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Stabilisation of locally Lipschitz non-linear systems under input saturation and quantisation

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Abstract: This study addresses the control of locally Lipschitz non-linear systems under quantisation and input saturation non-linearities. The non-linear dynamics of the systems are taken to be locally Lipschitz, rather than the conventional globally Lipschitz counterpart, to consider a generalised form of the Lipschitz non-linear systems. An ellipsoidal region containing the origin is constructed, for which the non-linear dynamics satisfy the Lipschitz condition in contrast to the conventional approaches. Saturation and quantisation non-linearities are dealt using a generalised local sector condition and a bound on the quantisation noise. A regional control strategy for the stabilisation of non-linear systems using state feedback is devised by employing these conditions, which is further extended to attain robustness against external perturbations. The proposed control strategies guarantee convergence of the states of a non-linear system inside a bounded reducible region in the neighbourhood of the origin. In contrast to the conventional approaches, the present study considers ellipsoidally Lipschitz non-linear systems, supports various types of quantisers, ensures attenuation of the disturbances, and provides a clear picture of the region of stability. An example of the application of the proposed control strategies for a modified Chua's circuit is demonstrated.

1 Introduction

Control of linear and non-linear systems under information processing and input constraints caused by the controller hardware and/or software and bounded-input limitations is an interesting research area. Conventional studies on the controller design of the linear and non-linear systems do not incorporate the input saturation and quantisation non-linearities while synthesising the control laws. When a controller is implemented through hardware, it faces quantisation non-linearity for the effective controller implementation using digital technologies, largely employed in the modern era [1–3]. This quantisation process, having various types (for instance, uniform, non-uniform, and adaptive quantisation), causes undesirable performance degradation and oscillations [4, 5]. Further, almost all control signals have to face the input saturation non-linearity caused by the bounded-input and actuator restrictions, resulting in the windup phenomenon [6–8]. The consequences of this phenomenon appear as the performance abasement, lags, undershoots, overshoots, and even instability of the constrained closed-loop system. Simultaneous dealing of these two non-linear input constraints is a challenging research issue, while synthesising a control law.

Stability analysis, stabilisation, and control of the linear systems under input saturation has remained an interesting research topic as seen in the works [6–10]. Recently, several extensions of the linear control systems to the non-linear plants have been reported in the papers [10–15] for dealing with the closed-loop performance degradation due to the actuator saturation. To design a controller for tracking control of a non-linear system in the presence of a reference model, two studies [13, 15] considering input saturation, disturbances, and input lags, via synchronisation of the master-slave processes, were recently reported using the state feedback controllers. Further, several studies on the one-step and the two-step control approaches of globally Lipschitz non-linear, quadratic non-linear, and feedback linearisable non-linear systems under various constraints (such as saturation, disturbances, and time-delays) have been explored to deal with the so-called windup effects. The approaches in [10, 12, 14] provide sufficient conditions for the semi-global stabilisation controller design in a step for the quadratic non-linear and Lipschitz non-linear systems by means of convex routines. Contrastingly, two-step global and semi-global control methodologies are designed in the work [11] for globally Lipschitz and linearisable non-linear models by taking two independent feedbacks of the output and saturation and by considering both feedback controller and compensator design in a single step. The one-step approaches can have better (even optimal) performance due to their consideration of the control law design in a single step, by imposing all the control constraints together.

As far as quantisation is concerned, several recent studies on the suppression of quantisation noise effects, at the state or output of a plant, are available in the literature. The work of Xue *et al.* [16] developed linear matrix inequality (LMI)-based conditions for controlling the linear systems in a communication network by application of a model predictive control system. In [17], a combined control strategy, based on model-based network control and event-triggered control, was applied for the linear systems under time-delay and quantisation effects. A high-precision current control strategy was presented in [18] for a non-linear surface permanent magnet synchronous motor using Kalman filtering by incorporating the quantisation consequences. Adaptive control strategies for the non-linear systems under input quantisation, with guaranteed asymptotic or L_2 stability of the tracking control error, subjected to unknown parameters, uncertain switching, external perturbations, and actuator faults, have been developed in the literature [19–21].

The above-mentioned research [6–15] and [16–21] addressed the control of the linear or the non-linear systems in the presence of either saturation or quantisation non-linearities. However, in practice, both saturation and quantisation non-linearities affect the performance of a control system when implemented via a digital technology, causing instability and performance degradation problems. The input saturation can result in the ‘windup’ phenomenon, leading to overshoot, undershoot, oscillations, lag, and instability. Likewise, the quantisation acts as a noise, which can cause the oscillatory closed-loop response and drift of the control parameters towards infinity. Both saturation and quantisation form complex input non-linearity, the combined effect of which can result in precarious closed-loop performance. Dealing with the combined non-linearities for stabilisation purposes is a difficult task because saturation does not allow global stability and performance and quantisation cannot be easily dealt with due to various types of quantisation with effects at various stages of the data acquisition and distribution processes in a closed-loop system.

Compared with the above-mentioned control schemes, some advanced research studies in [22–26] have successfully incorporated the effects of both of the non-linearities at the control signal while devising more reliable feedback control systems. The work in [22] addressed the output feedback control of a simple two-dimensional linear system to ensure uniformly ultimately bounded stability by incorporating input-output quantisation and saturation non-linearity. Control of linear systems under input delay, bounded quantisation, and input saturation was considered in [23] to ensure the convergence of the state vector into a bounded region. A regional control scheme for a category of continuous linear systems, with guaranteed regional stability, using a local sector condition, taking both input saturation and quantisation into account, was developed in [26]. Similar local methodologies for uncertain linear time-delayed plants and for Lipschitz systems have been studied in [24, 25], respectively, to attain convergence of the state vector into a bounded ellipsoidal region. Most of the developed approaches for quantised systems do not incorporate the effects of non-linear dynamics, as noted in [25].

It is notable that input saturation cannot be avoided in any practical control systems due to the bounded input restraint of the input signals [27]. In additionally, the quantisation of signals is always performed during the implementation of a controller via digital technologies due to the availability of a distinct number of signal states [28]. Most practical control systems encounter the quantisation phenomenon due to the use of digital technologies with the advantages of low cost, low power consumption, versatility in handling non-linearity, and programming flexibility. Due to the presence of non-linearities in almost all practical systems [29], further research work is needed to control the non-linear systems for saturation, quantisation, and inherent non-linear dynamics effects.

