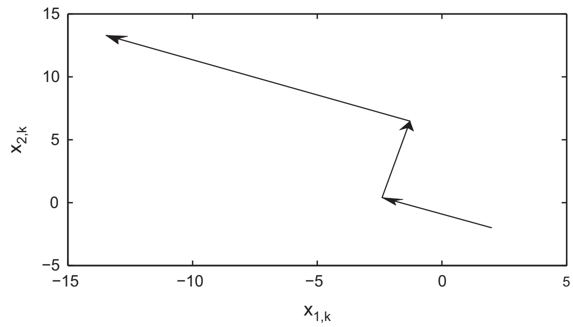

Fig. 1. Unstable phase portrait.

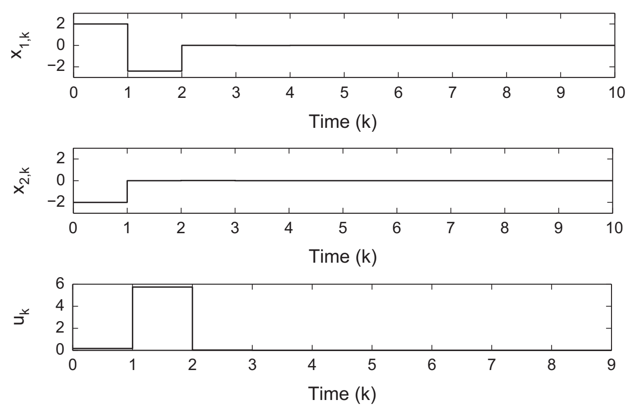

Fig. 2. Stabilization of a nonlinear system.

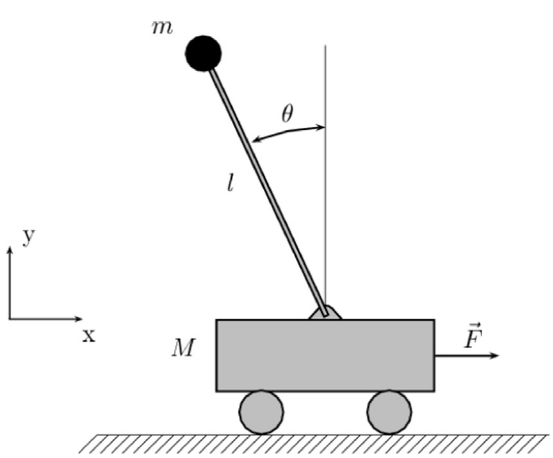

Fig. 3. Inverted pendulum on a cart.

$$+T\left(\frac{ml\omega_k^2\ \sin\ \theta_k - mg\ \sin\ \theta_k\ \cos\ \theta_k}{M+m\ \sin^2\ \theta_k}\right)$$

$$\theta_{k+1} = \theta_k + T\omega_k$$

$$\omega_{k+1} = \omega_k - \frac{T\ \cos\ \theta_k}{Ml+ml\ \sin^2\ \theta_k}\vec{F}_k - T\left(\frac{ml\omega_k^2\ \sin\ \theta_k\ \cos\ \theta_k}{Ml+ml\ \sin^2\ \theta_k}\right) + T\left(\frac{(M+m)g\ \sin\ \theta_k}{Ml+ml\ \sin^2\ \theta_k}\right) \quad (40)$$

where T is the sampling time.

Additionally, system (40) is disturbed by a vector $d_k = [0,\ d_{2,k},\ 0,\ d_{4,k}]^T$, which results in the general system presented as

