25

Multimodel Discrete Second Order Sliding Mode Control: Stability Analysis and Real Time Application on a Chemical Reactor

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1. Introduction

The variable structure control is principally characterized by its robustness with respect to the system's modeling uncertainties and external disturbances (Decarlo et al. (1988); Filippov (1960); Lopez & Nouri (2006); Utkin (1992)). Sliding Mode Systems are a particular case of the Variable Structure Systems (VSS). They are feedback systems with discontinuous gains switching the system' s structure according to the state evolution, in order to maintain the trajectory within some specified subspace called the sliding surface (Utkin (1992)). However, the application of this control law is confronted to a serious problem. In fact, sliding mode necessitates an infinite switching frequency which is impossible to realize in numerical applications because of the calculation time and of the sensors dynamics that can not be neglected. The discontinuous control generates in that case oscillations on the state and on the switching function (Utkin (1992)). Owing to the many advantages of the digital control strategy (Ben Abdennour et al. (2001)), the discretization of the sliding mode control (SMC) has become an interesting research field. Unfortunately, the chattering phenomenon is more obvious in this case, because the sampling rate is more reduced.

Many approaches have been suggested in order to resolve this last problem. Most of them propose a reduction in the oscillation amplitude at cost of the robustness of the control law (Utkin et al. (1999)). In the eighties, a new control technique, called high order sliding mode control, have been investigated. Its main idea is to reduce to zero, not only the sliding function, but also its high order derivatives. In the case of the r-order sliding mode control, the discontinuity is applied on the (r-1) derivative of the control. The effective control is obtained by (r-1) integrations and can, then, be considered as a continuous signal. In other words, the oscillations generated by the discontinuous control are transferred to the higher derivatives of the sliding function. This approach permits to reduce the oscillations amplitude, the notorious sliding mode systems robustness remaining intact (Levant (1993)).

Another problem of the SMC is its vulnerability to external disturbances, parametric variations and non linearity, essentially, during the reaching phase. A solution to this problem, based on the multimodel approach, was proposed by the authors in (Mihoub et al. (2009a)).

The combination of the multimodel approach and the second order discrete sliding mode control (2-DSMC) allows resolving both the chattering problem and the vulnerability during the reaching phase. A stability analysis of the multimodel discrete second order sliding mode control (MM-2-DSMC) is proposed in this work. The performances offered by the second order approach and by the multimodel approach are illustrated by a comparison between the experimental results on a chemical reactor of the first order DSMC, the 2-DSMC and the MM-2-DSMC (Mihoub et al. (2009a;b)).

2. Discrete second order sliding mode control

2.1 High order sliding mode approach

The high order sliding mode control concept have been introduced in the eighties at the aim of resolving the chattering phenomenon. Levantovsky (Levantovsky (1985)) and Emelyanov (Emelyanov et al. (1986)) proposed to transfer it on the higher derivatives of the control law. Therefore, the system's input becomes continuous.
Let's consider the non linear system defined by:

$$\dot{x} = f(t, x, u) \tag{1}$$

where :

- $x(t) = [x_1(t), ..., x_n(t)]^T \in X$ state vector, $X \subset R^n$.
- $u(t, x)$ is the control.
- $f(t, x, u)$ is a function supposed sufficiently differentiable.

We denote by $S(t, x)$ the sliding function. It is a differentiable function with its $(r-1)$ first derivatives relatively to the time depending only on the state $x(t)$ (that means they contain no discontinuities).
Definition **(Salgado (2004))**
A sliding mode is said "first order sliding mode" if and only if $S(t, x) = 0$ and $S(t, x)\dot{S}(t, x) < 0$
A sliding mode is said "r^{th} order sliding mode" if and only if:

$$S(t, x) = \dot{S}(t, x) = ... = S^{(r-1)}(t, x) = 0 \tag{2}$$

The aim of first order sliding mode control is to force the state to move on the switching surface $S(t, x) = 0$. In high order sliding mode control, the purpose is to force the state to move on the switching surface $S(t, x) = 0$ and to keep its $(r-1)$ first successive derivatives null (Salgado (2004)).
In the case of second order sliding mode control, we must verify:

$$S(t, x) = \dot{S}(t, x) = 0 \tag{3}$$

We introduce here the equivalent control approach for second order sliding mode control (Salgado (2004)).
The derivative of the sliding function is:

$$\frac{d}{dt}S(t, x) = \frac{\partial}{\partial t}S(t, x) + \frac{\partial}{\partial x}S(t, x)\frac{\partial x}{\partial t} \tag{4}$$

Considering the relation (1), we can write:

$$\dot{S}(t,x,u) = \frac{\partial}{\partial t}S(t,x) + \frac{\partial}{\partial x}S(t,x)f(t,x,u) \tag{5}$$

The second order derivative of $S(t,x)$ is:

$$\tfrac{d}{dt}\dot{S}(t,x,u) = \tfrac{\partial}{\partial t}\dot{S}(t,x,u) + \tfrac{\partial}{\partial x}\dot{S}(t,x,u)\tfrac{\partial x}{\partial t} + \tfrac{\partial}{\partial u}\dot{S}(t,x,u)\tfrac{\partial u}{\partial t} \tag{6}$$