In this paper, the stabilisation of non-linear systems containing ellipsoidally Lipschitz non-linearities subjected to quantisation and input saturation is addressed. By assuming that the non-linear component of the plant verifies the Lipschitz continuity in an ellipsoidal locality region as observed in [30], a condition for the controller design with a guaranteed region of stability is provided by taking a generalised local sector bound [31] for the input saturation non-linearity into account. It is worth mentioning that an ellipsoidal region of the Lipschitz condition can always be selected if the non-linearity is locally Lipschitz in an arbitrary region containing the origin. The results of the work in [25], not utilising the information of a region for the Lipschitz function, are based on maximisation of the allowable Lipschitz constant. Such tactics are not completely applicable to locally Lipschitz systems, cannot guarantee a region of stability when the Lipschitz constant varies from the allowable limit, and can provide large controller gains causing the sensitivity of the closed-loop system to noise. The ellipsoidally Lipschitz condition, in contrast, can more precisely represent a locally Lipschitz non-linear system and can be applied to a locally Lipschitz system satisfying the Lipschitz condition in an ellipsoid. To the best of the authors’ knowledge, the incorporation of the ellipsoidal region for controlling locally Lipschitz non-linear systems under input saturation and quantisation non-linearities is addressed for the first time to provide a guaranteed and proper region of stability. Controller gain for the proposed stabilisation technique can be computed via simple LMI-based convex routines. It is notable that obtainment of LMIs for locally Lipschitz non-linear systems, by considering the local region, is usually non-trivial.

Quantisation non-linearity is incorporated in the proposed design by taking the idea of the quantisation error bound. Therefore, the results of the proposed methodologies overcome the limitation of a specific selection of a quantiser, such as piece-wise constant mapping and logarithmic quantisers, as observed in [24–26]. The proposed control methodology ensures the convergence of the states of the closed-loop system into an ellipsoidal region, the size of which can be minimised according to the design requirements and depends upon the bound on the quantisation error. A novel result for Lipschitz non-linear systems is provided for the proposed method, which in contrast to [25] is applicable to continuous-time non-linear systems and allows a range of quantisers. Another contribution of the proposed methodology in contrast to the conventional methods is the derivation of the robust control synthesis condition for ellipsoidally Lipschitz non-linear systems subject to input saturation and quantisation non-linearities for dealing with bounded disturbances. The proposed robust controller ensures convergence of the states of the system into a bounded region by utilising the upper bound on the size of disturbances. A numerical simulation example is provided for the control of a modified Chua's circuit under quantisation, input saturation, disturbances, and a cubic locally Lipschitz non-linearity.

This paper is organised as follows: Section 2 establishes the system description and the problem formulation. The main results for the controller design of non-linear systems in the absence or presence of external perturbations are detailed in Section 3. In Section 4, a numerical simulation study for the modified Chua's circuit is provided. Conclusions are drawn in Section 5.

Standard notation is employed throughout this paper. The inequalities $Y > 0$ and $Z \geq 0$ represent that the matrices Y and Z are positive-definite and semi-positive-definite matrices, respectively. A block diagonal matrix is represented as $\mathrm{diag}(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ denote the corresponding diagonal blocks. For a matrix X, the ith row is denoted by $X_{(i)}$. For a vector υ, the saturation non-linearity is written as $\aleph_{\mathrm{sat}(i)}(\upsilon_{(i)}) = \mathrm{sgn}(\upsilon_{(i)}) \min\left(\sigma_{(i)}, \left|\upsilon_{(i)}\right|\right)$, where the constant vector σ contains the saturation bounds for the vector υ. The vector functions $\aleph_{\mathrm{dz}}(\cdot)$ and $\aleph_{\mathrm{quan}}(\cdot)$ are employed for the dead-zone and quantisation non-linearities, respectively. The L_2 norm and the Euclidean norm for a vector $x \in R^n$ are given by $\| x \|_2$ and $\| x \|$, respectively.

2 System description

Consider a non-linear system, given by

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= Ax(t) + \Upsilon(t, x) + B\aleph_{\mathrm{sat}}\left(\aleph_{\mathrm{quan}}(u(t))\right) + B_w w(t), \\ y(t) &= Cx(t) + D_w w(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $y(t) \in R^p$, $u(t) \in R^q$, and $w(t) \in R^m$ represent the state, output, control input, and exogenous input (containing noise and disturbance) vectors, respectively. $A \in R^{n\times n}$, $B \in R^{n\times q}$, $C \in R^{p\times n}$, $B_w \in R^{n\times m}$, and $D_w \in R^{p\times m}$ are the constant matrices. The vector function $\Upsilon(t, x) \in R^n$ represents the state non-linearity in the system. The input saturation and quantisation non-linearities are represented by $\aleph_{\mathrm{sat}}(\cdot)$ and $\aleph_{\mathrm{quan}}(\cdot)$, respectively.

Assumption 1: Let the function $\Psi(t, x)$ be described in the form $\Upsilon(t, x) = H\Psi(t, x)$ for a constant matrix H and a non-linear vector-function $\Psi(t, x)$, satisfying the ellipsoidally Lipschitz condition given by

$$\| \Psi(t, x) - \Psi(t, \bar{x}) \| \leq \| \Gamma(x - \bar{x}) \|, \tag{2}$$

$$x, \quad \bar{x} \in \chi^{\mathrm{T}} Z^{-1} \chi \leq 1, \quad Z = Z^{\mathrm{T}} > 0, \tag{3}$$

for all $\chi \in R^n$, where Γ is a constant matrix having appropriate dimensions.

Assumption 2: The amplitude of the disturbance is bounded and satisfies

$$w^{\mathrm{T}}(t)w(t) \leq \delta \tag{4}$$

for a positive scalar δ.

