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Adaptive controller design for lag-synchronization of two non-identical time-delayed chaotic systems with unknown parameters

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ABSTRACT

In this Letter, two adaptive controllers are proposed for the lag-synchronization of two non-identical time-delayed chaotic systems with fully unknown parameters. Based on Lyapunov-stability theorem and adaptive techniques, sufficient conditions for the lag-synchronization of these two systems are discussed. Finally, illustrative examples are given to verify the validity of the developed controllers.

1. Introduction

During last two decades, synchronization of chaotic systems has become one of the most interesting topics to engineering and science communities since the pioneering work of Pecora and Caroll [1]. It has been used in a variety of fields such as chemical reactions, secure communities, etc. Due to these applications, various control methods have been proposed for synchronization of chaotic systems such as active control [2], backstepping design [3], adaptive control [4–11], fuzzy control [12–14], sliding mode control [15–17], etc.

Most of the methods mentioned above synchronize two identical chaotic systems, but some often in real life applications, in cases such as laser array, biological systems, it is hardly the case that every component of the drive and response can be assumed to be identical. Most of these systems have model uncertainties, so one can expect that the chaotic systems can be represented by non-identical model [6–10].

In practice, some system's parameters cannot be exactly known in advance or some of them are completely unknown. The synchronization will be destroyed with the effects of these uncertainties. So, the design of adaptive controller for the synchronization of chaotic system with unknown parameters is an important issue [6–11].

It is well known that time-delay characteristics are frequently encountered in the most of engineering systems and delayed states appear in the dynamic of many physical and biological systems. These systems are difficult to achieve satisfactory performance. So, the stability issue of time-delay systems is of practical importance. Since Macky and Glass first found chaos in time-delayed systems [18], the time-delay chaotic systems have received more attention [13,19–22,26]. These systems can be used in secure communication systems based on chaotic systems. But, most existing studies focus on chaotic systems without time-delay states [1–17].

From the view point of engineering applications and characteristics of channels, due to signal propagation delays in the environment, it is reasonable to require the slave system at time (t) to synchronize the master system at time $(t-\tau)$, where τ is the propagation delay or channel time-delay. This kind of synchronization is called lag-synchronization [21–26].

Motivated by the above discussions, the aim of this Letter is to design a simple adaptive controller to synchronize the master system with the slave system, both with time-delay and fully unknown parameters in the presence of channel time-delay. Based on Lyapunov theory, the stability of the adaptive controller is proved.

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In addition, by the use of proper Lyapunov–Krasovskii functional and adaptive techniques, sufficient conditions for lag-synchronization of these two systems with unknown time-delay are achieved.

This Letter is organized as follows. In Section 2, the problem statement, drive-response scheme and some preliminaries are presented. Based on Lyapunov method, two adaptive controllers for the lag-synchronization of two mentioned systems are designed in Section 3. In Section 4, to show the effectiveness of the proposed methods, numerical examples are provided. Finally, concluding remarks are given in Section 5.

2. System description and problem formulation

In this section, some notations and definitions are firstly reviewed. $\|-\|_p$ denotes p-norm of the vector and $\|-\|$ represents the Euclidean norm of vectors and the norm of matrices. diagonal$(W_1, W_2, \ldots, W_m)$ denotes an $m \times m$ diagonal matrix with W_i $(i = 1, 2, \ldots, m)$ on its main diagonal. W^T represents the transpose of W. It is obvious that for any $W \in \Re^n$, we have $\|W\| \leqslant \|W\|_1$ [27].

A class of chaotic systems is described by the following differential equations:

$$\dot{x} = f_1(x) + F_1(x)\theta_1 + G_1\big(x(t - d_1)\big)\theta_2, \tag{1}$$

where $x \in R^n$ denotes the state vector, $f_1 : R^n \to R^n$, $F_1 : R^n \to R^{n\times m_1}$ and $G_1 : R^n \to R^{n\times q_1}$ are continuous nonlinear function matrices. $\theta_1 \in R^{m_1}$ and $\theta_2 \in R^{q_1}$ are the unknown parameter vectors. $d_1 > 0$ is the time-delay of the system (1). System (1) is called the drive system.

