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Brief paper

Global stabilization of feedforward nonlinear time-delay systems by bounded controls[☆]

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ABSTRACT

The problem of global stabilization for a family of feedforward nonlinear time-delay systems by bounded controls is considered. Based on a special canonical form of the considered nonlinear system, two types of new nonlinear control laws are proposed to achieve global stabilization. The new special canonical form used in this paper contains not only time delay in the input but also time delays in the state, which leads to natural cancellation in the recursive design. Moreover, some free parameters are introduced into these controllers. These advantages can help to simplify the proof for the global stability of the closed-loop system and improve the transient performance of the closed-loop system significantly. A practical example is given to illustrate the effectiveness of the proposed approaches.

1. Introduction

The existence of time delay can cause performance deterioration and even instability of the control system (Hale, 1977). Therefore, control of time-delay systems has been an active research topic for several decades, and many effective approaches have been established to handle various problems (see Du, Lam, and Shu, 2010; Koo and Choi, 2015, 2016; Krstic, 2010; Meng, Lam, Du, and Gao, 2010 and the references therein). On the other hand, practical control systems are subject to input saturation. Ignoring the saturation nonlinearity in controller design can degrade the system performance of the resulting closed-loop system when the saturation occurs and may even lead to instability. Thus, much research has been devoted to deal with saturation nonlinearity because of its significant influence (see Marchand & Hably, 2005; Sussmann, Sontag, & Yang, 1994; Teel, 1992; Wang, Xue, & Lu, 2015; Zhou & Duan, 2009 and the references therein). It is thus natural to consider the problem of stabilization of control systems by bounded and delayed controls, which, as we can expect, is even more difficult than the problems of stabilizing control systems by either bounded or delayed controls, and only few results are available in the literature (see Mazenc, Mondié, & Niculescu, 2003; Yakoubi & Chitour, 2007; Zhou, Duan, & Lin, 2010 and the references cited there).

Feedforward nonlinear systems, which have an upper triangular structure, are an important class of nonlinear systems (Francisco, Mazenc, & Mondié, 2007; Ye & Wang, 2007). For example, both the planar vertical takeoff and landing (PVTOL) aircraft model and the inertia wheel pendulum (IWP) model can be transformed into chains of integrators with nonlinear perturbations, which are special feedforward nonlinear systems. During the past two decades, many important stabilization results have been proposed for feedforward nonlinear systems (see Choi & Lim, 2010; Jo, Choi, & Lim, 2014; Ye, 2003; Zhang, Feng, & Sun, 2012; Zhang, Liu, Baron, & Boukas, 2011; Zhang, Liu, Feng, & Zhang, 2013 and the references therein). Motivated by Teel's forwarding design (Teel, 1992), Mazenc et al. firstly solved the global stabilization problem for the multiple integrators system by using bounded and delayed controls in Mazenc et al. (2003). Later on, these results were extended to a family of feedforward nonlinear systems with delay and saturation in the input in Mazenc, Mondié, and Francisco (2004). Also inspired by Teel's forwarding design (Teel, 1992), an adaptive stabilizer was proposed to solve the stabilization problem for feedforward nonlinear systems with time delays in Ye (2011), where the stabilizer consisted of a nested saturation feedback, and a set of switching logics were designed to tune online the saturation levels in a switching manner. Based on the transformed nonlinear system given by Mazenc et al. (2004), a new nonlinear control law consisting of cascade saturation functions was proposed in Ye, Jiang, Gui, and Yang (2012). Recently, in Ye

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(2014) a saturated and delayed controller was proposed to stabilize globally a class of feedforward nonlinear systems whose nominal dynamics is the cascade of multiple oscillators and multiple integrators. In Zhou and Yang (2016), we revisited the problem of global stabilization of multiple integrators system by using delayed and bounded controls, and three globally stabilizing nonlinear control laws were proposed based on some special canonical forms. Motivated by the results in Zhou and Yang (2016), the problem of global stabilization of the multiple oscillator system by using delayed and bounded controls was considered in Yang, Zhou, and Lam (2017), in which a nonlinear control law consisting of nested saturation functions was proposed.

In this paper, we consider the problem of bounded feedback global stabilization for a family of feedforward nonlinear time-delay systems. Motivated by our recent results in Zhou and Yang (2016), two classes of nonlinear control laws consisting of nested/cascade saturation functions are respectively proposed to solve the problem based on a new special canonical form, which contains both the current and the delayed state vectors. The main contribution of this paper and the significance of the obtained results can be stated as follows. Firstly, as pointed out in Zhou and Yang (2016), for the nominal dynamics of the considered feedforward nonlinear systems, the proposed special canonical form contains time delays in its state, which, because of the existence of time delay in the input, allows us to cancel all the other state components at every step of the recursive design so that only a scalar system decoupled from the other state components is required to be handled in every step. This is different from the design in Mazenc et al. (2004), which needs to consider a scalar time-delay system coupled with the former state components in every step of the recursive design. Moreover, the proposed controllers also contain some free parameters that can be well designed to improve the control performance. Secondly, the design approach proposed in this paper can deal with feedforward nonlinear systems whose nonlinearities contain not only the current states but also the delayed states, which are more general than the systems considered in Mazenc et al. (2004) and Ye et al. (2012). Finally, although the invertible state transformation used in this paper is similar to that in Zhou and Yang (2016), there still exist some difficulties because of the presence of the unknown nonlinearities. For example, due to the specific characteristic of the invertible transformation, the constraints imposed on the nonlinearities in the transformed system depend on the delayed state vectors (see Eq. (12)), which make the analysis more difficult than that in Zhou and Yang (2016). Moreover, compared with Zhou and Yang (2016), the recursive design in this paper is more challenging since the dynamics of the state y_i depends on the other states $y_j, j \geq i$ at the step i because of the presence of the nonlinearities, which leads to a quite complicated analysis. As a result, the resulting closed-loop system is still a nonlinear time-delay system and its stability should be verified carefully.

Notation: The notation used in this paper is fairly standard. For two integers p and q with $p \leq q$, the symbol $\mathbf{I}[p, q]$ refers to the set $\{p, p+1, \ldots, q\}$. Let $X_i = (x_i, x_{i+1}, \ldots, x_n, x_{n+1})^{\mathrm{T}}$, for any $i \in \mathbf{I}[1, n+1]$, where $x_{n+1} = u$. For a positive constant ε, $\sigma_\varepsilon(x) \triangleq \varepsilon \operatorname{sign}(x) \min\{|x/\varepsilon|, 1\}$ denotes the standard saturation function. The notation $|\cdot|$ refers to both the usual Euclidean norm for vectors and the induced 2-norm for matrices. For any two integers p and q and any functions g_i, $i \in \mathbf{I}[p, q]$, we denote $\Sigma_{i=p}^{q} g_i = 0$ if $q < p$. At last, for any constants a and b with $b \geq a$, we let $y_{[a,b]} = y(s), s \in [a, b]$ and $|y|_{[a,b]} \triangleq \sup_{s \in [a,b]} |y(s)|$.

2. Problem formulation and preliminaries

In this paper, we consider the following feedforward nonlinear system

$$\begin{cases} \dot{x}_1(t) = a_2 x_2(t - h_2) + \mathcal{L}_1\left((X_3)_{[t-h,t]}\right) + f_1, \\ \quad\vdots \\ \dot{x}_{n-1}(t) = a_n x_n(t - h_n) + \mathcal{L}_{n-1}\left((X_{n+1})_{[t-h,t]}\right) + f_{n-1}, \\ \dot{x}_n(t) = a_{n+1} x_{n+1}(t - h_{n+1}) + f_n, \end{cases} \tag{1}$$

in which $f_i = f_i((X_{i+1})_{[t-r,t]})$, $i \in \mathbf{I}[1, n]$, and $\mathcal{L}_k(\cdot)$, $k \in \mathbf{I}[1, n-1]$ are linear operators defined by

$$\mathcal{L}_k\left((X_{k+2})_{[t-h,t]}\right) = \sum_{j=k+2}^{n+1} \sum_{i=1}^{m_{kj}} a_{kji} x_j\left(t - h_{kji}\right),$$

where $n \geq 2$, $m_{kj} \geq 1$ are integers, a_i, a_{kji} are known constants satisfying $a_i \neq 0$, $\{h_i, h_{kji}\}$ are known non-negative numbers, $h = \max\{h_{kij}\}$, $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector, $x_{n+1} = u \in \mathbf{R}$ is the control input, and r is a non-negative constant that can be unknown. The functions $f_i(\cdot)$, $i \in \mathbf{I}[1, n]$ are continuous and satisfy the following assumption.

Assumption 1. There exist positive scalars ϕ_i, $i \in \mathbf{I}[1, n]$ such that

$$\left|f_i\left((X_{i+1})_{[t-r,t]}\right)\right| \leq \phi_i |X_{i+1}|^2_{[t-r,t]}, \tag{2}$$

whenever $|X_{i+1}|_{[t-r,t]} \leq 1$.

Our main work is to solve the following problem:

Problem 1. Find a state feedback control u satisfying $|u| \leq 1$ such that the closed-loop system is globally asymptotically stable and locally exponentially stable at the origin.

Remark 1. The upper bounds "1" in Assumption 1 and Problem 1 can be replaced by any given positive constant ρ. For example, if we study Problem 1 with $|u| \leq \rho$ for system (1) satisfying Assumption 1, then by the change of variable $v = u/\rho$, the system still satisfies Assumption 1 where the scalars ϕ_i are updated accordingly.