Remark 1: The condition in (2) and (3), recently investigated for the controller and observer design of non-linear systems [30], is a generalised form of the traditional Lipschitz condition (such as [32]). Mainly, this condition can be applied to a locally Lipschitz system if the non-linear dynamics do not satisfy the Lipschitz condition globally. If a non-linearity $\Upsilon(t, x)$ is locally Lipschitz in a region containing the origin, we can always select an ellipsoidal region $\chi^{\mathrm{T}} Z^{-1} \chi \leq 1$ for which the inequality (2) holds. It is worth mentioning that for $H = I$, Γ as a scalar, and z as a scalar with $Z = \lim_{z \to \infty} zI$, the condition in Assumption 1 represents the traditional global Lipschitz non-linearity as a specific case.

Remark 2: The system (1) can be employed to represent a class of highly non-linear systems due to consideration of the locally Lipschitz non-linearity $\Upsilon(t, x)$, input saturation non-linearity $\aleph_{\mathrm{sat}}(\cdot)$, and quantisation non-linearity $\aleph_{\mathrm{quan}}(\cdot)$. Saturation and quantisation non-linearities cannot be ignored because of the bounded-input restriction of actuators and utilisation of modern digital hardware for the controller implementation. Almost all practical systems depict non-linear behaviour; therefore, the state non-linearity $\Upsilon(t, x)$, representing a generalised class of (locally) Lipschitz non-linear functions, is employed in the present study. To the best of the authors' knowledge, stabilisation of the class of ellipsoidally Lipschitz non-linear systems under Assumptions 1–2, considered in this paper, has not been investigated in the previous studies.

The proposed stabilisation controller has the form

$$u(t) = Fx(t), \tag{5}$$

where $F \in R^{q \times n}$ is a constant state-feedback gain matrix to be determined for the stabilisation of the non-linear system (1). Note that $-\sigma_{(i)} \leq \upsilon_{(i)} - \aleph_{\mathrm{dz}(i)}(\upsilon_{(i)}) \leq \sigma_{(i)}$ always holds for $i = 1, 2, \ldots, q$, the following lemma may be useful in the derivation of the proposed control schemes.

Lemma 1 [31]: For a region, given by

$$\begin{aligned} S(\sigma) = \{ &\upsilon(t), \omega(t), \Phi(t) \in R^n; \\ &-\sigma + \bar{\Phi} \leq \upsilon(t) - \omega(t) + \Phi(t) \leq \sigma - \bar{\Phi} \}, \end{aligned} \tag{6}$$

$$\left| \Phi_{(i)}(t) \right| \leq \bar{\Phi}_{(i)}, \tag{7}$$

the sector condition

$$\aleph_{\mathrm{dz}}^{\mathrm{T}}(\upsilon(t)) W \left[\omega - \aleph_{\mathrm{dz}}(\upsilon(t)) \right] \geq 0 \tag{8}$$

for a diagonal matrix $W > 0$ is satisfied.

The condition in (8) holds along with $-\sigma_{(i)} \leq \upsilon_{(i)} - \aleph_{\mathrm{dz}(i)}(\upsilon_{(i)}) \leq \sigma_{(i)}$ for the region $S(\sigma)$. The present study aims to devise sufficient conditions for the design of an appropriate state-feedback control system (5) for stabilisation of the non-linear plant (1) under state, input saturation, and quantisation non-linearities in the absence or presence of external perturbations.

3 Controller design

To develop a control strategy, we need to employ the properties of the quantisation and saturation functions. The input signal $u(t)$ is affected by both of the non-linearities; therefore, controller design by considering the effects of both non-linearities together at the control signal is a perplexing research problem. Various research works have been carried out to deal with the saturation and quantisation individually; however, the investigation of the combined effects of these non-linearities for stabilisation controller design has been inadequately considered. Selection of the properties for derivation of a controller is an important factor for the stability analysis of a closed-loop system subjected to a combined paradigm of saturation and quantisation non-linearities. In addition, the resultant stability criteria should be simple in design, straightforward in realisation, and easy in application.

Let us define the quantisation error (or quantisation noise) as

$$\eta(t) = u(t) - \aleph_{\mathrm{quan}}(u(t)). \tag{9}$$

It is important to mention that the amplitude of the quantisation error is bounded by the (maximum) quantisation interval. Let $\bar{\eta}$ denote the vector containing the upper bounds on the quantisation error entities in $\eta(t)$. We can write

$$\left| \eta_{(i)}(t) \right| \leq \bar{\eta}_{(i)}, \quad \bar{\eta}_{(i)} > 0, \tag{10}$$

similar to [33]. Employing (5), the state (1) can be transformed into

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + \Upsilon(t, x) + B\aleph_{\mathrm{sat}}(u(t) - \eta(t)) + B_w w(t). \tag{11}$$

For a signal $v(t) \in R^q$, the dead-zone non-linearity $\aleph_{\mathrm{dz}}(v(t))$ is related with the saturation non-linearity as

$$\aleph_{\mathrm{dz}}(v(t)) = v(t) - \aleph_{\mathrm{sat}}(v(t)). \tag{12}$$

Selecting $v(t) = u(t) - \eta(t)$ in (12) and using it into (11), we obtain

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = &Ax(t) + \Upsilon(t, x) - B\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ &+ u(t) - \eta(t) + B_w w(t). \end{aligned} \tag{13}$$

By employing (5), we obtained

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = &(A + BF)x(t) + \Upsilon(t, x) - B\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ &- \eta(t) + B_w w(t). \end{aligned} \tag{14}$$

Remark 3: The dead-zone non-linearity in the closed-loop system (14) depends on the control input $u(t)$ and the quantisation error $\eta(t)$. The time profile of $\eta(t)$ is unknown at any time instant; therefore, the conventional sector conditions for the dead-zone functions (e.g. [6–8, 13–15] etc.) cannot be employed for the stability analysis. Recently, a generalised sector condition was developed in [31]. This new sector condition has been found to be suitable for conventional non-linear and adaptive control under input saturation consequences. In the present study, we explore the applicability of the generalised sector condition of [31] to the stabilisation of non-linear systems under both input saturation and quantisation non-linearities.

In the following theorem, we provide a condition to attain the stability of the closed-loop system (14) under zero disturbances by determining an appropriate controller gain matrix F subjected to the state non-linearity, dead-zone function, and quantisation effects.