The response system is given as follows:

$$\dot{y} = f_2(y) + F_2(y)\beta_1 + G_2\big(y(t - d_2)\big)\beta_2 + u, \tag{2}$$

where $y \in R^n$ denotes the state vector, $f_2 : R^n \to R^n$, $F_2 : R^n \to R^{n\times m_2}$ and $G_2 : R^n \to R^{n\times q_2}$ are continuous nonlinear function matrices. $\beta_1 \in R^{m_1}$ and $\beta_2 \in R^{q_1}$ are the unknown parameter vectors of the system. $d_2 > 0$ is a time-delay. u is the control input vector. Many time-delay chaotic systems investigated are in the form of system (1) and system (2).

Definition 1. System (1) and system (2) are lag-synchronized if there exists a scalar $\tau > 0$ such that the states of two systems are nearly identical, but one system lags in time to the other, i.e. $y(t) \cong x(t - \tau)$ with positive τ.

The control objective is to design the control input u such that lag-synchronization between system (1) and system (2) is achieved.

$$\|e\| = \big\|x(t - \tau) - y(t)\big\| \to 0 \quad \text{as } t \to \infty. \tag{3}$$

τ is the channel propagation delay or channel time-delay.

3. Controller design for lag-synchronization

In this section, two adaptive controllers have been proposed for lag-synchronizing systems (1) and (2) in the two cases:

1) The time-delays of system (1) and system (2), d_1 and d_2, are known a priori.
2) The time-delays of system (1) and system (2), d_1 and d_2, are unknown.

3.1. Controller design for systems with known time-delay

The channel propagation delay is considered as a function of time such that $\tau(t) \geqslant 0$ for all time t and $|\dot{\tau}| \leqslant \vartheta < \infty$ where ϑ is an unknown positive constant.

From (1), we have

$$\dot{x}\big(t - \tau(t)\big) = \big(1 - \dot{\tau}(t)\big)\big[f_1\big(x\big(t - \tau(t)\big)\big) + F_1\big(x\big(t - \tau(t)\big)\big)\theta_1 + G_1\big(x\big(t - d_1 - \tau(t)\big)\big)\theta_2\big]. \tag{4}$$

Subtracting (4) from (2), the following error dynamic is formed:

$$\begin{aligned} \dot{e} &= \dot{y}(t) - \dot{x}\big(t - \tau(t)\big) \\ &= f_2(y) + F_2(y)\beta_1 + G_2\big(y(t - d)\big)\beta_2 - \big(1 - \dot{\tau}(t)\big)\big[f_1\big(x\big(t - \tau(t)\big)\big) + F_1\big(x\big(t - \tau(t)\big)\big)\theta_1 + G_1\big(x\big(t - d_1 - \tau(t)\big)\big)\theta_2\big] + u. \end{aligned} \tag{5}$$

Theorem 1. *Response system (2) is synchronized with drive system (1) with propagation delay $\tau(t)$, if the controller is chosen as:*

$$\begin{aligned} u = {} & -Ke - f_2(y) + \big(1 - \dot{\tau}(t)\big)f_1\big(x\big(t - \tau(t)\big)\big) - F_2(y)\bar{\beta}_1 + \big(1 - \dot{\tau}(t)\big)F_1\big(x\big(t - \tau(t)\big)\big)\bar{\theta}_1 - G_2\big(y(t - d_2)\big)\bar{\beta}_2 \\ & + \big(1 - \dot{\tau}(t)\big)G_1\big(x\big(t - d_1 - \tau(t)\big)\big)\bar{\theta}_2, \end{aligned} \tag{6}$$

and the update laws are chosen as:

$$\dot{\bar{\theta}}_1 = -k_1\big(1 - \dot{\tau}(t)\big)F_1^T\big(x\big(t - \tau(t)\big)\big)e, \tag{7}$$

$$\dot{\bar{\theta}}_2 = -k_2\big(1 - \dot{\tau}(t)\big)G_1^T\big(x\big(t - d_1 - \tau(t)\big)\big)e, \tag{8}$$

$$\dot{\bar{\beta}}_1 = k_3F_2^T\big(y(t)\big)e, \tag{9}$$

$$\dot{\bar{\beta}}_2 = k_4 G_2^T\big(y(t-d_2)\big)e, \tag{10}$$

$$\dot{k}_j = a_j e_j^2, \quad j = 1, 2, \ldots, n, \tag{11}$$

in which k_i, $i = 1, 2, \ldots, 4$, *and* a_j, $j = 1, 2, \ldots, n$, *are positive constants and* $K = \mathrm{diagonal}(k_1, k_2, \ldots, k_n)$.