This last equation can be written as follow:

$$\frac{d}{dt}\dot{S}(t,x) = \theta(t,x) + \varsigma(t,x)\dot{u}(t) \tag{7}$$

with:

$$\begin{array}{l} \theta(t,x) = \frac{\partial}{\partial t}\dot{S}(t,x,u) + \frac{\partial}{\partial x}\dot{S}(t,x,u)f(t,x,u) \\ \varsigma(t,x) = \frac{\partial}{\partial u}\dot{S}(t,x,u) \end{array}. \tag{8}$$

Let's consider now the new system whose state variables are the sliding function $S(t,x)$ and its derivative $\dot{S}(t,x)$:

$$\begin{cases} y_1(t,x) = S(t,x) \\ y_2(t,x) = \dot{S}(t,x) \end{cases} \tag{9}$$

By using the equations (8) and (9), we can write:

$$\begin{cases} \dot{y}_1(t,x) = y_2(t,x) \\ \dot{y}_2(t,x) = \theta(t,x) + \varsigma(t,x)\dot{u}(t) \end{cases} \tag{10}$$

The system described by (10) is a second order system. For this new system a new sliding function can be proposed:

$$\begin{aligned} \sigma(t,x) &= y_2(t,x) + \alpha y_1(t,x) \\ &= \dot{S}(t,x) + \alpha S(t,x) \end{aligned} \tag{11}$$

The system whose input is $\dot{u}(t)$ and output $\sigma(t,x)$ has got a relative order equal to one and a sliding mode can be involved on $\sigma(t,x) = 0$ (Sira-Ramirez (1988)). The correspondent control law can be of the form:

$$\dot{u}(t) = \dot{u}_{eq(t)} - Msign(\sigma(t,x)) \tag{12}$$

The term $\dot{u}_{eq}(t)$ is deduced from:

$$\dot{\sigma}(t,x) = \dot{y}_2(t,x) + \alpha\dot{y}_1(t,x) = \ddot{S}(t,x) + \alpha\dot{S}(t,x) = 0 \tag{13}$$

with: $\ddot{S}(t,x) = C^T\ddot{x}(t)$.

The vector $\ddot{x}(t)$ can be deduced from the considered system:

$$\ddot{x}(t) = \frac{\partial}{\partial t}f(t,x,u) + \frac{\partial}{\partial x}f(t,x,u)\dot{x}(t) + \frac{\partial}{\partial u}f(t,x,u)\dot{u}(t) \tag{14}$$

The equivalent control for the new system is, then, written:

$$\dot{u}_{eq}(t) = -\frac{1}{C^T \frac{\partial}{\partial u} f(t,x,u)} \Big(C^T \frac{\partial}{\partial} f(t,x,u) + C^T \frac{\partial}{\partial x} f(t,x,u)\dot{x}(t) + \alpha \dot{S}(t,x) \Big) \tag{15}$$

The control input for the new system is:

$$\dot{u}(t) = \dot{u}_{eq}(t) + u_{dis}(t) \tag{16}$$

with $\quad u_{dis}(t) = -\, M\, sign\,(\sigma(t,x))$

The effective control to apply to the system (1) is obtained by integration:

$$u(t) = \int \dot{u}_{eq}(t)dt - \int u_{dis}(t)dt \tag{17}$$

If we consider a system whose output is the sliding function $S(t,x)$ with a relative order equal to one, the control algorithm, described above, is convergent if there exist positive constants Γ_m, Γ_M, Φ and s_0 such that, in a neighborhood $|S(t,x)| \leq s_0$, the following conditions are verified(Salgado (2004)):

$$\begin{aligned} &0 < \Gamma_m \leq \varsigma(t,x) \leq \Gamma_M \\ &|\theta(t,x)| \leq \Phi \end{aligned} \tag{18}$$

This approach requires the knowledge of a model of the system, and guaranties an asymptotic convergence of the sliding function to zero according to a desired dynamic.

2.2 Discrete second order sliding mode approach

Let's consider the following system :

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Hx(k) \end{aligned} \tag{19}$$

The sliding function relative to this system is taken in this linear form:

$$S(k) = C^T(x(k) - x_d(k)) \tag{20}$$

with $x_d(k)$ is the desired state vector and C is the sliding function's parameters' vector. A discrete first order sliding mode control can be given by the following expression (Gao et al. (1995)):

$$u(k) = (C^T B)^{-1}[\varphi\, S(k) - C^T Ax(k) - M\text{sign}(S(k))] \tag{21}$$

In order to develop a second order sliding mode controller, a fictive system whose state variables are $S(k+1)$ and $S(k)$ is considered. The new sliding function $\sigma(k)$ is defined by:

$$\sigma(k) = S(k+1) + \beta S(k) \tag{22}$$

with:

$$\begin{aligned} S(k+1) &= C^T(x(k+1) - x_d(k+1)) \\ &= C^T\,(Ax(k) + Bu(k) - x_d(k+1)) \end{aligned} \tag{23}$$

We note that β is chosen in the interval [0, 1[, in order to ensure the convergence of $\sigma(k)$.