Theorem 1: Consider a non-linear system in (1), satisfying the ellipsoidally Lipschitz condition in Assumption 1 under $w(t) = 0$. There exists a control system (5) that ensures the convergence of steady-state value of $x(t)$ into an ellipsoidal region given by the condition $(\bar{\eta}^{\mathrm{T}} \bar{\eta})^{-1} x^{\mathrm{T}}(t) Y^{-1} x(t) < 1$, for all the initial conditions $x^{\mathrm{T}}(0) Q^{-1} x(0) < 1$, if for matrices Q, Y, M, and N and a diagonal matrix U having appropriate dimensions, the following LMIs are satisfied:

$$Q = Q^{\mathrm{T}} > 0, \quad Y = Y^{\mathrm{T}} > 0, \quad U > 0, \tag{15}$$

$$\begin{bmatrix} Q & Q \\ * & Z \end{bmatrix} \geq 0, \tag{16}$$

$$\begin{bmatrix} Y & \sqrt{\bar{\eta}^{\mathrm{T}}\bar{\eta}}Q \\ * & Q \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} Q & M_{(i)}^{\mathrm{T}} - N_{(i)}^{\mathrm{T}} \\ * & \left(\sigma_{(i)} - \bar{\eta}_{(i)}\right)^2 \end{bmatrix} \geq 0, \tag{18}$$

$$\begin{bmatrix} AQ + QA^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}} & N^{\mathrm{T}} - BU & H & -I & Q & Q\Gamma^{\mathrm{T}} \\ * & -2U & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -Y & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0. \tag{19}$$

The controller gain F can be computed as $F = MQ^{-1}$.

Proof: Consider a Lyapunov function

$$V(t, x) = x^{\mathrm{T}}(t)Px(t), \quad P > 0. \tag{20}$$

The derivative of (20) along (14) becomes

$$\begin{aligned} \dot{V}(t, x) = {} & x^{\mathrm{T}}(t)\left(P(A + BF) + (A + BF)^{\mathrm{T}}P\right)x(t) \\ & + x^{\mathrm{T}}(t)P\Upsilon(t, x) - x^{\mathrm{T}}(t)PB\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & - x^{\mathrm{T}}(t)P\eta(t) + x^{\mathrm{T}}(t)PB_w w(t) + \Upsilon^{\mathrm{T}}(t, x)Px(t) \\ & - PB\aleph_{\mathrm{dz}}(u(t) - \eta(t))^{\mathrm{T}}B^{\mathrm{T}}Px(t) - \eta^{\mathrm{T}}(t)Px(t) \\ & + w^{\mathrm{T}}(t)B_w^{\mathrm{T}}Px(t). \end{aligned} \tag{21}$$

Incorporation of (2) obtains

$$\begin{aligned} \dot{V}(t, x) \leq {} & x^{\mathrm{T}}(t)\left(P(A + BF) + (A + BF)^{\mathrm{T}}P + \Gamma^{\mathrm{T}}\Gamma\right)x(t) \\ & + x^{\mathrm{T}}(t)P\Upsilon(t, x) - x^{\mathrm{T}}(t)PB\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & - x^{\mathrm{T}}(t)P\eta(t) + x^{\mathrm{T}}(t)PB_w w(t) + \Upsilon^{\mathrm{T}}(t, x)Px(t) \\ & - PB\aleph_{\mathrm{dz}}(u(t) - \eta(t))^{\mathrm{T}}B^{\mathrm{T}}Px(t) - \eta^{\mathrm{T}}(t)Px(t) \\ & + w^{\mathrm{T}}(t)B_w^{\mathrm{T}}Px(t) - \Psi^{\mathrm{T}}(t, x)\Psi(t, x). \end{aligned} \tag{22}$$

By setting $\upsilon(t) = u(t) - \eta(t)$ and $\Phi(t) = \eta(t)$, defining $\omega(t) = Gx(t)$ for an arbitrary matrix $G \in R^{q \times n}$, and, further, employing (10), the conditions (6) and (8) imply

$$S(\sigma) = \{-\sigma + \bar{\eta} \leq (F - G)x(t) \leq \sigma - \bar{\eta}\}, \tag{23}$$

$$\aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))W\left[Gx(t) - \aleph_{\mathrm{dz}}(u(t) - \eta(t))\right] \geq 0. \tag{24}$$

Note that $\omega(t) = Gx(t)$ is required to deduce a linear term $(F - G)x(t)$ in (23) for developing a convex region of stability and convex design constraint (18). To tackle the dead-zone function, we employ (22) and (24) and the upper bound on $\dot{V}(t, e)$ can be obtained as

$$\begin{aligned} \dot{V}(t, x) \leq {} & x^{\mathrm{T}}(t)\left(P(A + BF) + (A + BF)^{\mathrm{T}}P + \Gamma^{\mathrm{T}}\Gamma\right)x(t) \\ & + x^{\mathrm{T}}(t)P\Upsilon(t, x) - x^{\mathrm{T}}(t)PB\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & - x^{\mathrm{T}}(t)P\eta(t) + x^{\mathrm{T}}(t)PB_w w(t) + \Upsilon^{\mathrm{T}}(t, x)Px(t) \\ & - \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))B^{\mathrm{T}}Px(t) - \eta^{\mathrm{T}}(t)Px(t) \\ & + w^{\mathrm{T}}(t)B_w^{\mathrm{T}}Px(t) - \Psi^{\mathrm{T}}(t, x)\Psi(t, x) \\ & - 2\aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))W\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & + \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))WGx(t) \\ & + x^{\mathrm{T}}(t)G^{\mathrm{T}}W\aleph_{\mathrm{dz}}(u(t) - \eta(t)). \end{aligned} \tag{25}$$

Let us define

$$J(t, x) = \dot{V}(t, x) + x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t). \tag{26}$$