Proof. Define a Lyapunov function in the form of:

$$V = \frac{1}{2}e^T e + \frac{1}{2k_1}(\theta_1 - \bar{\theta}_1)^T(\theta_1 - \bar{\theta}_1) + \frac{1}{2k_2}(\theta_2 - \bar{\theta}_2)^T(\theta_2 - \bar{\theta}_2) + \frac{1}{2k_3}(\beta_1 - \bar{\beta}_1)^T(\beta_1 - \bar{\beta}_1) + \frac{1}{2k_4}(\beta_2 - \bar{\beta}_2)^T(\beta_2 - \bar{\beta}_2) + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{a_j}\big(k_j - k^*\big)^2, \tag{12}$$

where k^* is a positive scalar.

Differentiating the Lyapunov function along the trajectory of the error system (5) results in:

$$\dot{V} = e^T\Big(f_2(y) + F_2(y)\beta_1 + G_2\big(y(t-d_2)\big)\beta_2 - \big(1 - \dot{\tau}(t)\big)\big[f_1\big(x(t-\tau(t))\big) + F_1\big(x(t-\tau(t))\big)\theta_1 + G_1\big(x(t-d_1-\tau(t))\big)\theta_2\big] + u\Big) - \frac{1}{k_1}\dot{\bar{\theta}}_1^T(\theta_1 - \bar{\theta}_1) - \frac{1}{k_2}\dot{\bar{\theta}}_2^T(\theta_2 - \bar{\theta}_2) - \frac{1}{k_3}\dot{\bar{\beta}}_1^T(\beta_1 - \bar{\beta}_1) - \frac{1}{k_4}\dot{\bar{\beta}}_2^T(\beta_2 - \bar{\beta}_2) + \sum_{j=1}^{n}\frac{\dot{k}_j}{a_j}\big(k_j - k^*\big). \tag{13}$$

Substituting controller (6) into the above equality, we can obtain

$$\dot{V} = e^T\Big(-Ke + F_2(y)(\beta_1 - \bar{\beta}_1) + G_2\big(y(t-d_2)\big)(\beta_2 - \bar{\beta}_2) - \big(1 - \dot{\tau}(t)\big)\big[F_1\big(x(t-\tau(t))\big)(\theta_1 - \bar{\theta}_1) + G_1\big(x(t-d_1-\tau(t))\big)(\theta_2 - \bar{\theta}_2)\big]\Big) - \frac{1}{k_1}\dot{\bar{\theta}}_1^T(\theta_1 - \bar{\theta}_1) - \frac{1}{k_2}\dot{\bar{\theta}}_2^T(\theta_2 - \bar{\theta}_2) - \frac{1}{k_3}\dot{\bar{\beta}}_1^T(\beta_1 - \bar{\beta}_1) - \frac{1}{k_4}\dot{\bar{\beta}}_2^T(\beta_2 - \bar{\beta}_2) + \sum_{j=1}^{n}\frac{\dot{k}_j}{a_j}\big(k_j - k^*\big). \tag{14}$$

Using adaptive laws (7)–(11) in (14) yields

$$\dot{V} = -e^T k^* e \leqslant 0. \tag{15}$$

According to Barbalat's lemma [27], the error trajectory of dynamic model (5) will asymptotically converge to the origin. □

Remark 1. The control strength K is automatically adapted to a suitable strength depending on the initial values, which is significantly different from the usual linear feedback. In the usual linear feedback scheme a fixed strength is used no matter where the initial values start, thus the strength must be maximal, which means a kind of waste in practice. However, the final strength in the present method depends on the initial error, thus the strength must be of the lower order than those used in the constant gain schemes in the same idea with [28].

Let

$$-\big(1 - \dot{\tau}(t)\big)F_1\big(x(t-\tau(t))\big) = (h_1, h_2, \ldots, h_{m_1}), \tag{16}$$

$$-\big(1 - \dot{\tau}(t)\big)G_1\big(x(t-d_1-\tau(t))\big) = (h_{m_1+1}, h_{m_1+2}, \ldots, h_{m_1+q_1}), \tag{17}$$

$$F_2\big(x(t-\tau(t))\big) = (h_{m_1+q_1+1}, h_{m_1+q_1+2}, \ldots, h_{m_1+q_1+m_2}), \tag{18}$$

$$G_2\big(x(t-d_2-\tau(t))\big) = (h_{m_1+q_1+m_2+1}, h_{m_1+q_1+m_2+2}, \ldots, h_{m_1+q_1+m_2+q_2}), \tag{19}$$

where h_k $(k = 1, \ldots, m_1 + q_1 + m_2 + q_2)$ are vectors with n components.