In the absence of state delay, and when $a_i = 1$ and $a_{kji} = 0$, the nonlinear system (1) becomes

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t) + f_i\left(\underline{x}_{i+1}(t)\right), \ i \in \mathbf{I}[1, n-1], \\ \dot{x}_n(t) = u(t - h_{n+1}), \end{cases}$$

and, in this case, condition (2) reduces to $|f_i(\underline{x}_{i+1}(t))| \leq M|\underline{x}_{i+1}(t)|^2$, whenever $|\underline{x}_{i+1}(t)| \leq 1$, where $\underline{x}_i = (x_i, x_{i+1}, \ldots, x_n)^{\mathrm{T}}$ and M is a given constant. For this system the corresponding Problem 1 has been solved in Mazenc et al. (2004), where a nonlinear control law consisting of nested saturation functions was proposed. The controller established in Mazenc et al. (2004) satisfies $|u| \leq \varepsilon^*$ with $\varepsilon^* = 1/(20(kh_{n+1})^n)$, where $k = \max\{16n^3[4n\sqrt{n}(1 + n^2)^{n-1} + 1], 4(20)^{n+1}n(n+2)\}$. It follows that the saturation level decreases sharply as n increases (h_{n+1} is fixed), which indicates that the actuator capacity may not be fully utilized when n is relatively large. Based on a canonical form introduced in Mazenc et al. (2004), a new nonlinear control law consisting of cascade saturation functions was proposed in Ye et al. (2012) recently. In order to guarantee the stability of the closed-loop system, the saturation level of the controller established there is required to be sufficiently low.

In this paper, motivated by our recent results in Zhou and Yang (2016) and Yang et al. (2017), based on a novel canonical form for the feedforward nonlinear system (1), two new nonlinear control laws will be proposed to solve Problem 1. Compared with the canonical form introduced in Mazenc et al. (2004), the canonical form used in this paper contains not only a time delay in the input but also time delays in the state, which are essential in the recursive design since they lead to natural cancellation. As a result, the stability of the closed-loop system is more easy to test than that in Mazenc et al. (2004). The new canonical form also allows us to handle easily nonlinearities containing also the delayed states. This situation was not considered in Mazenc et al. (2004) and Ye et al. (2012). Moreover, the proposed controllers contain some free parameters that can be designed properly to improve the control performance.

We end this section with the following lemma, which is crucial in establishing our main results.

Lemma 1 (*Zhou and Yang, 2016*). *Let $\lambda > 0, \varepsilon > 0, \varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ be four given numbers. Consider the following scalar system*

$$\dot{x}(t) = u(t) - e_2(t), \ u(t) = -\sigma_\varepsilon(\lambda x(t) + e_1(t)), \ t \geq 0, \quad (3)$$

where $e_i(\cdot) : [0, \infty) \to \mathbf{R}$ are such that $|e_i| \leq \varepsilon_i, \ i = 1, 2, \forall t \geq 0$. If

$$2\varepsilon_1 + \varepsilon_2 < \varepsilon, \quad (4)$$

then there exists a finite number $T > 0$ such that $|\lambda x(t) + e_1(t)| \leq \varepsilon, \forall t \geq T$. Moreover, the function $u(t)$ can be simplified as $u(t) = -\lambda x(t) - e_1(t), \ \forall t \geq T$.

3. Solutions to the global stabilization problem

In order to present our solutions to Problem 1, we need to present a special state space description of system (1). To this end, we define

$$A = \begin{bmatrix} 0 & \lambda & \cdots & \lambda \\ & 0 & \ddots & \vdots \\ & & \ddots & \lambda \\ & & & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbf{R}^{n\times 1}, \quad (5)$$

where λ is a priori given positive number and

$$\tau \geq h_2 + h_3 + \cdots + h_{n+1} \geq 0. \quad (6)$$

Lemma 2. *Consider the following linear time-delay system*

$$\dot{y}(t) = Ay(t-\tau) + Bu(t-\tau). \quad (7)$$

Then there exists an invertible upper triangular transformation $y(t) = \mathcal{T}(x_{[t-\gamma_1,t]})$ (its associated inverse transformation is denoted by $x(t) = \mathcal{G}(y_{[t-\gamma_2,t+\gamma_2]})$) such that system (1) with $f_i(\cdot) = 0, i \in \mathbf{I}[1, n]$ is transformed into (7). Here

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} & \cdots & \mathcal{T}_{1n} \\ & \mathcal{T}_{22} & \ddots & \vdots \\ & & \ddots & \mathcal{T}_{n-1,n} \\ & & & \mathcal{T}_{nn} \end{bmatrix}, \ \mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \cdots & \mathcal{G}_{1n} \\ & \mathcal{G}_{22} & \ddots & \vdots \\ & & \ddots & \mathcal{G}_{n-1,n} \\ & & & \mathcal{G}_{nn} \end{bmatrix}, \quad (8)$$

where $\mathcal{T}_{ij}, \mathcal{G}_{ij}, j \in \mathbf{I}[i, n], i \in \mathbf{I}[1, n]$ are linear operators defined by

$$\begin{cases} \mathcal{T}_{ii}\left((x_i)_{[t-\gamma_1,t]}\right) &= \varphi_i(\lambda) x_i(t+\tau_i), \\ \mathcal{T}_{ij}\left((x_j)_{[t-\gamma_1,t]}\right) &= \sum_{k=1}^{p_{ij}} \varphi_{ijk}(\lambda) x_j(t+\tau_{ijk}), \end{cases} \quad (9)$$

and

$$\begin{cases} \mathcal{G}_{ii}\left((y_i)_{[t-\gamma_2,t+\gamma_2]}\right) &= \psi_i(\lambda) y_i(t+\kappa_i), \\ \mathcal{G}_{ij}\left((y_j)_{[t-\gamma_2,t+\gamma_2]}\right) &= \sum_{k=1}^{q_{ij}} \psi_{ijk}(\lambda) y_j(t+\kappa_{ijk}), \end{cases} \quad (10)$$

in which $p_{ij} \geq 0, q_{ij} \geq 0$ are some integers, $\varphi_i(\lambda) \neq 0, \varphi_{ijk}(\lambda) \neq 0$ are polynomial functions of λ, $\psi_i(\lambda) \neq 0, \psi_{ijk}(\lambda) \neq 0$ are polynomial functions of $1/\lambda$, $\tau_i \leq 0, \tau_{ijk} \leq 0, \kappa_i \geq 0, \kappa_{ijk}$ are polynomial functions of $\{h_i, h_{kji}\}$, and $\gamma_1 = \max\{|\tau_i|, |\tau_{ijk}|\} \geq 0$, $\gamma_2 = \max\{|\kappa_i|, |\kappa_{ijk}|\} \geq 0$.

The proof of the above lemma will be given in Appendix. The computation of the transformation can also be found in Appendix (see (66)). With the aid of above lemma, we have the following corollary, whose proof is also given in Appendix.

Corollary 1. *By the transformation $y(t) = \mathcal{T}(x_{[t-\gamma_1,t]})$ in Lemma 2, system (1) is transformed into*

$$\dot{y}(t) = Ay(t-\tau) + Bu(t-\tau) + H\left(Y_{[t-\mu,t]}\right), \quad (11)$$

where $H(Y_{[t-\mu,t]}) \triangleq [h_1((Y_2)_{[t-\mu,t]}), \ldots, h_n((Y_{n+1})_{[t-\mu,t]})]^{\mathrm{T}}$, satisfies, for some positive constants $d(\lambda) \leq 1$ and $\delta_i(\lambda), i \in \mathbf{I}[1, n]$,

$$\left|h_i\left((Y_{i+1})_{[t-\mu,t]}\right)\right| \leq \delta_i(\lambda) |Y_{i+1}|^2_{[t-\mu,t]}, \quad (12)$$

whenever $|Y_{i+1}|_{[t-\mu,t]} \leq d(\lambda) \leq 1$, where $Y_i = (y_i, y_{i+1}, \ldots, y_n, y_{n+1})^{\mathrm{T}}, i \in \mathbf{I}[1, n+1]$ with $y_{n+1} = u$, $Y = Y_1$, and $\mu \geq 0$ is a polynomial function of $\{r, h_i, h_{kji}\}$.

It follows from (10) that, if the y-system (11) is stabilized globally by a controller $u(t) = u(y(t))$, then the x-system (1) is also stabilized globally by the same controller. Moreover, in view of $\tau_i \leq 0$ and $\tau_{ijk} \leq 0$, the controller $u(t) = u(y(t))$ is implementable since $y(t)$ involves only the current and delayed information of the state $x(t)$. Therefore, it remains to design the stabilizing controller $u(t) = u(y(t))$ for the y-system (11). For future use, we define

$$A_n = \begin{bmatrix} -1 & & & \\ -1 & -1 & & \\ \vdots & \vdots & \ddots & \\ -1 & -1 & \cdots & -1 \end{bmatrix} \in \mathbf{R}^{n\times n}. \quad (13)$$

3.1. Control laws consisting of nested saturation functions

In this subsection we present our first solution to Problem 1.

Theorem 1. *Let $\beta \in \left(0, \frac{1}{2}\right)$ be a given positive constant, and λ be a given positive constant satisfying*

$$\lambda < \min\left\{\frac{1-2\beta}{2\tau}, \frac{1}{4\tau\left|A_n^{\mathrm{T}}A_n\right|}\right\}. \quad (14)$$

Then there exists a positive constant $\varepsilon^\natural = \varepsilon^\natural(\beta, \lambda) \in (0, 1)$ such that Problem 1 is solved by the controller $u(t) = -u_n(t)$, in which

$$\begin{cases} u_i(t) &= \sigma_{\varepsilon_i}(\lambda y_i(t) + u_{i-1}(t)), \ i \in \mathbf{I}[2, n], \\ u_1(t) &= \sigma_{\varepsilon_1}(\lambda y_1(t)), \end{cases} \quad (15)$$

where $\varepsilon_i, i \in \mathbf{I}[1, n]$, are some scalars satisfying

$$\varepsilon_i = \beta^{n-i}\varepsilon, \ \forall \varepsilon \in \left(0, \varepsilon^\natural\right), \ i \in \mathbf{I}[1, n]. \quad (16)$$