By analogy with the case of the first order discrete sliding mode control law (1-DSMC), the equivalent control that forces the system to evolute on the sliding function is deduced from :

$$\sigma(k+1) = \sigma(k) = 0 \tag{24}$$

The equations (22), (23) and (24) give:

$$S(k+1) + \beta S(k) = 0 \tag{25}$$

and

$$\begin{aligned} S(k+1) &= \sigma(k+1) - \beta S(k) \\ &= C^T(x(k+1) - x_d(k+1)) \\ &= C^T\left(Ax(k) + Bu_{eq}(k) - x_d(k+1)\right) \end{aligned} \tag{26}$$

Then:

$$u_{eq}(k) = (C^T B)^{-1}[-\beta\, S(k) - C^T Ax(k) + C^T(x_d(k+1)] \tag{27}$$

The robustness is ensured by the addition of a discontinuous term (sign of the new sliding function $\sigma(k)$). By analogy with the continuous-time case, we apply to the system (19) the integral of the discontinuous term which will be approximated by a first order transformation.

$$u_{dis}(k) = u_{dis}(k-1) - T_e M\, sign\,(\sigma(k)) \tag{28}$$

The control at the instant k is then (Mihoub et al. (2009b)):

$$u(k) = u_{eq}(k) + u_{dis}(k) \tag{29}$$

The integration of the discontinuous term of the control allows its use in the case of many applications where actuators can be damaged by the discontinuity of the 1-DSMC (gates, motoring...). However, this approach does not ameliorate the robustness of the system during the reaching phase (Mihoub et al. (2008)). To resolve this problem, the multimodel approach is exploited in the following paragraph.

3. A multimodel for the 2-DSMC

3.1 Multimodel approach

Instead of exploiting one global model of the system for the equivalent control calculation, the multimodel approach suggests the use of some partial models that express the process dynamics. Two problems must be resolved in this case: the construction of the partial models and the choice of the right one at the right time (Ltaief et al. (2003a;b; 2004); Mihoub et al. (2008; 2009a); Talmoudi et al. (2002a;b; 2003)). If the final model is built by the fusion technique, we must, of course, compute partial models validities.

3.1.1 Construction of the partial models

Some approaches have been proposed for the systematic determination of a generic models base. In (Lahmari (1999)), Ksouri L. proposed a models' base based on the Kharitonov's algebric approach. Four extreme models and a medium one can be exploited by the multimodel strategy. Ben Abdennour et al. (Ltaief et al. (2003a;b; 2004); Talmoudi et al.

(2002a;b; 2003)) have proposed two contributions for the systematic determination of the models'base. The first is based on the Chiu's approach for fuzzy classification (Chiu (1994)) and the second exploites the classification strategy based on the Kohenen's Neural Network.

3.1.2 The validities computing

The validities estimation can be insured, classically, by the residue approach:

$$v_i(k) = \frac{1 - \frac{r_i(k)}{\sum\limits_{c=1}^{md} r_c(k)}}{md - 1}, \quad i \in [1, md] \tag{30}$$

$$r_i(k) = |y(k) - y_i(k)| \tag{31}$$

with $y(k)$ is the system's output, $y_i(k)$ is the output of the i^{th} model and md is the models number.
In order to reduce the perturbation phenomenon due to the inadequate models, we reinforce the validities as follow:

$$v_i^{renf}(k) = v_i(k) \prod_{\substack{c=1 \\ i \neq c}}^{md} \left(1 - e^{-\left(\frac{r_c(k)}{g}\right)^2}\right) \tag{32}$$

with g is a positive coefficient. The normalized reinforced validities are given by:

$$v_{in}^{renf}(k) = \frac{v_i^{renf}(k)}{\sum\limits_{c=1}^{md} v_c^{renf}(k)} \tag{33}$$

3.2 The Multimodel 2-DSMC

As already mentioned, the 2-DSMC helps to reduce the chattering phenomenon by the integration of the discontinuous term which is used to guaranty the robustness of the control law. Unfortunately, this discontinuous term does not switch during the reaching phase (because the system has not reached the sliding surface yet). Consequently, during this phase the robustness is not guaranteed. A solution for this problem was proposed in (Mihoub et al. (2008; 2009a)) by combining the second order discrete sliding mode control and the multimodel approach.
The multimodel discrete second order sliding mode control (MM-2-DSMC) structure is given by the figure 1.
In our case, the partial models can be represented as follows :

$$\begin{array}{ll} \text{Modèle } 1 \ : & \begin{cases} x(k+1) = A_1 x(k) + B_1 u(k) \\ y(k) = Hx(k) \end{cases} \\ & \vdots \\ \text{Modèle } md: & \begin{cases} x(k+1) = A_{md} x(k) + B_{md} u(k) \\ y(k) = Hx(k) \end{cases} \end{array} \tag{34}$$