Remark 2. From proof of Theorem 1, we know that lag-synchronization error will converge to zero. From adaptive laws (7)–(10) we can see $\dot{\bar{\theta}}_1 = \dot{\bar{\theta}}_2 = \dot{\bar{\beta}}_1 = \dot{\bar{\beta}}_2 = 0$, when $e(t) = 0$, which means that $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\beta}_1$ and $\bar{\beta}_2$ approach to some constants and this doesn't elaborate that $\bar{\theta}_1 \to \theta_1$, $\bar{\theta}_2 \to \theta_2$, $\bar{\beta}_1 \to \beta_1$ and $\bar{\beta}_2 \to \beta_2$.

Remark 3. If h_k $(k = 1, \ldots, m_1 + q_1 + m_2 + q_2)$ are linearly independent on the lag-synchronization manifold $y(t) = x(t - \tau(t))$, then $\lim_{t\to\infty}(\bar{\theta}_1 - \theta_1) = \lim_{t\to\infty}(\bar{\theta}_2 - \theta_2) = \lim_{t\to\infty}(\bar{\beta}_1 - \beta_1) = \lim_{t\to\infty}(\bar{\beta}_2 - \beta_2) = 0$.

Proof. On the lag-synchronization manifold from (5) and (6), we have

$$-\big(1 - \dot{\tau}(t)\big)F_1\big(x(t-\tau(t))\big)(\theta_1 - \bar{\theta}_1) - \big(1 - \dot{\tau}(t)\big)G_1\big(x(t-d_1-\tau(t))\big)(\theta_2 - \bar{\theta}_2) + F_2\big(x(t-\tau(t))\big)(\beta_1 - \bar{\beta}_1) + G_2\big(x(t-d_2-\tau(t))\big)(\beta_2 - \bar{\beta}_2) = 0. \tag{20}$$

Then, we can derive

$$\sum_{k=1}^{m_1} h_k(\theta_{1_k} - \bar{\theta}_{1_k}) + \sum_{k=m_1}^{m_1+q_1} h_k(\theta_{2_k} - \bar{\theta}_{2_k}) + \sum_{k=m_1+q_1}^{m_1+q_1+m_2} h_k(\beta_{1_k} - \bar{\beta}_{1_k}) + \sum_{k=m_1+q_1+m_2}^{m_1+q_1+m_2+q_2} h_k(\beta_{2_k} - \bar{\beta}_{2_k}) = 0. \tag{21}$$

Since h_k $(k = 1, \ldots, m_1 + q_1 + m_2 + q_2)$ are linearly independent on the lag-synchronization manifold, the above equality holds if and only if $\theta_{1_k} = \bar{\theta}_{1_k}$, $k = 1, \ldots, m_1$, $\theta_{2_i} = \bar{\theta}_{2_i}$, $i = 1, \ldots, q_1$, $\beta_{1_j} = \bar{\beta}_{1_j}$, $j = 1, \ldots, m_2$, and $\beta_{2_z} = \bar{\beta}_{2_z}$, $z = 1, \ldots, q_2$, where $\theta_1 = [\theta_{1_1}, \theta_{1_2}, \ldots, \theta_{1_{m_1}}]^T$, $\theta_2 = [\theta_{2_1}, \theta_{2_2}, \ldots, \theta_{2_{q_1}}]^T$, $\beta_1 = [\beta_{1_1}, \beta_{1_2}, \ldots, \beta_{1_{m_2}}]^T$ and $\beta_2 = [\beta_{2_1}, \beta_{2_2}, \ldots, \beta_{2_{q_2}}]^T$. □

Condition expressed in Remark 2 is indeed the richness condition which is a necessary condition for identification algorithms.

Corollary 1. *Let the adaptive controller and adaptive laws be as follows*:

$$u = -Ke - f_2(y) + f_1\big(x(t-\tau)\big) - F_2(y)\bar{\beta}_1 + F_1\big(x(t-\tau)\big)\bar{\theta}_1 - G_2\big(y(t-d_2)\big)\bar{\beta}_2 + G_1\big(x(t-d_1-\tau)\big)\bar{\theta}_2, \tag{22}$$