$$x_{k+1} = f(x_k) + g(x_k)\vec{F}_k + d_k \quad (41)$$

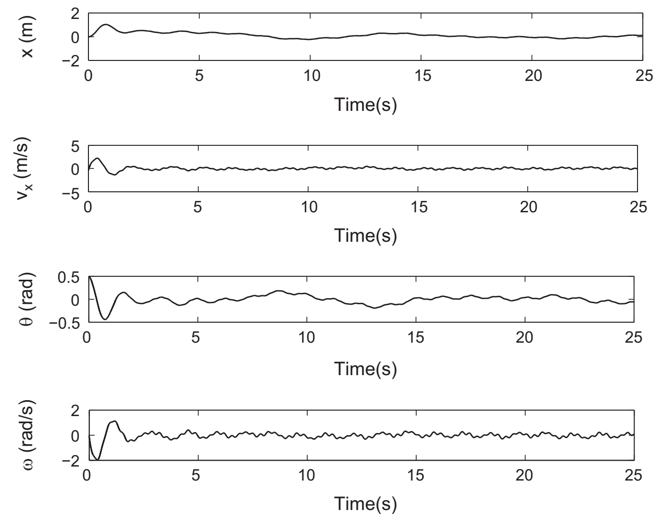

Fig. 4. Stabilized inverted pendulum time response.

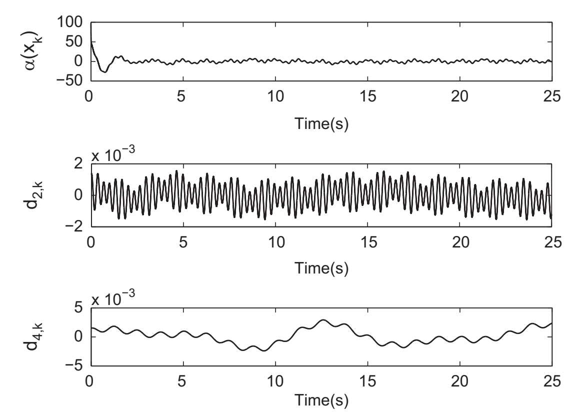

Fig. 5. Control signal and disturbance signals.

where $x_k = [x_k,\ v_{x,k},\ \theta_k,\ \omega_k]^T$ and the respective disturbance terms are

$$d_{2,k} = 0.005[0.3\ \sin(2kT) + 0.5\ \sin(\sqrt{13}kT) + 0.7\ \cos(15kT) + 1.9\ \cos(19kT)]$$

and

$$d_{4,k} = 0.005[0.2\ \cos(kT) + 0.3\ \sin(0.6kT) + 0.1\ \cos(5kT)].$$

Finally, the inverse optimal control law (8) is applied for this disturbed system as $\vec{F}_k = \alpha(x_k)$.

The parameters used for simulation are $M=3$ kg, $m=1$ kg, $l=0.5$ m, $g=9.81$ m/s^2 and the sampling time is $T=0.001$ s. The initial conditions are $[x_0,\ v_{x,0},\ \theta_0,\ \omega_0]^T = [0, -0.4, 0.5, 0]^T$.

For this simulation, matrix R in (8) is selected as $R=0.5$, whereas matrix P is determined using the PSO algorithm, which results in

$$P = \begin{bmatrix} 701501.30 & 450001.16 & 425020.23 & 301075.14 \\ 450001.16 & 701021.40 & 700101.11 & 697500.01 \\ 425020.23 & 700101.11 & 750022.10 & 725003.23 \\ 301075.14 & 697500.01 & 725003.23 & 750121.10 \end{bmatrix}$$

which is sufficient to ensure the stability at the upright position. Fig. 4 presents the time evolution of x_k, $v_{x,k}$, θ_k and ω_k. There, it can be seen that robust stabilization of the inverted pendulum is achieved. Fig. 5 displays the applied inverse optimal control law and the applied disturbance d_k.

5. Conclusions

This paper has established the inverse optimal control for discrete-time nonlinear systems. To avoid the solution of the Hamilton–Jacobi–Bellman equation, we propose a parameter-dependent CLF in a quadratic form, whose parameter is determined by using a PSO algorithm. Based on this CLF, an inverse optimal control law is synthesized. The proposed approach is extended to discrete-time disturbed nonlinear systems, which results in a robust inverse optimal control, avoiding the solution of the Hamilton–Jacobi–Isaacs equation. Simulation results illustrate that the proposed controller ensures stabilization of nonlinear systems and minimizes a cost functional.

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