Proof. For simplicity, in this proof we denote $\epsilon_k = \frac{\varepsilon_k - \varepsilon_{k-1}}{\lambda}, k \in \mathbf{I}[1, n]$ where $\varepsilon_0 = 0$. We first consider the nth subsystem of (11), namely,

$$\begin{aligned} \dot{y}_n(t) &= -u_n(t-\tau) + h_n\left((Y_{n+1})_{[t-\mu,t]}\right) \\ &= -\sigma_{\varepsilon_n}(\lambda y_n(t-\tau) + u_{n-1}(t-\tau)) \\ &\quad + h_n\left((Y_{n+1})_{[t-\mu,t]}\right), \end{aligned} \quad (17)$$

where $t \geq \mu$. Taking integration on both sides of (17) from $t-\tau$ to t gives $y_n(t) - y_n(t-\tau) = \int_{t-\tau}^{t}(-u_n(s-\tau) + h_n((Y_{n+1})_{[s-\mu,s]}))ds$, namely, for all $t \geq \tau + \mu$, $y_n(t-\tau) = y_n(t) + \int_{t-\tau}^{t}(u_n(s-\tau) - h_n((Y_{n+1})_{[s-\mu,s]}))\mathrm{d}s$, substitution of which into (17) gives

$$\dot{y}_n(t) = -\sigma_{\varepsilon_n}(\lambda y_n(t) + \upsilon_n(t)) + h_n\left((Y_{n+1})_{[t-\mu,t]}\right), \tag{18}$$

in which $\upsilon_n(t) = \lambda\int_{t-\tau}^{t}(u_n(s-\tau) - h_n((Y_{n+1})_{[s-\mu,s]}))\mathrm{d}s + u_{n-1}(t-\tau)$. It follows from (15) that, if ε is sufficiently small,

$$|u_i(t)| \leq \varepsilon_i \leq d(\lambda) \leq 1, \ \forall t \geq 0, \ i \in \mathbf{I}[1,n]. \tag{19}$$

Hence, we can get from (12) and (19) that $|h_n(Y_{n+1})_{[t-\mu,t]}| \leq \delta_n(\lambda)|Y_{n+1}|^2_{[t-\mu,t]} \leq \delta_n(\lambda)\varepsilon_n^2$, $\forall t \geq \tau+\mu$, and $|\upsilon_n(t)| \leq \lambda\tau(\varepsilon_n + \delta_n(\lambda)\varepsilon_n^2) + \varepsilon_{n-1}$. Notice that system (18) is exactly in the form of (3). Hence, if $2(\lambda\tau(\varepsilon_n + \delta_n(\lambda)\varepsilon_n^2) + \varepsilon_{n-1}) + \delta_n(\lambda)\varepsilon_n^2 < \varepsilon_n$, namely, $(2\lambda\tau+1)\delta_n(\lambda)\varepsilon < 1-(2\lambda\tau+2\beta)$, which can be guaranteed by (14) if ε is sufficiently small, by using Lemma 1, there exists a finite time $T_n \geq \tau+\mu$ such that $|\lambda y_n(t-\tau)+u_{n-1}(t-\tau)| = |\lambda y_n(t)+\upsilon_n(t)| \leq \varepsilon_n$, namely, $|y_n(t-\tau)| \leq \epsilon_n$, is satisfied for all $t \geq T_n$. Consequently, $u_n(t)$ can be simplified as $u_n(t) = \lambda y_n(t) + u_{n-1}(t)$, $\forall t \geq T_n$, and system (11) reduces to

$$\begin{cases} \dot{y}_1(t) = \sum_{j=2}^{n-1}\lambda y_j(t-\tau) - u_{n-1}(t-\tau) + h_1, \\ \quad\vdots \\ \dot{y}_{n-1}(t) = -u_{n-1}(t-\tau) + h_{n-1}, \\ \dot{y}_n(t) = -\lambda y_n(t-\tau) - u_{n-1}(t-\tau) + h_n \end{cases} \tag{20}$$

where $h_i = h_i((Y_{i+1})_{[t-\mu,t]})$, $i \in \mathbf{I}[1,n]$. Now we consider the $(n-1)$th subsystem of (20), namely,

$$\begin{aligned} \dot{y}_{n-1}(t) &= -u_{n-1}(t-\tau) + h_{n-1}\left((Y_n)_{[t-\mu,t]}\right) \\ &= -\sigma_{\varepsilon_{n-1}}(\lambda y_{n-1}(t-\tau) + u_{n-2}(t-\tau)) \\ &\quad + h_{n-1}\left((Y_n)_{[t-\mu,t]}\right), \end{aligned} \tag{21}$$

where $t \geq T_n + \mu$. Taking integration on both sides of (21) from $t-\tau$ to t gives $y_{n-1}(t) - y_{n-1}(t-\tau) = \int_{t-\tau}^{t}(-u_{n-1}(s-\tau) + h_{n-1}((Y_n)_{[s-\mu,s]}))\mathrm{d}s$, namely, for all $t \geq T_n+\tau+\mu$, $y_{n-1}(t-\tau) = y_{n-1}(t) + \int_{t-\tau}^{t}(u_{n-1}(s-\tau) - h_{n-1}((Y_n)_{[s-\mu,s]}))\mathrm{d}s$, substitution of which into (21) gives $\dot{y}_{n-1}(t) = -\sigma_{\varepsilon_{n-1}}(\lambda y_{n-1}(t) + \upsilon_{n-1}(t)) + h_{n-1}((Y_n)_{[t-\mu,t]})$, $\forall t \geq T_n+\tau+\mu$, in which $\upsilon_{n-1}(t) = \lambda\int_{t-\tau}^{t}(u_{n-1}(s-\tau) - h_{n-1}((Y_n)_{[s-\mu,s]}))\mathrm{d}s + u_{n-2}(t-\tau)$. If $\epsilon_n \leq d(\lambda)$, which can be guaranteed if ε is sufficiently small, by using (12) and (19), we have $|h_{n-1}(Y_n)_{[t-\mu,t]}| \leq \delta_{n-1}(\lambda)|Y_n|^2_{[t-\mu,t]} \leq 2\delta_{n-1}(\lambda)(\epsilon_n^2 + \varepsilon_n^2)$, where $t \geq T_n+\tau+\mu$, and $|\upsilon_{n-1}(t)| \leq \lambda\tau(\varepsilon_{n-1} + 2\delta_{n-1}(\lambda)(\epsilon_n^2+\varepsilon_n^2)) + \varepsilon_{n-2}$. Hence, if $2(\lambda\tau(\varepsilon_{n-1} + 2\delta_{n-1}(\lambda)(\epsilon_n^2+\varepsilon_n^2)) + \varepsilon_{n-2}) + 2\delta_{n-1}(\lambda)(\epsilon_n^2+\varepsilon_n^2) < \varepsilon_{n-1}$, namely, $(4\lambda\tau+2)\delta_{n-1}(\lambda)((\frac{1-\beta}{\lambda})^2 + 1)\varepsilon < \beta(1-(2\lambda\tau+2\beta))$, which can be guaranteed by (14) if ε is sufficiently small, by using Lemma 1 again, there exists a finite time $T_{n-1} \geq T_n+\tau+\mu$ such that $|\lambda y_{n-1}(t-\tau) + u_{n-2}(t-\tau)| = |\lambda y_{n-1}(t) + \upsilon_{n-1}(t)| \leq \varepsilon_{n-1}$, namely, $|y_{n-1}(t-\tau)| \leq \epsilon_{n-1}$ is satisfied for all $t \geq T_{n-1}$. Therefore, $u_{n-1}(t) = \lambda y_{n-1}(t) + u_{n-2}(t)$, $\forall t \geq T_{n-1}$. The closed-loop system can be simplified accordingly. By repeating the above procedure for $u_j(t)$, $j = n-2, n-3, \ldots, i$, where $i \in \mathbf{I}[1, n-2]$, we finally arrive at the y_i-system

$$\begin{aligned} \dot{y}_i(t) &= -u_i(t-\tau) + h_i\left((Y_{i+1})_{[t-\mu,t]}\right) \\ &= -\sigma_{\varepsilon_i}(\lambda y_i(t-\tau) + u_{i-1}(t-\tau)) \\ &\quad + h_i\left((Y_{i+1})_{[t-\mu,t]}\right). \end{aligned} \tag{22}$$

Similar to the above analysis, if there holds

$$\epsilon_j \leq d(\lambda), \ j = n, n-1, \ldots, i+1, \tag{23}$$

and $2(\lambda\tau(\varepsilon_i + 2\delta_i(\lambda)(\Sigma_{j=i+1}^{n}\epsilon_j^2 + \varepsilon_n^2)) + \varepsilon_{i-1}) + 2\delta_i(\lambda)(\Sigma_{j=i+1}^{n}\epsilon_j^2 + \varepsilon_n^2) < \varepsilon_i$, namely,

$$\begin{aligned} &(4\lambda\tau+2)\delta_i(\lambda)\left(\sum_{j=i+1}^{n}\left(\frac{\beta^{n-j}-\beta^{n-j+1}}{\lambda}\right)^2 + 1\right)\varepsilon \\ &< \beta^{n-i}(1-(2\lambda\tau+2\beta)), \end{aligned} \tag{24}$$

by using Lemma 1, there exists a finite time $T_i \geq T_{i+1}+\tau+\mu$ such that $|\lambda y_i(t-\tau) + u_{i-1}(t-\tau)| = |\lambda y_i(t) + \upsilon_i(t)| \leq \varepsilon_i$, namely, $|y_i(t-\tau)| \leq \epsilon_i$, is satisfied for all $t \geq T_i$. Therefore, $u_i(t) = \lambda y_i(t) + u_{i-1}(t)$, $\forall t \geq T_i$. It is clear from (16) and (14) that both (23) and (24) can be guaranteed if ε is sufficiently small. Hence, the closed-loop system reduces to

$$\begin{cases} \dot{y}_1(t) = \sum_{j=2}^{i}\lambda y_j(t-\tau) - u_i(t-\tau) + h_1, \\ \quad\vdots \\ \dot{y}_i(t) = -u_i(t-\tau) + h_i, \\ \dot{y}_{i+1}(t) = -\lambda y_{i+1}(t-\tau) - u_i(t-\tau) + h_{i+1}, \\ \quad\vdots \\ \dot{y}_n(t) = -\sum_{j=i+1}^{n}\lambda y_j(t-\tau) - u_i(t-\tau) + h_n. \end{cases} \tag{25}$$