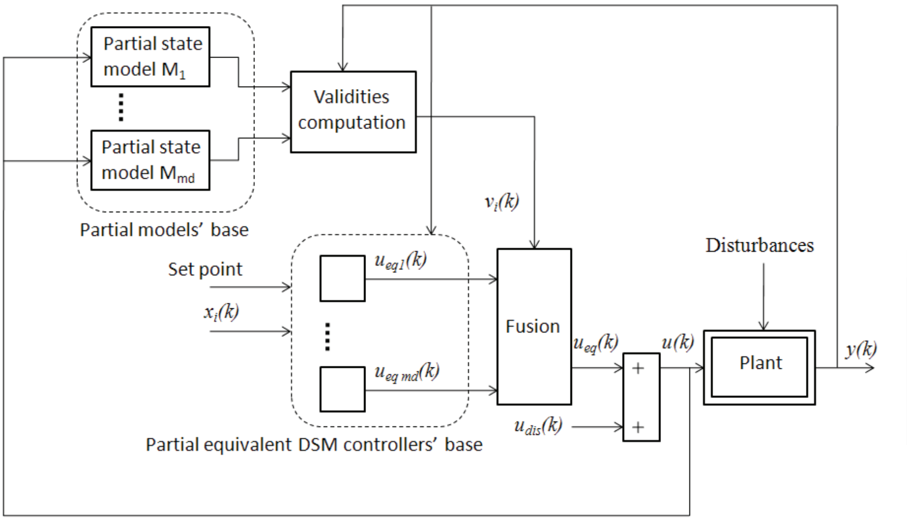


Fig. 1. The structure of a multimodel discrete second order sliding mode control (MM-2-DSMC).

where *md* is the number of the partial models. The control applied to the system is given by the following relation:

$$u(k) = v_1(k)u_{1eq}(k) + v_2(k)u_{2eq}(k) + ... + v_{md}(k)u_{mdeq}(k) + u_{dis}(k); \tag{35}$$

with

- $v_i(k)$: the validity of the i^{th} local state model,
- $u_{ieq}(k)$: the partial equivalent term of the 2-DSMC calculated using the i^{th} local state model,
- $u_{dis}(k)$: the discontinuous term of the control.

$$u_{eqi}(k) = (C^T B_i)^{-1} \left(\alpha\, S(k) - C^T A_i x(k) + C^T (x_d(k+1)\right)$$

A_i et B_i are the matrixes of the i^{th} partial state model. The discontinuous term is given by the following expression:

$$u_{dis}(k) = u_{dis}(k-1) - M\, sign\,(\sigma(k))$$

The multimodel discrete second order sliding mode control (MM-2-DSMC) is, then, given by:

$$u(k) = \sum_{i=1}^{m_d} v_i(k)u_{eqi}(k) + u_{dis}(k) \tag{36}$$

A stability analysis of this last control law is proposed in the following paragraph.

3.3 Stability analysis of the MM-2-DSMC

Let's consider the following non stationary system:

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) + \Gamma(k) \\ y(k) &= Hx(k) \end{aligned} \tag{37}$$

$\Gamma(k)$ represents eventual non linearities and external disturbances.

Considering the following notations: $A_m = \sum_{i=1}^{md} v_i A_i$ and $B_m = \sum_{i=1}^{md} v_i B_i$,

we obtain the following model:

$$\begin{aligned} x(k+1) &= A_m x(k) + B_m u(k) \\ y(k) &= Hx(k) \end{aligned} \tag{38}$$

which is the multimodel approximation of the system (37). This last system can be, then, written in the following form:

$$\begin{aligned} x(k+1) &= (A_m + \Delta A_m)\, x(k) + (B_m + \Delta B_m)\, u(k) + \Gamma(k) \\ y(k) &= Hx(k) \end{aligned} \tag{39}$$

such that:

$$\begin{aligned} A_d &= A_m + \Delta A_m \\ B_d &= B_m + \Delta B_m \end{aligned} \tag{40}$$

We note:

$$\Delta(k) = \Delta A_m x(k) + \Delta B_m u(k) + \Gamma(k) \tag{41}$$

The system (37) can, in this case, be written as follows:

$$\begin{aligned} x(k+1) &= A_m x(k) + B_m u(k) + \Delta(k) \\ y(k) &= Hx(k) \end{aligned} \tag{42}$$

The control law given by (36) is applied to the system. In the case of the multimodel, the equivalent term $\sum_{i=1}^{m_d} v_i(k) u_{eqi}(k)$ of (36) is written as follow:

$$u_{eq}(k) = (C^T B_m)^{-1}[-\beta\, S(k) - C^T A_m x(k)] \tag{43}$$

In this case, the sliding function dynamics are given by the following expression:

$$C^T x(k+1) = S(k+1) = -\beta S(k) + C^T \Delta(k) + (C^T B_m) u_{dis}(k) \tag{44}$$

The sliding function variation $[S(k+1) - S(k)]$ is given by the following relation:

$$\begin{aligned} S(k+1) - S(k) = &- \beta(S(k) - S(k-1) + C^T\left(\Delta(k) - \Delta(k-1)\right) \\ &- (C^T B_m) M sign\left(S(k) + \beta S(k-1)\right) \end{aligned} \tag{45}$$

In what follows, the quantity $(C^T B_m)M$ will be noted M^*.