For $J(t, x) < 0$, the following two cases arise: (i) If $x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t) \geq 0$, $\dot{V}(t, x) < 0$ is implied. It further entails that $V(t, x) < V(0, x(0)) \leq 1$, and for all $x^{\mathrm{T}}(0)Px(0) \leq 1$, the state $x(t)$ converges to the ellipsoidal region $x^{\mathrm{T}}(t)Y^{-1}x(t) < \bar{\eta}^{\mathrm{T}}\bar{\eta}$, because $\eta^{\mathrm{T}}(t)\eta(t) \leq \bar{\eta}^{\mathrm{T}}\bar{\eta}$. (ii) If $x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t) < 0$, the state vector lies in the region $x^{\mathrm{T}}(t)Y^{-1}x(t) < \bar{\eta}^{\mathrm{T}}\bar{\eta}$. In both cases, the region $x^{\mathrm{T}}(t)Px(t) < 1$ remains valid if the ellipsoidal region $\left(\bar{\eta}^{\mathrm{T}}\bar{\eta}\right)^{-1}x^{\mathrm{T}}(t)Y^{-1}x(t) < 1$ is included into $x^{\mathrm{T}}(t)Px(t) < 1$ through $(\bar{\eta}^{\mathrm{T}}\bar{\eta})^{-1}Y^{-1} > P$, which leads to

$$\begin{bmatrix} (\bar{\eta}^{\mathrm{T}}\bar{\eta})^{-1}Y^{-1} & I \\ * & P \end{bmatrix} > 0. \tag{27}$$

It further produces the LMI (17) in Theorem 1 by application of the congruence transformation through $\mathrm{diag}\left(\sqrt{\bar{\eta}^{\mathrm{T}}\bar{\eta}}Y, Q\right)$, where $Q = P^{-1}$. Using (25), (26), and $w(t) = 0$, we have

$$\begin{aligned} J(t, x) \leq {} & x^{\mathrm{T}}(t)\left(P(A + BF) + (A + BF)^{\mathrm{T}}P + \Gamma^{\mathrm{T}}\Gamma\right)x(t) \\ & + x^{\mathrm{T}}(t)P\Upsilon(t, x) - x^{\mathrm{T}}(t)PB\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & - x^{\mathrm{T}}(t)P\eta(t) + \Upsilon^{\mathrm{T}}(t, x)Px(t) \\ & - \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))B^{\mathrm{T}}Px(t) \\ & - \eta^{\mathrm{T}}(t)Px(t) - \Psi^{\mathrm{T}}(t, x)\Psi(t, x) \\ & - 2\aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))W\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & + \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t))WGx(t) \\ & + x^{\mathrm{T}}(t)G^{\mathrm{T}}W\aleph_{\mathrm{dz}}(u(t) - \eta(t)) \\ & + x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t), \end{aligned} \tag{28}$$

which implies

$$J(t, x) \leq \xi_1^{\mathrm{T}}\Lambda_1\xi_1, \tag{29}$$

by substituting $\Upsilon(t, x) = H\Psi(t, x)$, where

$$\xi_1^{\mathrm{T}} = \left[x(t)^{\mathrm{T}} \quad \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t) - \eta(t)) \quad \Psi^{\mathrm{T}}(t, x) \quad \eta^{\mathrm{T}}(t)\right], \tag{30}$$

$$\Lambda_1 = \begin{bmatrix} \Omega_1 & G^{\mathrm{T}}W - PB & PH & -P \\ * & -2W & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix}, \tag{31}$$

where $\Omega_1 = P(A + BF) + (A + BF)^{\mathrm{T}}P + Y^{-1} + \Gamma^{\mathrm{T}}\Gamma$.

To ensure $J(t,x) < 0$, $\Lambda_1 < 0$ is required. Applying the congruence transformation using $\mathrm{diag}(Q, U, I, I)$ and substituting $M = FQ$ and $N = GQ$, where $Q = P^{-1}$ and $U = W^{-1}$, we have

$$\Lambda_2 = \begin{bmatrix} \Omega_2 & N^{\mathrm{T}} - BU & H & -I \\ * & -2U & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{32}$$

where $\Omega_2 = AQ + QA^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}} + QY^{-1}Q + Q\Gamma^{\mathrm{T}}\Gamma Q$. Employing the Schur complement to (32) obtains the LMI (19). To ensure the condition (3) and $S(\sigma)$, the ellipsoidal region of stability, given by $x^{\mathrm{T}}(t)Px(t) < 1$, is included into the regions $x^{\mathrm{T}}Z^{-1}x \le 1$ and $-\sigma + \bar{\eta} \le (F - G)x(t) \le \sigma - \bar{\eta}$ through the constraints

$$P \ge Z^{-1}, \quad P \ge \left(\sigma_{(i)} - \bar{\eta}_{(i)}\right)^{-2}\left(F_{(i)} - G_{(i)}\right)^{\mathrm{T}}\left(F_{(i)} - G_{(i)}\right). \tag{33}$$

The constraints in (33) result in (16) and (18) by means of the Schur complement, congruence transformation (using $\mathrm{diag}(Q, I)$), and substituting $N = GQ$. This completes the proof. □

Remark 4: The local stabilisation condition of Theorem 1 for the non-linear system in (1) has been derived by applying the rigorous local stability and Lyapunov redesign analyses. It is worth mentioning that the stabilisation of locally Lipschitz non-linear systems subject to input saturation and quantisation is an important research area because of the non-linear nature of practical systems, bounded-input constraint of plants, and quantisation of control signals due to digital hardware constraints. Unfortunately, this research problem has not been clearly addressed in the previous studies, such as [22–24, 26], and the present work is an effort to fill this research gap.

The following corollary can be obtained from Theorem 1 by taking $Z = \lim_{z \to \infty} zI$ and $\Gamma = \sqrt{\lambda} I$, where the scalar λ is the Lipschitz constant for a globally Lipschitz non-linearity $\Psi(t,x)$.

Corollary 1: Consider a non-linear system in (1) with $\| \Psi(t,x) - \Psi(t,\bar{x}) \| \le \lambda \| (x - \bar{x}) \|$ under $w(t) = 0$. There exists a control system (5) that ensures convergence $x(t)$ into an ellipsoidal region $(\bar{\eta}^{\mathrm{T}}\bar{\eta})^{-1}x^{\mathrm{T}}(t)Y^{-1}x(t) < 1$, for all initial conditions $x^{\mathrm{T}}(0)Q^{-1}x(0) < 1$, if for matrices Q, Y, M, and N and a diagonal matrix U having appropriate dimensions, the LMIs (15), (17), (18), and

$$\begin{bmatrix} AQ + QA^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}} & N^{\mathrm{T}} - BU & H & -I & Q & Q\sqrt{\lambda} \\ * & -2U & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -Y & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{34}$$

are satisfied. The controller gain F and the matrix G can be computed as $F = MQ^{-1}$ and $G = NQ^{-1}$, respectively.