By repeating the above process for the sub-controllers $u_j(t)$, $j = i-1, i-2, \ldots, 1$, we conclude that there exists a finite number $T_1 \geq T_2+\tau+\mu$ such that, for all $t \geq T_1$, there holds

$$\dot{y}(t) = \lambda A_n y(t-\tau) + H\left(Y_{[t-\mu,t]}\right). \tag{26}$$

In the following, we will analyze the stability of the closed-loop system (26). Let $P = I_n$. Then

$$\lambda A_n^{\mathrm{T}}P + \lambda PA_n + \lambda P = -\lambda BB^{\mathrm{T}}.$$

Consider the positive definite quadratic function $V(y(t)) = y^{\mathrm{T}}(t)Py(t)$. Then

$$\begin{aligned} \dot{V}(y(t)) &= -\lambda|y(t)|^2 - \lambda\left|B^{\mathrm{T}}y(t)\right|^2 \\ &\quad + 2\lambda y^{\mathrm{T}}(t)A_n(y(t-\tau) - y(t)) \\ &\quad + 2y^{\mathrm{T}}(t)H\left(Y_{[t-\mu,t]}\right) \\ &\leq -\frac{\lambda}{2}|y(t)|^2 - \lambda\left|B^{\mathrm{T}}y(t)\right|^2 \\ &\quad + 4\lambda\left|A_n^{\mathrm{T}}A_n\right||y(t-\tau) - y(t)|^2 \\ &\quad + \frac{4}{\lambda}\left|H\left(Y_{[t-\mu,t]}\right)\right|^2. \end{aligned} \tag{27}$$

On the one hand, it follows from (12) and $u = -\lambda B^{\mathrm{T}}y$ that, for all $t \geq T_1$, there holds

$$\begin{aligned} \left|H\left(Y_{[t-\mu,t]}\right)\right|^2 &= \sum_{i=1}^{n}\left|h_i\left((Y_{i+1})_{[t-\mu,t]}\right)\right|^2 \\ &\leq \sum_{i=1}^{n}\left(\delta_i(\lambda)|Y_{i+1}|^2_{[t-\mu,t]}\right)^2 \\ &\leq q_0(\lambda)|y|^4_{[t-\mu,t]}, \end{aligned} \tag{28}$$

where $q_0(\lambda) > 0$ is a constant dependent on λ. On the other hand, as

$$\begin{aligned} &|y(t-\tau) - y(t)| \\ &\leq \int_{t-\tau}^{t}\left|\lambda A_n y(s-\tau) + H\left(Y_{[s-\mu,s]}\right)\right|\mathrm{d}s, \end{aligned} \tag{29}$$

by using the Jensen inequality (Gu, 2000), we can obtain

$$\begin{aligned}&|y(t-\tau)-y(t)|^2\\ \le&\tau\int_{t-\tau}^{t}\left|\lambda A_n y(s-\tau)+H\left(Y_{[s-\mu,s]}\right)\right|^2\mathrm{d}s\\ \le&2\tau\int_{t-\tau}^{t}\left(\lambda^2\left|A_n^{\mathrm T}A_n\right||y(s-\tau)|^2+q_0(\lambda)|y|^4_{[s-\mu,s]}\right)\mathrm{d}s,\end{aligned}\tag{30}$$

where we have used (28). Substituting (28) and (30) into (27) gives

$$\begin{aligned}\dot V(y(t))\le&-\frac{\lambda}{2}|y(t)|^2-\lambda\left|b_n^{\mathrm T}y(t)\right|^2+\frac{4q_0(\lambda)}{\lambda}|y|^4_{[t-\mu,t]}\\&+8\tau\lambda\left|A_n^{\mathrm T}A_n\right|\int_{t-\tau}^{t}\left(\lambda^2\left|A_n^{\mathrm T}A_n\right||y(s-\tau)|^2\right.\\&\left.+q_0(\lambda)|y|^4_{[s-\mu,s]}\right)\mathrm{d}s.\end{aligned}\tag{31}$$

Hence, under the condition that $V(y(t+\theta))<pV(y(t))$, $\forall\theta\in[-\tau-\mu,0]$, where $t\ge T_1$ and $p>1$ is any given scalar, inequality (31) can be continued as

$$\begin{aligned}&\dot V(y(t))\\ \le&-\frac{\lambda}{2}|y(t)|^2-\lambda\left|b_n^{\mathrm T}y(t)\right|^2+8\tau^2\lambda^3p\left|A_n^{\mathrm T}A_n\right|^2|y(t)|^2\\&+\frac{4}{\lambda}q_0(\lambda)p^2|y(t)|^4+8\tau^2\lambda q_0(\lambda)p^2\left|A_n^{\mathrm T}A_n\right||y(t)|^4\\ =&-\rho(t)V(y(t))-\lambda\left|b_n^{\mathrm T}y(t)\right|^2\end{aligned}\tag{32}$$

where $\rho(t)=\frac{\lambda}{2}-8\tau^2\lambda^3p|A_n^{\mathrm T}A_n|^2-(\frac{4}{\lambda}q_0(\lambda)p^2+8\tau^2\lambda q_0(\lambda)p^2|A_n^{\mathrm T}A_n|)|y(t)|^2$. Now, notice that, for all $t\ge T_1$, there holds

$$|y(t)|^2\le\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left(\frac{\beta^{n-i}-\beta^{n-i+1}}{\lambda}\right)^2\varepsilon^2.\tag{33}$$

Thus, we can deduce from (32) and (33) that, if ε is sufficiently small and $\frac{\lambda}{2}-8\tau^2\lambda^3p|A_n^{\mathrm T}A_n|^2>0$, which can be guaranteed by (14), there exists a $\gamma>0$ such that $\rho(t)>\gamma$, $\forall t\ge T_1$. Therefore,

$$\dot V(y(t))<-\gamma V(y(t)),\ \forall t\ge T_1.\tag{34}$$

The closed-loop system (26) is thus uniformly exponentially stable by virtue of the Razumikhin stability theorem (Hale, 1977). The proof is complete. ■

Remark 2. We give some explanations on the proof of Theorem 1. First, it is easy to see that system (1) cannot diverge to infinity in finite time. Thus, for global stability analysis there is no need to make a special analysis of the first delay interval, which is however very important for local stability analysis (Fridman, 2014; Liu & Fridman, 2014). Second, we can see that the index "2" on the right hand side of inequality (2) can be replaced by any constant $q>1$. Third, inequality (2) can be replaced by

$$\left|f_i\left((X_{i+1})_{[t-r,t]}\right)\right|\le\phi_i|X_{i+1}|^2_{[t-r,t]}+\varepsilon|X_{i+1}|_{[t-r,t]},\tag{35}$$

whenever $|X_{i+1}|_{[t-r,t]}\le 1$, and $t\ge T_i^*$, where $\varepsilon>0$ can be arbitrarily small. In this case, the corresponding inequality in (12) is given by $|h_i((Y_{i+1})_{[t-\mu,t]})|\le\delta_i(\lambda)(|Y_{i+1}|^2_{[t-\mu,t]}+\varepsilon|Y_{i+1}|_{[t-\mu,t]})$, whenever $|Y_{i+1}|_{[t-\mu,t]}\le d(\lambda)\le 1$. Hence, in the first part of the proof of Theorem 1, the corresponding function $h_i((Y_{i+1})_{[t-\mu,t]})$ satisfies $|h_i((Y_{i+1})_{[t-\mu,t]})|\le\delta_i^*(\lambda)\varepsilon^2$, $\forall t\ge T_{i+1}+\tau+\mu$, for some $\delta_i^*(\lambda)>0$, namely, the corresponding function $h_i((Y_{i+1})_{[t-\mu,t]})$ is also of order 2 of ε. Thus, when ε is sufficiently small, (26) can be also obtained for all $t\ge T_1$. On the other hand, the corresponding function $H(Y_{[t-\mu,t]})$ satisfies $|H(Y_{[t-\mu,t]})|^2\le q_0(\lambda)(|y|^4_{[t-\mu,t]}+\varepsilon|y|^3_{[t-\mu,t]}+\varepsilon^2|y|^2_{[t-\mu,t]})$, for some $q_0(\lambda)>0$ (see (28)). Following the second part of the proof of Theorem 1, inequality (32) is finally obtained where $|y(t)|^2$ in $\rho(t)$ is replaced by $|y(t)|^2+\varepsilon|y(t)|+\varepsilon^2$. Hence, if ε is sufficiently small, we can also obtain (34), which ensures asymptotic stability of the closed-loop system.

3.2. Control laws consisting of cascade saturation functions

In this subsection, we present another solution to Problem 1.

Theorem 2. *Let $\beta\in(0,1)$ satisfy*

$$2\beta-\beta^n-1<0,\tag{36}$$

and λ be a given positive constant satisfying

$$\lambda<\min\left\{\frac{2\beta-\beta^n-1}{2\tau(\beta^n-1)},\frac{1}{4\tau\left|A_n^{\mathrm T}A_n\right|}\right\}.\tag{37}$$

Then there exists a positive constant $\varepsilon^\dagger=\varepsilon^\dagger(\beta,\lambda)\in(0,1)$ such that Problem 1 is solved by the controller $u(t)=-u_n(t)$, in which

$$\begin{cases}u_i(t)=\sigma_{\varepsilon_i}(\lambda y_i(t))+u_{i-1}(t),\ i\in\mathbf{I}[2,n],\\ u_1(t)=\sigma_{\varepsilon_1}(\lambda y_1(t)),\end{cases}\tag{38}$$

where ε_i, $i\in\mathbf{I}[1,n]$, are some scalars satisfying

$$\varepsilon_i=\beta^{n-i}\varepsilon,\ \forall\varepsilon\in\left(0,\varepsilon^\dagger\right),\ i\in\mathbf{I}[1,n].\tag{39}$$