Which gives:

$$S(k+1) + \beta S(k) = \left(S(k) + \beta S(k-1) + C^T \left(\Delta(k) - \Delta(k-1)\right) - M^* sign\left(S(k) + \beta S(k-1)\right)\right) \tag{46}$$

The relation (46) can be written:

$$\sigma(k+1) = \sigma(k) + C^T \left(\Delta(k) - \Delta(k-1)\right) - M^* sign\left(\sigma(k)\right) \tag{47}$$

In discrete time sliding mode control, instead of the sliding mode, a quasi sliding-mode is considered in the vicinity of the sliding surface, such that $|\sigma(k)| < \varepsilon$, where $\sigma(k)$ is the sliding function and ε is a positive constant called the quasi-sliding-mode band width.
Bartoszewicz, in (Bartoszewicz (1998)), gave the following sufficient and necessary condition for a system to satisfy a convergent quasi sliding mode:

$$\begin{cases} \sigma(k) > \varepsilon \Rightarrow -\varepsilon \leq \sigma(k+1) < \sigma(k) \\ \sigma(k) < -\varepsilon \Rightarrow \sigma(k) < \sigma(k+1) \leq \varepsilon \\ |\sigma(k)| < \varepsilon \Rightarrow |\sigma(k+1)| \leq \varepsilon \end{cases} \tag{48}$$

$\forall k$, $C^T \left(\Delta(k) - \Delta(k-1)\right)$ is supposed to be bounded such that:

$$\left| C^T \left(\Delta(k) - \Delta(k-1)\right) \right| < \Delta_0 \tag{49}$$

with Δ_0 being a positive constant.

Théorème 0.1. *Let's consider the system (37) to which the MM-2-DSMC given by (36) is applied. If the discontinuous term amplitude M is chosen such that:*

$$M^* > \Delta_0 \tag{50}$$

where $M^* = (C^T B_m)M$ and Δ_0 is the external disturbances and system's parameters' variation bound given by (49), then, the MM-2-DSMC of (36) results in a convergent quasi sliding mode.

Proof.
ε is chosen equal to $M^* + \Delta_0$.
To prove the convergence of the proposed control technique, we must, then, check the following three conditions:

$$\sigma(k) > M^* + \Delta_0 \Rightarrow -\left(M^* + \Delta_0\right) \leq \sigma(k+1) < \sigma(k) \tag{51}$$

$$\sigma(k) < -\left(M^* + \Delta_0\right) \Rightarrow \sigma(k) < \sigma(k+1) \leq M^* + \Delta_0 \tag{52}$$

$$|\sigma(k)| < M^* + \Delta_0 \Rightarrow |\sigma(k+1)| \leq M^* + \Delta_0 \tag{53}$$

1. Let's begin by the condition (51):

$$\sigma(k) > M^* + \Delta_0 \Rightarrow -M^* - \Delta_0 < \sigma(k+1) < \sigma(k)$$

   * The inequality

$$\sigma(k+1) < \sigma(k) \tag{54}$$

can be written as follows:

$$\sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* sign\,(\sigma(k)) < \sigma(k) \quad (55)$$

Knowing that $\sigma(k) > 0$, the inequality (55) becomes:

$$\sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* < \sigma(k) \quad (56)$$

By substructing $\sigma(k)$ from both sides of this last inequality, we obtain:

$$C^T(\Delta(k) - \Delta(k-1)) - M^* < 0 \quad (57)$$

This last inequality is true because M^* is chosen such that $M^* > \Delta_0$

* The inequality

$$-M^* - \Delta_0 < \sigma(k+1) \quad (58)$$

can be written as follows:

$$-M^* - \Delta_0 < \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* \quad (59)$$

which gives:

$$-\Delta_0 - C^T(\Delta(k) - \Delta(k-1)) < \sigma(k) \quad (60)$$

This last inequality is true, knowing that $\sigma(k) > M^* + \Delta_0 > 0$ and $-\Delta_0 - C^T(\Delta(k) - \Delta(k-1)) < 0$

2. Let's consider condition (52):

$$\sigma(k) < -M^* - \Delta_0 \Rightarrow \sigma(k) < \sigma(k+1) < M^* + \Delta_0$$

By replacing $\sigma(k+1)$ by its expression, we obtain:

$$\sigma(k) < \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^* < M^* + \Delta_0 \quad (61)$$

* The inequality

$$\sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^* < M^* + \Delta_0 \quad (62)$$

can be written as follows:

$$\sigma(k) + C^T(\Delta(k) - \Delta(k-1)) < \Delta_0 \quad (63)$$

which gives:

$$\sigma(k) < \Delta_0 - C^T\,(\Delta(k) - \Delta(k-1)) \quad (64)$$

This last inequality is true because $\Delta_0 - C^T\,(\Delta(k) - \Delta(k-1)) > 0$ and $\sigma(k) < 0$.