Proof: Applying $\Gamma = \sqrt{\lambda} I$ to (19) produces (34). It can be validated that (16) is equivalent to $Q \ge 0$ by means of the Schur complement and substituting $Z = \lim_{z \to \infty} zI$. The constraint $Q \ge 0$ can be ignored if (15) is ensured. □

Remark 5: The work on control law derivation for the case of non-linear systems subject to input saturation and quantisation is currently inadequate. The results in Corollary 1 are novel and can be utilised for the state feedback stabilisation of globally Lipschitz non-linear systems subject to input saturation and quantisation non-linearities. The approach of Corollary 1 has been derived as a special case of the main control methodology in Theorem 1, and such control schemes for the class of globally Lipschitz non-linear systems are deficient in the literature.

Remark 6: An equivalent form of the specific result of (34), by application of the congruence transform through $\mathrm{diag}(I, I, I, I, I, \lambda^{-1/2})$, is given by

$$\begin{bmatrix} AQ + QA^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}} & N^{\mathrm{T}} - BU & H & -I & Q & Q \\ * & -2U & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -Y & 0 \\ * & * & * & * & * & -\bar{\lambda} I \end{bmatrix} < 0,$$

where $\bar{\lambda} = \lambda^{-1}$. Similar to the case of [25], this inequality can be employed for locally Lipschitz non-linear systems by minimising $\bar{\lambda}$ to obtain a maximum allowable bound of λ. However, the local region inside which $\Psi(t,x)$ satisfies the Lipschitz condition is not employed. Consequently, the region of stability may not be guaranteed when the allowable value of the Lipschitz constant is violated. Second, maximisation of λ may result in a large gain controller, which can be very sensitive to measurement noises. Therefore, the proposed methodology in Theorem 1 is less conservative than the present result of Corollary 1 and the approach in [25].

Now we provide a controller design condition for the stabilisation of ellipsoidally Lipschitz non-linear systems under saturated input and quantisation constraints to attain robustness against external disturbances and perturbations.

Theorem 2: Consider a non-linear system in (1), satisfying the ellipsoidally Lipschitz condition in Assumption 1 and disturbances bound in Assumption 2. There exists a control system (5) that ensures convergence of $x(t)$ into an ellipsoidal region $(\bar{\eta}^{\mathrm{T}}\bar{\eta} + \delta)^{-1}x^{\mathrm{T}}(t)Y^{-1}x(t) < 1$, for all initial conditions $x^{\mathrm{T}}(0)Q^{-1}x(0) < 1$, if for matrices Q, Y, M, and N and a diagonal matrix U having appropriate dimensions, the LMIs (15), (16), (18), and

$$\begin{bmatrix} Y & \sqrt{\bar{\eta}^{\mathrm{T}}\bar{\eta} + \delta}\, Q \\ * & Q \end{bmatrix} > 0, \tag{35}$$

$$\begin{bmatrix} \Omega_3 & N^{\mathrm{T}} - BU & H & -I & B_w & Q & Q\Gamma^{\mathrm{T}} \\ * & -2U & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -Y & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{36}$$

where $\Omega_3 = AQ + QA^{\mathrm{T}} + BM + M^{\mathrm{T}}B^{\mathrm{T}}$, are satisfied. The controller gain F and the matrix G can be computed as $F = MQ^{-1}$ and $G = NQ^{-1}$, respectively.

Proof: Let us define

$$J_w(t,x) = \dot{V}(t,x) + x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t) - w^{\mathrm{T}}(t)w(t). \tag{37}$$

Similar to the proof of Theorem 1, we can observe that $J(t,x) < 0$ ensures convergence of the vector $x(t)$ to the ellipsoidal region $x^{\mathrm{T}}(t)Y^{-1}x(t) < \bar{\eta}^{\mathrm{T}}\bar{\eta} + \delta$ for the selection of the initial condition from the region $x^{\mathrm{T}}(0)Px(0) \le 1$. By including the region $x^{\mathrm{T}}(t)Y^{-1}x(t) < \bar{\eta}^{\mathrm{T}}\bar{\eta} + \delta$ into $x^{\mathrm{T}}(t)Px(t) < 1$, we obtain (35). Inserting (25) into (37), we have

$$
\begin{aligned}
J_w(t,x) = {} & x^{\mathrm{T}}(t)\big(P(A+BF)+(A+BF)^{\mathrm{T}}P+\Gamma^{\mathrm{T}}\Gamma\big)x(t)\\
& + x^{\mathrm{T}}(t)P\Upsilon(t,x) - x^{\mathrm{T}}(t)PB\aleph_{\mathrm{dz}}(u(t)-\eta(t))\\
& - x^{\mathrm{T}}(t)P\eta(t) + x^{\mathrm{T}}(t)PB_w w(t) + \Upsilon^{\mathrm{T}}(t,x)Px(t)\\
& - \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t)-\eta(t))B^{\mathrm{T}}Px(t) - \eta^{\mathrm{T}}(t)Px(t)\\
& + w^{T}(t)B_w^{T}Px(t) - \Psi^{\mathrm{T}}(t,x)\Psi(t,x)\\
& - 2\aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t)-\eta(t))W\aleph_{\mathrm{dz}}(u(t)-\eta(t))\\
& + \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t)-\eta(t))WGx(t)\\
& + x^{\mathrm{T}}(t)G^{\mathrm{T}}W\aleph_{\mathrm{dz}}(u(t)-\eta(t))\\
& + x^{\mathrm{T}}(t)Y^{-1}x(t) - \eta^{\mathrm{T}}(t)\eta(t) - w^{\mathrm{T}}(t)w(t)\,.
\end{aligned}
\tag{38}
$$

This can be simplified as

$$
J_w(t,x) \le \xi_2^{\mathrm{T}}\Lambda_3\xi_2, \tag{39}
$$

$$
\xi_2^{\mathrm{T}} = \begin{bmatrix} x(t)^{\mathrm{T}} & \aleph_{\mathrm{dz}}^{\mathrm{T}}(u(t)-\eta(t)) & \Psi^{\mathrm{T}}(t,x) & \eta^{\mathrm{T}}(t) & w^{\mathrm{T}}(t)\end{bmatrix}, \tag{40}
$$

$$
\Lambda_3 = \begin{bmatrix}
\Omega_1 & G^{\mathrm{T}}W - PB & PH & -P & PB_w\\
* & -2W & 0 & 0 & 0\\
* & * & -I & 0 & 0\\
* & * & * & -I & 0\\
* & * & * & * & -I
\end{bmatrix}. \tag{41}
$$

The LMI (36) is obtained through congruence transformation, $\mathrm{diag}(Q,U,I,I)$, by taking $M = FQ$, $N = GQ$, $Q = P^{-1}$, and $U = W^{-1}$ and by utilising the Schur complement to $\Lambda_3 < 0$ for validating $J_w(t,x) < 0$. LMIs (15), (16), and (18) are obtained in the same way as observed in the proof of Theorem 1. This completes the proof. □

For a globally Lipschitz non-linear system, the following corollary can be derived from Theorem 2.