Proof. For simplicity, in this proof we denote $\epsilon_j=\Sigma_{i=1}^{j}\varepsilon_i$ and $\eta_j=\Sigma_{i=1}^{j}\beta^{n-i}$, $j\in\mathbf{I}[1,n]$. We first consider the nth subsystem of (11), namely,

$$\begin{aligned}\dot y_n(t)=&-u_n(t-\tau)+h_n\left((Y_{n+1})_{[t-\mu,t]}\right)\\=&-\sigma_{\varepsilon_n}(\lambda y_n(t-\tau))-u_{n-1}(t-\tau)\\&+h_n\left((Y_{n+1})_{[t-\mu,t]}\right),\end{aligned}\tag{40}$$

where $t\ge\mu$. Similarly to the proof of Theorem 1, we get $y_n(t-\tau)=y_n(t)+\int_{t-\tau}^{t}(u_n(s-\tau)-h_n((Y_{n+1})_{[s-\mu,s]}))\mathrm{d}s$, substitution of which into (40) gives

$$\begin{aligned}\dot y_n(t)=&-\sigma_{\varepsilon_n}(\lambda y_n(t)+v_n(t))-u_{n-1}(t-\tau)\\&+h_n\left((Y_{n+1})_{[t-\mu,t]}\right),\forall t\ge\tau+\mu,\end{aligned}\tag{41}$$

where $v_n(t)=\lambda\int_{t-\tau}^{t}(u_n(s-\tau)-h_n((Y_{n+1})_{[s-\mu,s]}))\mathrm{d}s$. If ε is sufficiently small, it follows from (38) and (39) that

$$|u_i(t)|\le\epsilon_i\le d(\lambda)\le 1,\ \forall t\ge 0,\ i\in\mathbf{I}[1,n].\tag{42}$$

We then get from (12) and (42) that $|h_n((Y_{n+1})_{[t-\mu,t]})|\le\delta_n(\lambda)\epsilon_n^2$, and $|v_n(t)|\le\lambda\tau(\epsilon_n+\delta_n(\lambda)\epsilon_n^2)$. By using Lemma 1, if $2\lambda\tau(\epsilon_n+\delta_n(\lambda)\epsilon_n^2)+(\epsilon_n+\delta_n(\lambda)\epsilon_n^2)<\varepsilon_n$, namely, $(2\lambda\tau+1)\delta_n(\lambda)\eta_n^2\varepsilon<1-\eta_{n-1}-2\lambda\tau\eta_n$, which can be guaranteed by

$$\lambda<\frac{1-\eta_{n-1}}{2\tau\eta_n},\tag{43}$$

if ε is sufficiently small, there exists a finite time $T_n\ge\tau+\mu$ such that $|\lambda y_n(t-\tau)|=|\lambda y_n(t)+v_n(t)|\le\varepsilon_n$, $\forall t\ge T_n$, namely, $|y_n(t-\tau)|\le\varepsilon_n/\lambda$, $\forall t\ge T_n$, and $u_n(t)=\lambda y_n(t)+u_{n-1}(t)$, $\forall t\ge T_n$. As a result, system (11) reduces to (20), where $u_{n-1}(t)$ satisfies (38).

Now we consider the $(n-1)$th subsystem of (20), namely,

$$\begin{aligned}\dot y_{n-1}(t)=&-u_{n-1}(t-\tau)+h_{n-1}\left((Y_n)_{[t-\mu,t]}\right)\\=&-\sigma_{\varepsilon_{n-1}}(\lambda y_{n-1}(t-\tau))-u_{n-2}(t-\tau)\\&+h_{n-1}\left((Y_n)_{[t-\mu,t]}\right),\end{aligned}\tag{44}$$

where $t\ge T_n+\mu$. Similarly to the proof of Theorem 1, we obtain $y_{n-1}(t-\tau)=y_{n-1}(t)+\int_{t-\tau}^{t}(u_{n-1}(s-\tau)-h_{n-1}((Y_n)_{[s-\mu,s]}))\mathrm{d}s$, substitution of which into (44) gives

$$\dot{y}_{n-1}(t) = -\sigma_{\varepsilon_{n-1}}(\lambda y_{n-1}(t) + v_{n-1}(t)) - u_{n-2}(t-\tau) + h_{n-1}\left((Y_n)_{[t-\mu,t]}\right), \ \forall t \geq T_n + \tau + \mu,$$

where $v_{n-1}(t) = \lambda \int_{t-\tau}^{t}(u_{n-1}(s-\tau) - h_{n-1}((Y_n)_{[s-\mu,s]}))\mathrm{d}s$. If $\varepsilon_n/\lambda \leq d(\lambda)$, which can be guaranteed if ε is sufficiently small, similarly to the proof of Theorem 1, by using (12) and (42), we have $|v_{n-1}(t)| \leq \lambda\tau(\epsilon_{n-1} + 2\delta_{n-1}(\lambda)((\varepsilon_n/\lambda)^2 + \epsilon_n^2))$ and $|u_{n-2}(t-\tau) - h_{n-1}| \leq \epsilon_{n-2} + 2\delta_{n-1}(\lambda)((\varepsilon_n/\lambda)^2 + \epsilon_n^2)$. By using Lemma 1, if $2\lambda\tau(\epsilon_{n-1} + 2\delta_{n-1}(\lambda)(\varepsilon_n^2/\lambda^2+\epsilon_n^2)) + \epsilon_{n-2} + 2\delta_{n-1}(\lambda)(\varepsilon_n^2/\lambda^2+\epsilon_n^2) < \varepsilon_{n-1}$, namely, $(4\lambda\tau+2)\delta_{n-1}(\lambda)((1/\lambda)^2 + \eta_n^2)\varepsilon < \beta - \eta_{n-2} - 2\lambda\tau\eta_{n-1}$, which can be guaranteed by

$$\lambda < \frac{\beta - \eta_{n-2}}{2\tau\eta_{n-1}}, \tag{45}$$

if ε is sufficiently small, there exists a finite time $T_{n-1} \geq T_n + \tau + \mu$ such that, for all $t \geq T_{n-1}$, $|\lambda y_{n-1}(t-\tau)| = |\lambda y_{n-1}(t) + v_{n-1}(t)| \leq \varepsilon_{n-1}$, namely, $|y_{n-1}(t-\tau)| \leq \varepsilon_{n-1}/\lambda$, $\forall t \geq T_{n-1}$, and $u_{n-1}(t) = \lambda y_{n-1}(t) + u_{n-2}(t)$, $\forall t \geq T_{n-1}$. The closed-loop system can be simplified accordingly. By repeating the above procedure for $u_j(t), j = n-2, n-3, \ldots, i$, where $i \in \mathbf{I}[1, n-2]$, we arrive at the y_i-system

$$\begin{aligned}\dot{y}_i(t) &= -u_i(t-\tau) + h_i\left((Y_{i+1})_{[t-\mu,t]}\right) \\ &= -\sigma_{\varepsilon_i}(\lambda y_i(t-\tau)) - u_{i-1}(t-\tau) \\ &\quad + h_i\left((Y_{i+1})_{[t-\mu,t]}\right).\end{aligned} \tag{46}$$

Similar to the above analysis, if $\varepsilon_j/\lambda \leq d(\lambda)$, $j = n, n-1, \ldots, i+1$, which can be guaranteed if ε is sufficiently small in view of (39), and $2\lambda\tau(\epsilon_i + 2\delta_i(\lambda)(\Sigma_{j=i+1}^n(\varepsilon_j/\lambda)^2 + \epsilon_n^2)) + \epsilon_{i-1} + 2\delta_i(\lambda)(\Sigma_{j=i+1}^n(\varepsilon_j/\lambda)^2 + \epsilon_n^2) < \varepsilon_i$, namely,

$$\begin{aligned}&(4\lambda\tau+2)\,\delta_i(\lambda)\left(\sum_{j=i+1}^{n}\left(\frac{\beta^{n-j}}{\lambda}\right)^2 + \eta_n^2\right)\varepsilon \\ &<\beta^{n-i} - \eta_{i-1} - 2\lambda\tau\eta_i,\end{aligned} \tag{47}$$

then there exists a finite time $T_i \geq T_{i+1} + \tau + \mu$ such that, $|\lambda y_i(t-\tau)| = |\lambda y_i(t) + v_i(t)| \leq \varepsilon_i$, $\forall t \geq T_i$, namely, $|y_i(t-\tau)| \leq \varepsilon_i/\lambda$, $\forall t \geq T_i$, and $u_i(t) = \lambda y_i(t) + u_{i-1}(t)$, $\forall t \geq T_i$. As a result, the closed-loop system reduces to (25), where $u_{i-1}(t)$ satisfies (38). Notice that (47) can be guaranteed if ε is sufficiently small and

$$\lambda < \frac{\beta^{n-i} - \eta_{i-1}}{2\tau\eta_i}. \tag{48}$$

Similarly to the proof of Theorem 1, by repeating the above process for the controllers $u_j(t), j = i-1, i-2, \ldots, 1$, we claim that, if λ satisfies

$$\lambda < \min_{i\in\mathbf{I}[2,n]}\left\{\frac{1}{2\tau}, \frac{\beta^{n-i} - \eta_{i-1}}{2\tau\eta_i}\right\}, \tag{49}$$

there exists a finite number $T_1 \geq T_2 + \tau + \mu$ such that, for all $t \geq T_1$, the closed-loop system becomes (26). However, (49) can be guaranteed by (37). The rest is similar to the proof of Theorem 1 and is omitted. The proof is finished. ■

For arbitrary given delays $\{h_i, h_{kji}\}$, Theorems 1 and 2 provide two classes of nonlinear control laws consisting of nested and cascade saturation functions, respectively. Due to the novel canonical form introduced in Corollary 1, the linearized dynamics of the closed-loop system is of a lower triangular form, making the stability of the closed-loop system easy to test. This is different from Mazenc et al. (2004), in which the linearized dynamics of the corresponding closed-loop system is not in a triangular form, making the stability analysis rather complicated and the resulting stability conditions quite conservative. Moreover, some free parameters are also introduced into these two kinds of controllers to improve the control performance, which will be illustrated by a practical example in the next section.