* Besides, it is evident that $\sigma(k) < \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^*$, knowing that:

$$M^* > \Delta_0 > C^T(\Delta(k) - \Delta(k-1)) \quad (65)$$

3. Let's consider condition (53):

$$|\sigma(k)| < M^* + \Delta_0 \Rightarrow |\sigma(k+1)| < M^* + \Delta_0$$

* If $\sigma(k) > 0$, then, the inequality

$$|\sigma(k)| < M^* + \Delta_0 \tag{66}$$

becomes:

$$0 < \sigma(k) < M^* + \Delta_0 \tag{67}$$

which gives:

$$\begin{aligned}
C^T(\Delta(k) - \Delta(k-1)) - M^* &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* \\
&\qquad < M^* + \Delta_0 + C^T(\Delta(k) - \Delta(k-1)) - M^* \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* < M^* + \Delta_0 + \Delta_0 - M^* \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* < \Delta_0 + \Delta_0 \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) - M^* < M^* + \Delta_0 \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k+1) < M^* + \Delta_0
\end{aligned}$$

Then,

$$|\sigma(k+1)| < M^* + \Delta_0 \tag{68}$$

* If $\sigma(k) < 0$, then,

$$|\sigma(k)| < M^* + \Delta_0 \tag{69}$$

becomes

$$-M^* - \Delta_0 < \sigma(k) < 0 \tag{70}$$

then,

$$\begin{aligned}
C^T(\Delta(k) - \Delta(k-1)) + M^* - M^* - \Delta_0 &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^* \\
&\qquad < C^T(\Delta(k) - \Delta(k-1)) + M^* \\
\Rightarrow -\Delta_0 - \Delta_0 &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^* < M^* + \Delta_0 \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k) + C^T(\Delta(k) - \Delta(k-1)) + M^* < M^* + \Delta_0 \\
\Rightarrow -\Delta_0 - M^* &< \sigma(k+1) < M^* + \Delta_0
\end{aligned}$$

so,

$$|\sigma(k+1)| < M^* + \Delta_0 \tag{71}$$

The verification of the three conditions (51), (52) and (53) proves the existence of the convergent quasi sliding mode. Therefore, the controller given by (29) is stable. □

Note
The bound of $\Delta(k)$ can be determined by studying the uncertainties of the different partial models, how much they cover the different real system's dynamics and the method used for the calculation of the validities degrees. The linear matrixes inequalities (LMI) approach can be used in these conditions.

4. Experimentation on a chemical reactor

4.1 Process description

The semi-batch reactor control provides a very challenging problem for the process control engineer, due to the high non linearity that characterizes its dynamic behavior. Therefore, we choose to apply the proposed control laws for temperature control of the chemical reactor presented in this section (a photo of the considered reactor is given by figure 2).

This process is used to esterify olive oil. The produced ester is widely used for the

Fig. 2. The esterification reactor used for the experiments.

manufacture of cosmetic products. A specific temperature profile sequence must be followed in order to guarantee an optimal exploitation of the involved reagents' quantities. The olive oil contains, essentially, a mono-unsaturated fatty acid that react with alcohol to give water and ester as shown by the following reaction equation:

$$Acid + Alcohol \underset{2}{\overset{1}{\rightleftharpoons}} Ester + Water \tag{72}$$

The final solution contains all the reagents and products in certain proportions. To drive the reaction equilibrium in the way 1 and, consequently, increase the ester's proportion, we should take away water from the solution. This is done by vaporization. The fatty acid (oleic acid) and the ester ebullition temperatures are approximately $300°C$. The chosen alcohol (1-butanol) is characterized by an ebullition temperature of $118°C$. Consequently, heating the reactor to a temperature slightly over $100°C$ will result in the vaporization of water only (which is evacuated through the condenser).

The reactor is heated by circulating a coolant fluid through the reactor jacket. This fluid is, in turn, heated by three resistors located in the heat exchanger (Figure 3). The reactor temperature control loop monitors temperature inside the reactor and manipulates the power delivered to the resistors. It is, also, possible to cool the coolant fluid by circulating cold water through a coil in the heat exchanger. Cooling is, normally, done when the reaction is over, in order to accelerate the reach of ambient temperature.

The process can be considered as a single input - single output system. The input is the heating power $P(W)$. The output is the reactor temperature $TR(°C)$. The interface between the process and the calculator is ensured by a data acquisition card of the type RTI 810. The data acquisition card ensures the conversion of the analog measures of the temperature to digital values and the conversion of the digital control value to an analog electric signal proportional the power applied to the heating resistors.