$$\dot{\bar{\theta}}_1 = -k_1 F_1^T\big(x(t-\tau)\big)e, \tag{23}$$

$$\dot{\bar{\theta}}_2 = -k_2 G_1^T\big(x(t-d_1-\tau)\big)e, \tag{24}$$

$$\dot{\bar{\beta}}_1 = k_3 F_2^T\big(y(t)\big)e, \tag{25}$$

$$\dot{\bar{\beta}}_2 = k_4 G_2^T\big(y(t-d_2)\big)e, \tag{26}$$

$$\dot{k}_j = a_j e_j^2, \quad j = 1, 2, \ldots, n, \tag{27}$$

where k_i, $i = 1, 2, \ldots, 4$, and a_j, $j = 1, 2, \ldots, n$, are positive constants and $K = \mathrm{diagonal}(k_1, k_2, \ldots, k_n)$. Then, the lag-synchronization of system (1) and system (2) with constant time-lag $\tau > 0$ is achieved.

Proof. Let $\dot{\tau} = 0$ in Theorem 1. □

3.2. Controller design for systems with unknown time-delay

In this sub-section, we will construct an adaptive controller to lag-synchronized (1) and (2), when the time-delays of these systems are unknown.

In this case, system (1) and system (2) require satisfying the following assumptions.

Assumption 1. The nonlinear sections G_1 and G_2 satisfy the following inequalities,

$$\big\|G_1\big(x(t-\alpha_1)\big) - G_1\big(x(t-\alpha_2)\big)\big\| \leqslant l_1 \big\|x(t-\alpha_1) - x(t-\alpha_2)\big\|, \tag{28}$$

$$\big\|G_2\big(x(t-\alpha_1)\big) - G_2\big(x(t-\alpha_2)\big)\big\| \leqslant l_2 \big\|x(t-\alpha_1) - x(t-\alpha_2)\big\|, \tag{29}$$

where α_1, α_2, l_1 and l_2 are positive constants. This assumption is like Lipschitz condition.

Assumption 2. The unknown parameters θ_2 and β_2 are norm bounded by two unknown positive scalars, δ_{θ_2} and δ_{β_2}, respectively.

Theorem 2. *If the adaptive controller is chosen as*

$$\begin{aligned} u = {} & -Ke - \Lambda\,\mathrm{sign}(e) - f_2(y) + \big(1-\dot{\tau}(t)\big) f_1\big(x\big(t-\tau(t)\big)\big) - F_2(y)\bar{\beta}_1 + \big(1-\dot{\tau}(t)\big)F_1\big(x\big(t-\tau(t)\big)\big)\bar{\theta}_1 - G_2\big(y(t)\big)\bar{\beta}_2 \\ & + \big(1-\dot{\tau}(t)\big)G_1\big(x\big(t-\tau(t)\big)\big)\bar{\theta}_2, \end{aligned} \tag{30}$$

and the update laws are chosen as:

$$\dot{\bar{\theta}}_1 = -k_1\big(1-\dot{\tau}(t)\big)F_1^T\big(x\big(t-\tau(t)\big)\big)e, \tag{31}$$

$$\dot{\bar{\theta}}_2 = -k_2\big(1-\dot{\tau}(t)\big)G_1^T\big(x\big(t-\tau(t)\big)\big)e, \tag{32}$$

$$\dot{\bar{\beta}}_1 = k_3 F_2^T\big(y(t)\big)e, \tag{33}$$

$$\dot{\bar{\beta}}_2 = k_4 G_2^T\big(y(t)\big)e, \tag{34}$$

$$\dot{k}_j = a_j e_j^2,\ j = 1, 2, \ldots, n, \tag{35}$$

$$\dot{\gamma}_j = b_j |e_j|,\ j = 1, 2, \ldots, n, \tag{36}$$

where k_i, $i = 1, 2, \ldots, 4$, a_j, $j = 1, 2, \ldots, n$, and b_j, $j = 1, 2, \ldots, n$, are positive constants, $K = \mathrm{diagonal}(k_1, k_2, \ldots, k_n)$ and $\Lambda = \mathrm{diagonal}(\gamma_1, \gamma_2, \ldots, \gamma_n)$, then lag-synchronization occurs between system (1) and system (2) with channel propagation $\tau(t) \geqslant 0$.