4. A practical example

In this section, we will use the PVTOL aircraft control system as an example to illustrate the theory developed in this paper. The PVTOL system in the presence of time delays can be expressed as (see (6)–(8) in Zavalario, Fantoni, and Lozano, 2003)

$$\begin{cases}\ddot{x}_{\mathrm{p}} = -u_1(t-\nu_1)\sin\left(\theta_{\mathrm{p}}(t-\nu_2)\right), \\ \ddot{z}_{\mathrm{p}} = u_1(t-\nu_1)\cos\left(\theta_{\mathrm{p}}(t-\nu_2)\right) - 1, \\ \ddot{\theta}_{\mathrm{p}} = u_2(t-\nu_3),\end{cases} \tag{50}$$

where x_{p} and z_{p} denote respectively the horizontal and vertical positions, θ_{p} is the roll angle of the aircraft with the horizon, u_1 and u_2 denote respectively the thrust and angular acceleration, and $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu_3 \geq 0$ are the delays (the weak coupling coefficient in the equation was neglected since it is in general very small. See Zavalario et al., 2003). Here $(x_{\mathrm{p}}, z_{\mathrm{p}}, \theta_{\mathrm{p}})$ are state variables and (u_1, u_2) are control variables. Let $x_1 = -x_{\mathrm{p}}, x_2 = -\dot{x}_{\mathrm{p}}, x_3 = \theta_{\mathrm{p}}, x_4 = \dot{\theta}_{\mathrm{p}}, z_1 = z_{\mathrm{p}}$ and $z_2 = \dot{z}_{\mathrm{p}}$. Then (50) can be rewritten as

$$\begin{cases}\dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u_1(t-\nu_1)\sin(x_3(t-\nu_2)),\end{cases} \tag{51}$$

$$\begin{cases}\dot{x}_3(t) = x_4(t), \\ \dot{x}_4(t) = u_2(t-\nu_3),\end{cases} \tag{52}$$

$$\begin{cases}\dot{z}_1(t) = z_2(t), \\ \dot{z}_2(t) = u_1(t-\nu_1)\cos(x_3(t-\nu_2)) - 1.\end{cases} \tag{53}$$

Denote $\nu = \nu_2 + \nu_3$ and assume that $\nu \geq \nu_1$. We only illustrate the method in Theorem 1. Our design consists of the following four steps.

Step 1: Design of the sub-system (52). Since it is in the form of (1), the control law u_2 can be designed as

$$u_2(t) = -\sigma_{\varepsilon_{24}}\left(\lambda_2 y_4(t) + \sigma_{\varepsilon_{23}}\left(\lambda_2 y_3(t) + \sigma_{\varepsilon_{22}}(U_2(t))\right)\right), \tag{54}$$

in which $\lambda_2 < \min\{(1-2\beta_2)/(2\nu), 1/(4\nu|A_4^{\mathrm{T}}A_4|)\}$, $\varepsilon_{22} = \beta_2\varepsilon_{23} = \beta_2^2\varepsilon_{24}$, $\beta_2 \in (0, 1/2)$, $U_2(t)$ is to be determined, and

$$\begin{bmatrix} y_3(t) \\ y_4(t)\end{bmatrix} = \begin{bmatrix}\lambda_2 x_3(t-r_3) + x_4(t-r_4) \\ x_4(t-r_4)\end{bmatrix}, \tag{55}$$

in which $r_3 = 2\nu_2 + \nu_3$ and $r_4 = \nu_2$. For the closed-loop system consisting of (52) and (54), similar to the analysis of Theorem 1, when ε_{22} is sufficiently small, there exists a finite time $T \geq 2\nu$ such that

$$|x_3(t)| \leq \frac{\pi}{4}, \ \forall t \geq T. \tag{56}$$

Step 2: Simplification of the z-system. Similar to the treatment in Francisco et al. (2007), we set

$$u_1(t) = \frac{1 + U_1(t)}{\cos(\sigma_1(\chi_3(t-(\nu-\nu_1))))}, \tag{57}$$

where $\chi_3(t) = x_3(t) + \nu_3 x_4(t) + \int_{t-\nu_3}^{t}\int_{t-\nu_3}^{s} u_2(l)\mathrm{d}l\mathrm{d}s = x_3(t) + \nu_3 x_4(t) + \int_{t-\nu_3}^{t}(t-l)u_2(l)\mathrm{d}l$, and $U_1(t)$ is to be determined. From the second equation in (52) we have $x_4(s) = x_4(t) + \int_t^s u_2(l-\nu_3)\mathrm{d}l$, by which and the first equation in (52), we further have, for all $t \geq 2\nu_3$ (see Francisco et al., 2007),

$$x_3(t+\nu_3) = x_3(t) + \int_t^{t+\nu_3} x_4(s)\,\mathrm{d}s = \chi_3(t). \tag{58}$$

It then follows from (56), (58) and the definition of $\sigma_1(\cdot)$ that, for all $t \geq T + 2\nu_3$,

$$u_1(t-\nu_1) = \frac{1 + U_1(t-\nu_1)}{\cos(x_3(t-\nu_2))}, \tag{59}$$

which implies that, for all $t \geq T + 2\nu_3$, system (53) can be simplified as

$$\begin{cases} \dot{z}_1(t) = z_2(t), \\ \dot{z}_2(t) = U_1(t-\nu_1). \end{cases} \tag{60}$$

Step 3: Design of $U_1(t)$ for the z-system. Since (60) is in the form of (1), by Theorem 1, a globally stabilizing controller $U_1(t)$ for (60) can be designed as

$$U_1(t) = -\sigma_{\varepsilon_{12}}\left(\lambda_1 w_2(t) + \sigma_{\varepsilon_{11}}(\lambda_1 w_1(t))\right), \tag{61}$$

where $\lambda_1 < \min\{(1-2\beta_1)/2\nu_1, 1/(4\nu_1|A_2^{\mathrm{T}}A_2|)\}$, $\varepsilon_{11} = \beta_1\varepsilon_{12}$, $\beta_1 \in (0, 1/2)$, $\varepsilon_{11} \leq \varepsilon^{\natural} = \varepsilon^{\natural}(\beta_1, \lambda_1) \in (0, 1)$, and

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1(t-\nu_1) + z_2(t) \\ z_2(t) \end{bmatrix}.$$

Step 4: Complete the design of the x-system. In view of (59), for all $t \geq T + 2\nu_3$, systems (51) and (52) can be expressed as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t-\nu_2) + f_2\left((x_3)_{[t-v,t]}\right), \\ \dot{x}_3(t) = x_4(t), \\ \dot{x}_4(t) = x_5(t-\nu_3), \end{cases} \tag{62}$$

which is in the form of (1), where $x_5 = u_2$ and $f_2((x_3)_{[t-v,t]}) = x_3(t-\nu_2) - \tan(x_3(t-\nu_2)) - U_1(t-\nu_1)\tan(x_3(t-\nu_2))$. For all $x_3 \in [-1, 1]$ (which is guaranteed by (56)), we have $|\tan(x_3) - x_3| \leq 0.6x_3^2$ (Francisco et al., 2007). From Step 3, we know that $|z_1(t)|$ and $|z_2(t)|$ converge to zero. Thus, it follows from $|\tan(x_3(t))| \leq 1, \forall t \geq T$, that $|U_1(t-\nu_1)\tan(x_3(t-\nu_2))|$ converges to zero. Hence, for any $\varepsilon_0 > 0$, there exists a number $T_1 = T_1(\varepsilon_0) \geq T + 2\nu_3$ such that, for all $t \geq T_1$, $|f_2((x_3)_{[t-v,t]})| \leq \varepsilon_0|x_3(t-\nu_2)|$, which implies that $f_2((x_3)_{[t-v,t]})$ satisfies (35). Thus, in view of Theorem 1 and Remark 2, the control law u_2 can be further designed as

$$\begin{aligned} u_2(t) &= -\sigma_{\varepsilon_{24}}\left(\lambda_2 y_4(t) + \sigma_{\varepsilon_{23}}\left(\lambda_2 y_3(t) + \sigma_{\varepsilon_{22}}(U_2(t))\right)\right) \\ &= -\sigma_{\varepsilon_{24}}(\lambda_2 y_4(t) + \sigma_{\varepsilon_{23}}(\lambda_2 y_3(t) + \sigma_{\varepsilon_{22}}(\lambda_2 y_2(t) \\ &\quad + \sigma_{\varepsilon_{21}}(\lambda_2 y_1(t))))), \end{aligned} \tag{63}$$

where $\varepsilon_{21} = \beta_2\varepsilon_{22}$, and

$$\begin{cases} y_1(t) = \lambda_2^3 x_1(t-r_1)\, 3\lambda_2^2 x_2(t-r_2) \\ \qquad + 3\lambda_2 x_3(t-r_3) + x_4(t-r_4), \\ y_2(t) = \lambda_2^2 x_2(t-r_2) + 2\lambda_2 x_3(t-r_3) \\ \qquad + x_4(t-r_4), \end{cases}$$

in which $r_1 = 3v$ and $r_2 = 2v$.

In conclusion, the closed-loop system consisting of (51), (52), (53) and the controllers (57), (61) and (63) is globally uniformly asymptotically and locally exponentially stable.

Finally, we perform simulations for the closed-loop system consisting of (51), (52), (53), (57) and (63). Let $\nu_1 = 1.0$, $\nu_2 = \nu_3 = 0.5$, $\beta_1 = 0.45$, $\lambda_1 = 0.049$, $\varepsilon_1 = 0.50$, $\beta_2 = 0.45$, $\lambda_2 = 0.049$, $\varepsilon_2 = 0.40$. For a given initial condition $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0.2$, $z_1(0) = z_2(0) = 0.3$, the trajectories of the six state variables and the two control inputs are respectively plotted in Fig. 1. It follows that the states converge to the origin and the inputs are bounded, which indicates the effectiveness of the proposed methods.

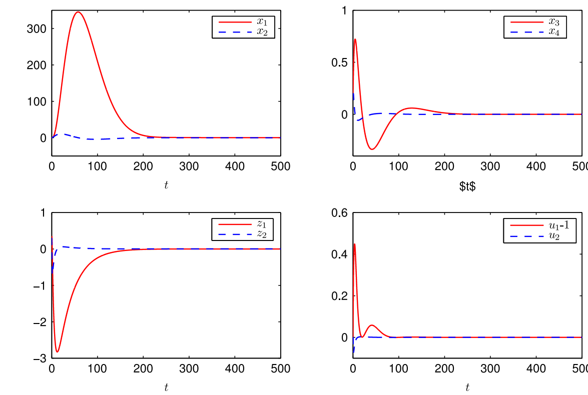

Fig. 1. The trajectories of the six state variables and the two control inputs.