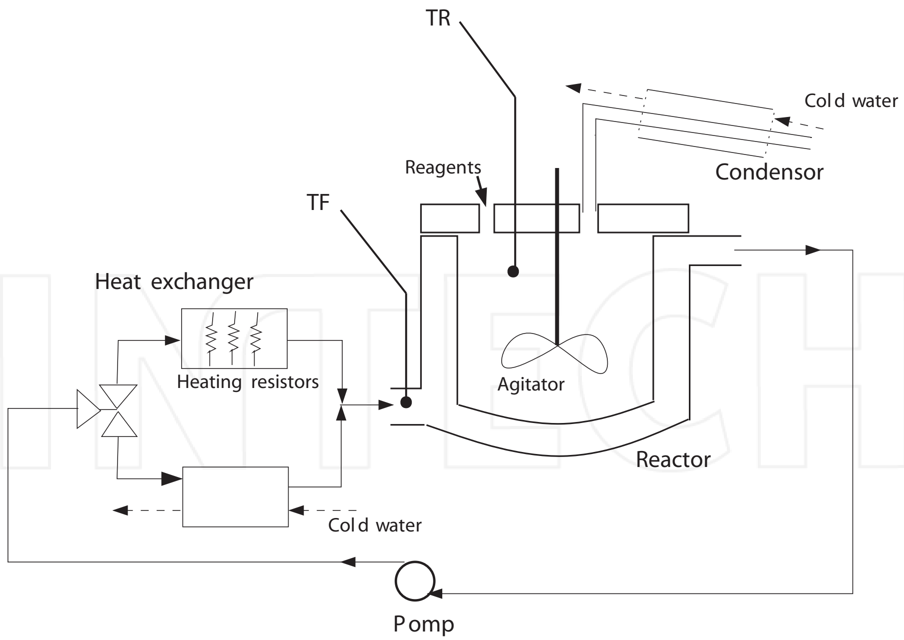


Fig. 3. Synoptic scheme of the real process.

The control law must carry out the following three stages :

- Bring the reactor's temperature TR to $105°C$.
- Keep the reactor's temperature to this value until the reaction is over (no more water dripping out of the condenser).
- Lower the reactor's temperature.

We chose, therefore, the set point given by figure 4.

We represent, in figure 5, the static characteristic of the system. The different coordinates are taken relatively to the three stages of the process. We notice that the system can be considered as a linear one, though, with some approximations. According to the step responses of the system, the retained sampling step is equal to $180s$. A Pseudo-Random, Binary input Signal (PRBS) is applied to the real system. An identification of the system structure, based on the instrumental determinants ratio method (Ben Abdennour et al. (2001)), led to a discrete second order linear model.

Due to the nature of the control law to be applied to the reactor, the needed model is a state model. The considered state variables are the reactor's temperature $TR(°C)$ (noted $x_1(k)$), which is at the same time the system's output, and the coolant fluid temperature $TF(°C)$ (noted $x_2(k)$).

The state variables sequences $x_1(k)$ and $x_2(k)$ relative to the PRBS excitation input are measured and used for the parametric identification of the system. The least square method leads to the following nominal model:

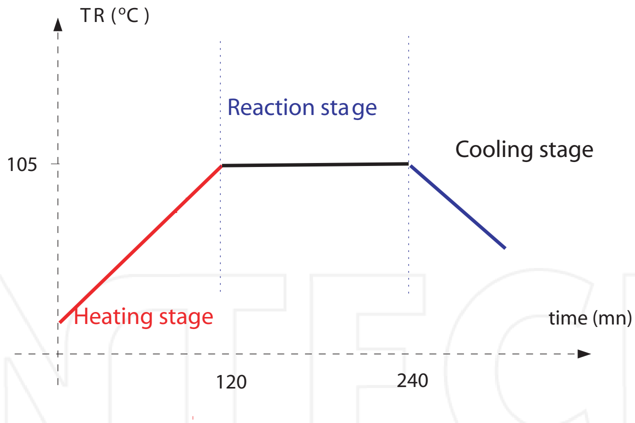


Fig. 4. Desired reactor temperature.

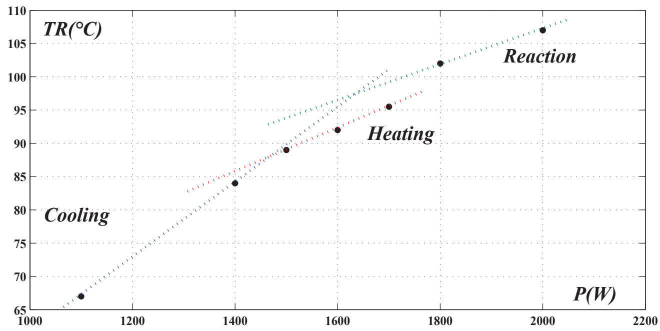


Fig. 5. Static characteristic of the reactor.

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.6158 & 0.3600 \\ 0.0409 & 0.9118 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0033 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\end{aligned}
\tag{73}
$$

The application of the multimodel approach and by using the least square method applied on the input/states sequence relative to each reaction stage leads to three partial models of the form:

$$
\begin{aligned}
x(k+1) &= A_i x(k) + B_i u(k) \\
y(k) &= Hx(k) \qquad\qquad i \in [1,3]
\end{aligned}
\tag{74}
$$

with for:

- the heating stage:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.4712 & 0.4953 \\ -0.1296 & 1.0651 \end{bmatrix} \; ; \quad B_1 = \begin{bmatrix} 0 \\ 0.0036 \end{bmatrix} \\
H &= \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{aligned}
\tag{75}
$$

- the reaction stage:

$$A_2 = \begin{bmatrix} 0.4114 & 0.5482 \\ -0.0358 & 0.9787 \end{bmatrix} \; ; \quad B_2 \begin{bmatrix} 0 \\ 0.0034 \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{76}$$