Corollary 2: Consider a non-linear plant (1), validating the Lipschitz condition $\| \Psi(t,x) - \Psi(t,\bar{x}) \| \le \lambda \| (x-\bar{x}) \|$ and disturbances bound in Assumption 2. There exists a control system (5) that ensures the convergence of $x(t)$ into an ellipsoidal region $(\bar{\eta}^{\mathrm{T}}\bar{\eta}+\delta)^{-1}x^{\mathrm{T}}(t)Y^{-1}x(t) < 1$, for the initial conditions $x^{\mathrm{T}}(0)Q^{-1}x(0) < 1$, if for matrices Q, Y, M, and N and a diagonal matrix U having appropriate dimensions, the LMIs (15), (18), (35), and

$$
\begin{bmatrix}
\Omega_3 & N^{\mathrm{T}} - BU & H & -I & B_w & Q & Q\sqrt{\lambda}\\
* & -2U & 0 & 0 & 0 & 0 & 0\\
* & * & -I & 0 & 0 & 0 & 0\\
* & * & * & -I & 0 & 0 & 0\\
* & * & * & * & -I & 0 & 0\\
* & * & * & * & * & -Y & 0\\
* & * & * & * & * & * & -I
\end{bmatrix} < 0 \tag{42}
$$

are satisfied. The controller gain F and the matrix G can be computed as $F = MQ^{-1}$ and $G = NQ^{-1}$, respectively.

Remark 7: The stabilisation methodologies provided in Theorem 1 and Corollary 1 cannot be applied to the robust control of non-linear systems under uncertainties and perturbations, which are unavoidable in real-time systems as observed in [34–36]. Compared with the approach in Theorem 1, the proposed scheme in Theorem 2 is more pragmatic owing to its application of disturbance rejection for ellipsoidally Lipschitz non-linear systems under input saturation and quantisation non-linearities. Similarly, the novel approach in Corollary 2, derived from the main control scheme of Theorem 2, is better than Corollary 1, when applied to control constrained globally Lipschitz systems under perturbations. A recent approach in [25] considered the stabilisation of discrete-time non-linear systems in the presence of input saturation and logarithmic quantisation. Contrastingly, our specific result of Corollary 2 supports various forms of quantisers, considers the control of continuous-time non-linear systems, and can be employed to attain robustness against disturbances.

Remark 8: The design conditions provided in Theorems 1 and 2 and Corollaries 1 and 2, obtained through rigorous regional and Lyapunov redesign analysis, can be employed for multi-object control synthesis for non-linear systems by incorporating additional constraints. For instance, to minimise the effect of disturbances and quantisation noise, the ellipsoidal region $(\bar{\eta}^{\mathrm{T}}\bar{\eta}+\delta)^{-1}x^{\mathrm{T}}(t)Y^{-1}x(t) < 1$ can be minimised. One way to reduce the region is to introduce a constraint $Y \le \rho_1 I$ and minimise the scalar ρ_1, where ρ_1 represents an upper bound on the largest eigenvalue of Y. Similarly, the region of stability can be enlarged by maximising the region $x^{\mathrm{T}}(t)Q^{-1}x(t) < 1$ through the maximisation of the minor diameter of $x^{\mathrm{T}}(t)Q^{-1}x(t) = 1$. This can be accomplished by introducing additional LMIs $Q \ge \rho_2 I$ and $\rho_2 > 0$ and by minimising $-\rho_2$, where ρ_2 represents the lower bound on the smallest eigenvalue of Q.

The present study can be further extended for considering the additional control constraints, like state-constraints and actuator faults [37], unknown parameters [38], and overflow constraints on dynamic control systems [39–41], to enhance the scope of the proposed constrained control approach. In addition, further studies can be taken into account for obtaining the necessary and sufficient conditions for the solvability of the controller design constraints under input saturation and quantisation non-linearities.

4 Simulation results

Chua's circuit, demonstrating a variety of non-linear phenomena, has many useful and appealing applications in the fields of chaos, oscillatory behavioural investigation, and secure communication. Consider a modified Chua's circuit [15, 42, 43], given by

$$
\begin{aligned}
A &= \begin{bmatrix} 10/7 & 10 & 0\\ 1 & -1 & 1\\ 0 & -100/7 & 0\end{bmatrix}, \quad \Upsilon(t,x) = \begin{bmatrix} -(20/7)x_1^3\\ 0\\ 0\end{bmatrix},\\
B &= \begin{bmatrix} 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}, \quad B_w = \begin{bmatrix} 0.9 & 0\\ 0 & 0\\ 0 & 0.8\end{bmatrix}.
\end{aligned}
\tag{43}
$$

The Chua's circuit model (43) exhibits complex chaotic behaviour. The non-linearity $\Upsilon(t,x)$ can be expressed as $\Upsilon(t,x) = H\Psi(t,x)$, by selecting

$$
H = [1 \quad 0 \quad 0]^{\mathrm{T}}, \quad \Psi(t,x) = -(20/7)x_1^3. \tag{44}
$$

The function $\Psi(t,x)$ is a cubic non-linearity, which does not validate the conventional global Lipschitz condition.