5. Conclusion

This paper has considered the problem of global stabilization of a class of feedforward nonlinear time-delay systems by bounded controls. With the aid of a special canonical form of the considered system, two types of nonlinear controllers, which use not only the current states but also the delayed states, were proposed. Some free parameters were introduced into these two types of controllers to improve the control performance. The effectiveness of the developed methods was illustrated by a practical example.

Appendix

A.1. Proof of Lemma 2

Taking Laplace transformations on systems (1) with $f_i(\cdot) = 0$, $i \in \mathbf{I}[1, n]$ and system (7) to obtain

$$\begin{cases} sX(s) = A_1(s)X(s) + B_1(s)U(s), \\ sY(s) = A_2(s)Y(s) + B_2(s)U(s), \end{cases} \tag{64}$$

where $X(s)$, $Y(s)$ and $U(s)$ denote respectively the Laplace transformations of $x(t)$, $y(t)$ and $u(t)$, $A_2(s) = A\mathrm{e}^{-\tau s}$, $B_2(s) = B\mathrm{e}^{-\tau s}$, and $(A_1(s), B_1(s))$ is given by

$$A_1(s) = \begin{bmatrix} 0 & a_2\mathrm{e}^{-h_2 s} & a_{13}(s) & \ldots & a_{1n}(s) \\ & 0 & a_3\mathrm{e}^{-h_3 s} & \ldots & a_{2n}(s) \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_n\mathrm{e}^{-h_n s} \\ & & & & 0 \end{bmatrix}, \tag{65}$$

and $B_1(s) = [a_{1,n+1}(s), \ldots, a_{n-1,n+1}(s), a_{n+1}\mathrm{e}^{-h_{n+1}s}]^{\mathrm{T}}$, where $a_{kj}(s) = \Sigma_{i=1}^{m_{kj}} a_{kji}\mathrm{e}^{-h_{kji}s}, j \in \mathbf{I}[k+2, n+1], k \in \mathbf{I}[1, n-1]$. Similar to the analysis of Lemma 1 in Zhou and Yang (2016), these two systems in (64) are related by some invertible transformation $Y(s) = T(s)X(s)$ (its inverse transformation is denoted by $X(s) = G(s)Y(s)$) if and only if

$$T(s) = Q_2(s)Q_1^{-1}(s) \quad \left(G(s) = Q_1(s)Q_2^{-1}(s)\right), \tag{66}$$

where $Q_i(s) = [B_i(s), A_i(s)B_i(s), \ldots, A_i^{n-1}(s)B_i(s)]$, $i = 1, 2$. It can be shown that $Q_i(s)$, $i = 1, 2$ are triangular matrices given by

$$Q_1(s) = \begin{bmatrix} c_{11}(s) & \cdots & c_{1,n-1}(s) & c_1\mathrm{e}^{\alpha_1 s} \\ \vdots & \iddots & \iddots & \\ c_{n-1,1}(s) & c_{n-1}\mathrm{e}^{\alpha_{n-1}s} & & \\ c_n\mathrm{e}^{\alpha_n s} & & & \end{bmatrix}, \tag{67}$$

$$Q_2(s)=\begin{bmatrix} f_{11}e^{-\tau s} & \cdots & f_{1n}e^{-n\tau s} \\ \vdots & \iddots & \\ f_{n1}e^{-\tau s} & & \end{bmatrix}, \tag{68}$$

where $c_{ij}(s)=\Sigma_{k=1}^{r_{ij}}c_{ijk}e^{\alpha_{ijk}s}, j\in \mathbf{I}[1,n-i], i\in \mathbf{I}[1,n-1]$, in which $c_i\neq 0, c_{ijk}\neq 0$ are some constants, $r_{ij}\geq 0$ are some integers, $\alpha_{ijk}\leq 0$ are polynomial functions of $\{h_i,h_{kji}\}$, $\alpha_i=-\Sigma_{j=i+1}^{n+1}h_j$, and $f_{ij}=f_{ij}(\lambda)>0$ are polynomial functions of λ. Since $a_i\neq 0$ and $\lambda\neq 0$, $Q_i, i=1,2$ are all invertible and are also triangular matrices, given by

$$Q_1^{-1}(s)=\begin{bmatrix} & & & d_1e^{\beta_1 s} \\ & & d_2e^{\beta_2 s} & d_{2n}(s) \\ & \iddots & \iddots & \vdots \\ d_ne^{\beta_n s} & d_{n2}(s) & \cdots & d_{nn}(s) \end{bmatrix}, \tag{69}$$

$$Q_2^{-1}(s)=\begin{bmatrix} & & & l_{1n}e^{\tau s} \\ & & l_{2,n-1}e^{2\tau s} & l_{2n}e^{2\tau s} \\ & \iddots & \iddots & \vdots \\ l_{n1}e^{n\tau s} & l_{n2}e^{n\tau s} & \cdots & l_{nn}e^{n\tau s} \end{bmatrix}, \tag{70}$$

where $d_{ij}(s)=\Sigma_{k=1}^{g_{ij}}d_{ijk}e^{\beta_{ijk}s}, j\in \mathbf{I}[n+2-i,n], i\in \mathbf{I}[2,n]$, in which $d_i\neq 0, d_{ijk}\neq 0$ are some constants, $g_{ij}\geq 0$ are some integers, β_{ijk} are polynomial functions of $\{h_i,h_{kji}\}$, $\beta_i=\Sigma_{j=n+2-i}^{n+1}h_j$, and $l_{ij}=l_{ij}(\lambda)\neq 0$ are polynomial functions of $1/\lambda$. From (67)–(70) we have

$$Q_2(s)Q_1^{-1}(s)=\begin{bmatrix} \varphi_1e^{\tau_1 s} & \varphi_{12}(s) & \cdots & \varphi_{1n}(s) \\ & \varphi_2e^{\tau_2 s} & \ddots & \vdots \\ & & \ddots & \varphi_{n-1,n}(s) \\ & & & \varphi_ne^{\tau_n s} \end{bmatrix}, \tag{71}$$

where $\varphi_{ij}(s)=\varphi_{ij}(\lambda,s)=\Sigma_{k=1}^{p_{ij}}\varphi_{ijk}(\lambda)e^{\tau_{ijk}s}, j\in[i+1,n], i\in \mathbf{I}[1,n-1]$, $p_{ij}\geq 0$ are some integers, τ_{ijk} are polynomial functions of $\{h_i,h_{kji}\}$, $\varphi_i=\varphi_i(\lambda)\neq 0, \varphi_{ijk}(\lambda)\neq 0$ are polynomial functions of λ, and $\tau_i=-(n+1-i)\tau+\Sigma_{j=i+1}^{n+1}h_j\leq 0, i\in \mathbf{I}[1,n]$. Similarly,

$$Q_1(s)Q_2^{-1}(s)=\begin{bmatrix} \psi_1e^{\kappa_1 s} & \psi_{12}(s) & \cdots & \psi_{1n}(s) \\ & \psi_2e^{\kappa_2 s} & \ddots & \vdots \\ & & \ddots & \psi_{n-1,n}(s) \\ & & & \psi_ne^{\kappa_n s} \end{bmatrix}, \tag{72}$$

where $\psi_{ij}(s)=\psi_{ij}(\lambda,s)=\Sigma_{k=1}^{q_{ij}}\psi_{ijk}(\lambda)e^{\kappa_{ijk}s}, j\in[i+1,n], i\in \mathbf{I}[1,n-1]$, $q_{ij}\geq 0$ are some integers, κ_{ijk} are polynomial functions of $\{h_i,h_{kji}\}$, $\psi_i=\psi_i(\lambda)\neq 0, \psi_{ijk}(\lambda)\neq 0$ are polynomial functions of $1/\lambda$, and $\kappa_i=(n+1-i)\tau-\Sigma_{j=i+1}^{n+1}h_j\geq 0, i\in \mathbf{I}[1,n]$. It follows from (71) and (72) that the transformations in (66) are well defined and can be exactly expressed as $y(t)=\mathcal{T}(x_{[t-\gamma_1,t]})$ and $x(t)=\mathcal{G}(y_{[t-\gamma_2,t+\gamma_2]})$ in (8)–(10) in the time domain. Hence it remains to show that τ_{ijk} in (71) are non-positive.

For any function $c_{ij}(s)=\Sigma_{k=1}^{r_{ij}}c_{ijk}e^{\alpha_{ijk}s}$, denote $\Delta\left(c_{ij}(s)\right)=\max_{k\in\mathbf{I}[1,r_{ij}]}\left\{\alpha_{ijk}\right\}\triangleq\alpha_{ij}$. Then

$$\Delta(Q_1(s))=\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1,n-1} & \alpha_1 \\ \vdots & \iddots & \iddots & \\ \alpha_{n-1,1} & \alpha_{n-2} & & \\ \alpha_n & & & \end{bmatrix},$$

$$\Delta\left(Q_2^{-1}(s)\right)=\begin{bmatrix} & & & \tau \\ & & 2\tau & 2\tau \\ & \iddots & \iddots & \vdots \\ n\tau & n\tau & \cdots & n\tau \end{bmatrix},$$

and

$$\Delta(T(s))=\begin{bmatrix} \tau_1 & \tau_{12} & \cdots & \tau_{1n} \\ & \tau_2 & \ddots & \vdots \\ & & \ddots & \tau_{n-1,n} \\ & & & \tau_n \end{bmatrix},$$

$$\Delta(G(s))=\begin{bmatrix} \kappa_1 & \kappa_{12} & \cdots & \kappa_{1n} \\ & \kappa_2 & \ddots & \vdots \\ & & \ddots & \kappa_{n-1,n} \\ & & & \kappa_n \end{bmatrix}, \tag{73}$$

where $\tau_{ij}=\Delta(\varphi_{ij}(s))$ and $\kappa_{ij}=\Delta(\psi_{ij}(s))$. For any real matrices $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$, define $A\boxtimes B=(\max_{k\in\mathbf{I}[1,n]}(a_{ik}+b_{kj}))_{n\times n}$. Then, for any $n\times n$ matrices $P(s)$ and $Q(s)$ whose elements are in the form of $c_{ij}(s)$, it is easy to verify that

$$\Delta(P(s)Q(s))\preceq\Delta(P(s))\boxtimes\Delta(Q(s)),$$

where $\preceq$ is defined in an element-wise manner. Thus

$$\Delta(G(s))=\Delta\left(Q_1(s)Q_2^{-1}(s)\right)\preceq\Delta(Q_1(s))\boxtimes\Delta\left(Q_2^{-1}(s)\right). \tag{74}$$