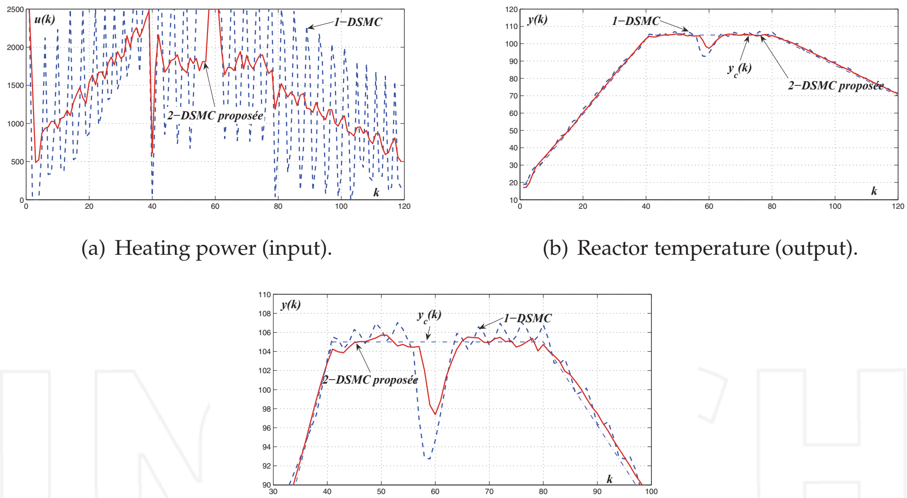
- the cooling stage:

$$A_3 = \begin{bmatrix} 0.6914 & 0.2877 \\ -0.0386 & 0.0.9912 \end{bmatrix} \; ; \quad B_3 = \begin{bmatrix} 0 \\ 0.0032 \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{77}$$

The control performance and robustness of the previously mentioned control laws, with respect to the model-system mismatch and external disturbance, are illustrated and compared through the experimental results given in the following paragraph.

4.2 Experimental results

In this paragraph, the performance of the MM-2-DSMC is shown by an experimentation on the chemical reactor. Firstly, the chattering reduction, obtained by exploiting the second order sliding mode control, is illustrated by a comparison between the results obtained by the first order discrete sliding mode control with those realized by the 2-DSMC (Mihoub et al. (2009b)). The nominal model (73) is used for both the DSMC and the 2-DSMC.



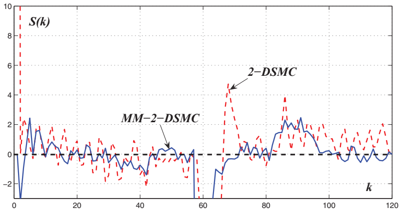
(a) Heating power (input).

(b) Reactor temperature (output).

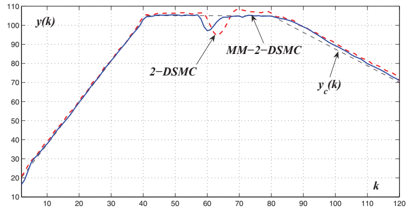
(c) Zoom on the reactor temperature evolution.

Fig. 6. Comparison between DSMC and 2-DSMC

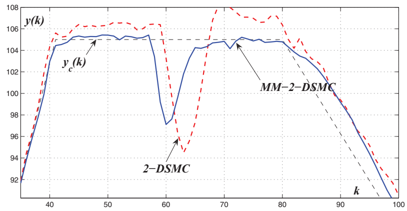
We observe that the chattering of the control ($u(k)$) is remarkably reduced (figure 6.a). A better set point tracking is, consequently, obtained as shown by figures 6.b and 6.c, which represent, respectively, the evolution of the reactor temperature and a zooming of this last one in the neighborhood of 105° C. As mentioned above, the reaction takes place essentially during this phase. If the temperature reactor overshoots 105° C, a large amount of alcohol is evaporated and wasted and if it does not reach 105° C, the reaction kinetics are slowed down. So, the 2-DSMC results in a better efficiency relatively to the first order DSMC.



(a) Sliding function.



(b) Reactor temperature (output).



(c) Zoom of the reactor temperature evolution.

Fig. 7. Comparison between 2-DSMC and MM-DSMC

Secondly, the multimodel approach is combined with the 2-DSMC in order to enhance the reaching phase. The MM-2-DSMC and the 2-DSMC are represented together in figure 7 (Mihoub et al. (2009a)). It can be observed that the sliding function overshoots due to a bad reaching phase in the case of the 2-DSMC are reduced thanks to the multimodel approach (see figure 7.a). A better set point tracking is then obtained, as shown by figures 7.b and 7.c. An amelioration of the efficiency of the chemical reactor is, consequently, obtained.

5. Conclusion

In this work, the problems of the discrete sliding mode control are discussed. A solution to the chattering problem can be given by the second order sliding mode. To enhance the reaching phase, the multimodel approach is exploited. A combination of the 2-DSMC and the multimodel approach is, then, used. A stability analysis of the multimodel second order discrete sliding mode control is proposed in this work. An experimentation on a chemical reactor is considered. On the one hand, a comparison between the results obtained by the first order DSMC and those obtained by the 2-DSMC showed the chattering reduction offered by the second order approach. On the other hand, a comparison between the results of the 2-DSMC and those of the MM-2-DSMC, illustrated both an enhancement of the reaching phase and a notable reduction of the chattering phenomenon. A better efficiency of the reactor is, therefore, obtained.

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