The existing control and stabilisation approaches in [22–24, 26] are based on linear systems, besides the other reason regarding the type of quantisation. Therefore, these methods fail for the present case of the modified Chua's circuit. The approach in [25] considers the non-linear systems, but it fails for the system in (43) and (44) for two reasons: First, the work of [25] considers a global Lipschitz function $\Psi(t,x)$; however, the non-linear function given by $\Psi(t,x) = -(20/7)x_1^3$ is a local Lipschitz function in the present case. Second, the results of [25] are limited to a specific logarithmic quantiser. However, the present study is independent from the type of quantiser and applicable to all uniform or non-uniform quantisers with bounded quantisation error. Here, we select the uniform quantisation of the control signal for the verification of the present study.

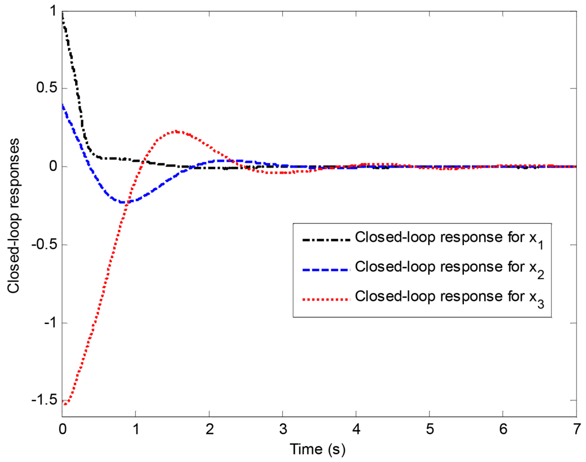

Fig. 1 *Closed-loop response of modified Chua's circuit under saturation, quantisation, and disturbance by application of proposed controller*

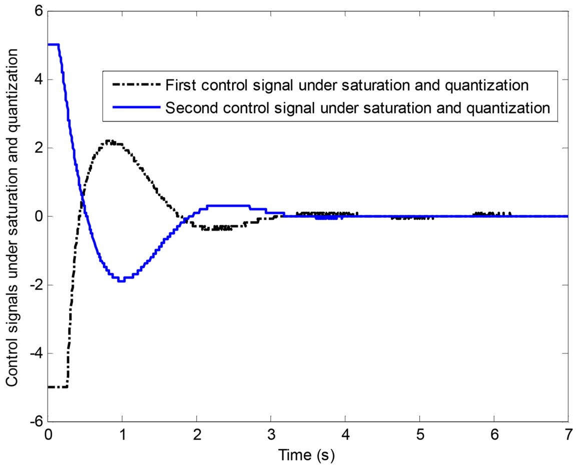

Fig. 2 *Proposed controller response for modified Chua's circuit under saturation, quantisation and disturbance*

Note $\partial\Psi(t,x)/\partial x_1 = -(60/7)x_1^2$. For $x_1 \in [-1 \quad 1]$, the absolute of the maximum value of $\partial\Psi(t,x)/\partial x_1$ gives the locally Lipschitz constant for $\Psi(t, x_1)$ as 60/7. Consequently, we have

$$\begin{aligned} \| \Psi(t,x) - \Psi(t,\bar{x}) \| &\le (60/7)\| (x_1 - \bar{x}_1) \| \\ &+ 0\| (x_2 - \bar{x}_2) \| + 0\| (x_3 - \bar{x}_3) \| . \end{aligned} \tag{45}$$

Therefore, the matrix Γ in (2) can be selected as

$$\Gamma = \begin{bmatrix} \sqrt{60/7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

If we choose

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \tag{47}$$

in the region (3), the condition $x_1 \in [-1 \quad 1]$ remains valid. Hence, $\Psi(t,x)$ is locally Lipschitz in the ellipsoidal region $x, \bar{x} \in \chi^{\mathrm{T}} Z^{-1} \chi \le 1, Z = Z^{\mathrm{T}} > 0$. The saturation non-linearity limits and quantisation error bounds are selected as $\bar{u} = [5 \quad 5]^{\mathrm{T}}$ and $\bar{\eta} = [0.05 \quad 0.05]^{\mathrm{T}}$, respectively. By application of the proposed methodology in Theorem 2, the controller gain matrix for $\delta = 0.5$ is computed as

$$F = \begin{bmatrix} -12.671 & -11.624 & -0.442 \\ 1.472 & 9.388 & -1.215 \end{bmatrix}. \tag{48}$$

To test the proposed control scheme for the modified Chua's circuit (43), we select the disturbance as $d(t) = [0.1\sin 220t \quad 0.3\sin 180t]^{\mathrm{T}}$. The states of the closed-loop response under input saturation, quantisation, and disturbance are plotted in Fig. 1. The corresponding control signal under the saturation and quantisation constraints is provided in Fig. 2. The states of the Chua's circuit are converging under the effects of the non-linearities and disturbance, which demonstrates the reliability of the proposed methods under the non-linear constraints and robustness against disturbances. The control signals are well recovered from the saturation for stabilisation of the Chua's circuit. The effect of quantisation is observed on the control signal, which adjusts itself for the stabilisation of the states. Hence, the resultant control strategy can be used for attaining the objective of stabilising locally Lipschitz non-linear systems under the non-linear effects of input saturation and quantisation and the undesirable effects of disturbances.

5 Conclusions

This paper studied stabilisation controller design for non-linear systems containing ellipsoidally Lipschitz non-linearities by incorporating the bounded quantisation error and input saturation. A sufficient controller design condition based on convex routines was established by employing the bound on quantisation non-linearity, a generalised local sector condition for saturation, Lyapunov stability theory, regional analysis, and Lyapunov redesign. This condition for the input saturation-constrained non-linear system under quantisation was further extended to attain robustness against external perturbations. Novel results for controlling the quantised and constrained non-linear plants by considering the globally Lipschitz non-linearities in the absence and presence of disturbances were inferred from the main results as particular scenarios. In contrast to the conventional approaches, the resultant methodologies can be applied to locally Lipschitz non-linear systems and guarantee a precise estimate of the region of stability. Further, the proposed methods can be used for various quantisation schemes and ensure robustness against quantisation noise and external perturbations. A numerical simulation study on the control of a modified Chua's circuit by incorporating control signal quantisation and saturation was demonstrated to show the effectiveness of the proposed methodologies. Further work is required for the analysis of the design constraints, for the incorporation of state constraints, to consider adaptive controllers, and to include finite word-length effects.

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