Since $\alpha_{ijk}\leq 0$ and $\alpha_i=-\Sigma_{j=i+1}^{n+1}h_j$, we can compute

$$\Delta(Q_1(s))\boxtimes\Delta\left(Q_2^{-1}(s)\right)=\begin{bmatrix} \kappa_1 & \kappa_1 & \cdots & \kappa_1 \\ & \kappa_2 & \cdots & \kappa_2 \\ & & \ddots & \vdots \\ & & & \kappa_n \end{bmatrix}. \tag{75}$$

Then it follows from (73), (74) and (75) that $\kappa_{ij}\leq\kappa_i, j\in\mathbf{I}[i+1,n], i\in\mathbf{I}[1,n-1]$, which implies

$$\kappa_{ijk}\leq\kappa_i,\ k\in\mathbf{I}\left[1,q_{ij}\right], j\in\mathbf{I}[i+1,n], i\in\mathbf{I}[1,n-1]. \tag{76}$$

Furthermore, we can rewrite (72) as $G(s)=Q_1(s)Q_2^{-1}(s)=\mathrm{diag}\{\psi_1e^{\kappa_1 s},\psi_2e^{\kappa_2 s},\ldots,\psi_ne^{\kappa_n s}\}G_0(s)$ with

$$G_0(s)=\begin{bmatrix} 1 & \psi_{12}^+(s) & \cdots & \psi_{1n}^+(s) \\ & 1 & \ddots & \vdots \\ & & \ddots & \psi_{n-1,n}^+(s) \\ & & & 1 \end{bmatrix}, \tag{77}$$

where $\psi_{ij}^+(s)=\psi_{ij}^+(\lambda,s)=\Sigma_{k=1}^{q_{ij}}\psi_{ijk}^+(\lambda)e^{\kappa_{ijk}^+s}, j\in[i+1,n], i\in\mathbf{I}[1,n-1]$, in which $\psi_{ijk}^+(\lambda)=\psi_{ijk}(\lambda)/\psi_i\neq 0$ and $\kappa_{ijk}^+=\kappa_{ijk}-\kappa_i\leq 0$ (see (76)). It follows that

$$G_0^{-1}(s)=\begin{bmatrix} 1 & \psi_{12}^-(s) & \cdots & \psi_{1n}^-(s) \\ & 1 & \ddots & \vdots \\ & & \ddots & \psi_{n-1,n}^-(s) \\ & & & 1 \end{bmatrix},$$

in which $\psi_{ij}^-(s)=\psi_{ij}^-(\lambda,s)=\Sigma_{k=1}^{v_{ij}}\psi_{ijk}^-(\lambda)e^{\kappa_{ijk}^-s}, j\in[i+1,n], i\in\mathbf{I}[1,n-1]$, $v_{ij}\geq 0$ are some integers, κ_{ijk}^- are polynomial functions of $\{h_i,h_{kji}\}$, $\psi_{ijk}^-(\lambda)\neq 0$ are polynomial functions of $1/\lambda$. Since $\psi_{ij}^-(s)$ are functions of $\psi_{ij}^+(s)$ via multiplication, addition, and subtraction, it follows from $\kappa_{ijk}^+\leq 0$ that $\kappa_{ijk}^-\leq 0$. Then, as $T(s)=G^{-1}(s)=G_0^{-1}(s)\mathrm{diag}\{\frac{e^{-\kappa_1 s}}{\psi_1},\frac{e^{-\kappa_2 s}}{\psi_2},\ldots,\frac{e^{-\kappa_n s}}{\psi_n}\}$, we get

$$\Delta(T(s))\preceq\begin{bmatrix} -\kappa_1 & -\kappa_2 & \cdots & -\kappa_n \\ & -\kappa_2 & \ddots & \vdots \\ & & \ddots & -\kappa_n \\ & & & -\kappa_n \end{bmatrix}=\begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \\ & \tau_2 & \ddots & \vdots \\ & & \ddots & \tau_n \\ & & & \tau_n \end{bmatrix}, \tag{78}$$

where we have noticed $\tau_i = -\kappa_i = -(n+1-i)\tau + \Sigma_{j=i+1}^{n+1} h_j \le 0$. Then it follows from (73) and (78) that $\tau_{ij} \le \tau_j \le 0, i \in \mathbf{I}[1, j-1], j \in \mathbf{I}[2, n]$, which implies

$$\tau_{ijk} \le \tau_j \le 0, k \in \mathbf{I}\left[1, p_{ij}\right], i \in \mathbf{I}\left[1, j-1\right], j \in \mathbf{I}\left[2, n\right]. \tag{79}$$

The proof is finished.

A.2. Proof of Corollary 1

By Lemma 2, system (1) is transformed into (11) by $y(t) = \mathcal{T}(x_{[t-\gamma_1,t]})$, where $H(Y_{[t-\mu,t]}) = \mathcal{T}(f(\cdot))$ with $f(\cdot) = [f_1((X_2)_{[t-r,t]}), \ldots, f_n((X_{n+1})_{[t-r,t]})]^{\mathrm{T}}$. By virtue of the structure of $\mathcal{T}$ in (8) and (9), we have, for $i \in \mathbf{I}[1, n]$,

$$\begin{aligned} & h_i\left((Y_{i+1})_{[t-\mu,t]}\right) \\ =& \varphi_i(\lambda) f_i\left((X_{i+1})_{[t-r+\tau_i,t+\tau_i]}\right) \\ &+ \sum_{j=i+1}^{n}\sum_{k=1}^{p_{ij}} \varphi_{ijk}(\lambda) f_j\left(\left(X_{j+1}\right)_{[t-r+\tau_{ijk},t+\tau_{ijk}]}\right). \end{aligned} \tag{80}$$

In the following, we will show (12). By the inverse transformation $x(t) = \mathcal{G}(y_{[t-\gamma_2,t+\gamma_2]})$ in (8) and (10), we find that, for any $i \in \mathbf{I}[1, n]$, there exists a positive constant $d(\lambda) \le 1$ such that $|Y_{i+1}|_{[t-\mu+\kappa_i,t+\kappa_i]} \le d(\lambda) \Rightarrow |X_{i+1}|_{[t-r-\gamma_1-\tau_i,t]} \le 1$, where $\mu \ge 0$ is a (sufficiently large) number that is a polynomial function of $\{r, h_i, h_{kji}\}$. Thus

$$\begin{aligned} & |Y_{i+1}|_{[t-\mu,t]} = |Y_{i+1}|_{[t+\kappa_i+\tau_i-\mu,t+\kappa_i+\tau_i]} \le d(\lambda) \\ \Rightarrow & |X_{i+1}|_{[t-r-\gamma_1,t+\tau_i]} \le 1, \forall i \in \mathbf{I}[1, n]. \end{aligned}$$

Hence, for any $i \in \mathbf{I}[1, n]$, it follows from (80), (2) and (79) that there exist positive scalars $\delta_i(\lambda)$, such that, whenever, $|Y_{i+1}|_{[t-\mu,t]} \le d(\lambda)$,

$$\begin{aligned} & \left|h_i\left((Y_{i+1})_{[t-\mu,t]}\right)\right| \\ \le & |\varphi_i(\lambda)|\, \phi_i |X_{i+1}|^2_{[t-r+\tau_i,t+\tau_i]} \\ & + \sum_{j=i+1}^{n}\sum_{k=1}^{p_{ij}} \left|\varphi_{ijk}(\lambda)\right| \phi_j \left|X_{j+1}\right|^2_{[t-r+\tau_{ijk},t+\tau_{ijk}]} \\ \le & |\varphi_i(\lambda)|\, \phi_i |X_{i+1}|^2_{[t-r-\gamma_1,t+\tau_i]} \\ & + \sum_{j=i+1}^{n}\left(\sum_{k=1}^{p_{ij}} \left|\varphi_{ijk}(\lambda)\right|\right) \phi_j \left|X_{j+1}\right|^2_{[t-r-\gamma_1,t+\tau_j]} \\ \le & \delta_i^*(\lambda) \sum_{j=i}^{n} \left|X_{j+1}\right|^2_{[t-r-\gamma_1,t+\tau_j]}. \end{aligned} \tag{81}$$

On the other hand, by (76) we get

$$\Delta(G(s)) \preceq \begin{bmatrix} \kappa_1 & \kappa_1 & \cdots & \kappa_1 \\ & \kappa_2 & \ddots & \vdots \\ & & \ddots & \kappa_{n-1} \\ & & & \kappa_n \end{bmatrix},$$

where $\kappa_j \ge \kappa_{j+1}, j \in \mathbf{I}[1, n-1]$. Since $x(t) = \mathcal{G}(y_{[t-\gamma_2,t+\gamma_2]})$ and $\mathcal{G}$ possesses the triangular structure (8), it yields

$$\begin{aligned} \left|X_{j+1}(t)\right| \le & \delta_i^{\ddagger}(\lambda) \left|Y_{j+1}\right|_{[t-\gamma_2,t+\kappa_{j+1}]} \\ \le & \delta_i^{\ddagger}(\lambda) \left|Y_{j+1}\right|_{[t-\gamma_2,t+\kappa_j]}, j \in \mathbf{I}[1, n]. \end{aligned}$$

Inserting these inequalities into (81) gives

$$\begin{aligned} \left|h_i\left((Y_{i+1})_{[t-\mu,t]}\right)\right| \le & \delta_i^{\dagger}(\lambda) \sum_{j=i}^{n} \left|Y_{j+1}\right|^2_{[t-r-\gamma_1-\gamma_2,t+\tau_j+\kappa_j]} \\ \le & \delta_i(\lambda) \left|Y_{j+1}\right|^2_{[t-\mu,t]}. \end{aligned}$$

This completes the proof.

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