Proof. Choose a candidate Lyapunov function as follows

$$V = \frac{1}{2}e^T e + \frac{1}{2k_1}(\theta_1 - \bar{\theta}_1)^T(\theta_1 - \bar{\theta}_1) + \frac{1}{2k_2}(\theta_2 - \bar{\theta}_2)^T(\theta_2 - \bar{\theta}_2) + \frac{1}{2k_3}(\beta_1 - \bar{\beta}_1)^T(\beta_1 - \bar{\beta}_1)$$
$$+ \frac{1}{2k_4}(\beta_2 - \bar{\beta}_2)^T(\beta_2 - \bar{\beta}_2) + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{a_j}\left(k_j - k^*\right)^2 + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{b_j}\left(\gamma_j - \gamma^*\right)^2 + \frac{1}{2}\int_t^{t-d_2}\left\|e(\lambda)\right\|^2 d\lambda, \tag{37}$$

where k^* and γ^* are positive constants to be determined.

Then, the derivative of V is

$$\dot{V} = e^T\big(f_2(y) + F_2(y)\beta_1 + G_2\big(y(t-d_2)\big)\beta_2 - \big(1-\dot{\tau}(t)\big)\big[f_1\big(x\big(t-\tau(t)\big)\big) + F_1\big(x\big(t-\tau(t)\big)\big)\theta_1$$
$$+ G_1\big(x\big(t-d_1-\tau(t)\big)\big)\theta_2\big] + u\big) - \frac{1}{k_1}\dot{\bar{\theta}}_1^T(\theta_1 - \bar{\theta}_1) - \frac{1}{k_2}\dot{\bar{\theta}}_2^T(\theta_2 - \bar{\theta}_2) - \frac{1}{k_3}\dot{\bar{\beta}}_1^T(\beta_1 - \bar{\beta}_1)$$
$$- \frac{1}{k_4}\dot{\bar{\beta}}_2^T(\beta_2 - \bar{\beta}_2) + \sum_{j=1}^{n}\frac{\dot{k}_j}{a_j}\left(k_j - k^*\right) + \sum_{j=1}^{n}\frac{\dot{\gamma}_j}{b_j}\left(\gamma_j - \gamma^*\right) + \frac{1}{2}\left\|e(t)\right\|^2 - \frac{1}{2}\left\|e(t-d_2)\right\|^2. \tag{38}$$

Substituting controller (30) into (38), it can be derived that

$$\dot{V} = e^T\big(F_2(y)(\beta_1 - \bar{\beta}_1) + \big(G_2\big(y(t-d_2)\big) - G_2\big(y(t)\big)\big)\beta_2 + G_2\big(y(t)\big)(\beta_2 - \bar{\beta}_2) - \big(1-\dot{\tau}(t)\big)\big[F_1\big(x\big(t-\tau(t)\big)\big)(\theta_1 - \bar{\theta}_1)$$
$$+ \big(G_1\big(x\big(t-d_1-\tau(t)\big)\big) - G_1\big(x\big(t-\tau(t)\big)\big)\big)\theta_2 + G_1\big(x\big(t-\tau(t)\big)\big)(\theta_2 - \bar{\theta}_2)\big] - Ke - \Lambda\,\mathrm{sign}(e)\big) - \frac{1}{k_1}\dot{\bar{\theta}}_1^T(\theta_1 - \bar{\theta}_1)$$
$$- \frac{1}{k_2}\dot{\bar{\theta}}_2^T(\theta_2 - \bar{\theta}_2) - \frac{1}{k_3}\dot{\bar{\beta}}_1^T(\beta_1 - \bar{\beta}_1) - \frac{1}{k_4}\dot{\bar{\beta}}_2^T(\beta_2 - \bar{\beta}_2) + \sum_{j=1}^{n}\frac{\dot{k}_j}{a_j}\left(k_j - k^*\right) + \sum_{j=1}^{n}\frac{\dot{\gamma}_j}{b_j}\left(\gamma_j - \gamma^*\right) + \frac{1}{2}\left\|e(t)\right\|^2$$
$$- \frac{1}{2}\left\|e(t-d_2)\right\|^2. \tag{39}$$

It is noted that the condition $\|x(t)\| \leqslant q < \infty$ can be easily satisfied since one of the properties of chaotic system is that the chaotic trajectory is limitary where q is an unknown and sufficiently constant.

From Assumptions 1 and 2, the following inequalities hold

$$\begin{aligned}
&\big(1-\dot{\tau}(t)\big)e^T\big(G_1\big(x\big(t-d_1-\tau(t)\big)\big) - G_1\big(x\big(t-\tau(t)\big)\big)\big)\theta_2 \\
&\quad\leqslant \left|1-\dot{\tau}(t)\right|\|e\|\left\|G_1\big(x\big(t-d_1-\tau(t)\big)\big) - G_1\big(x\big(t-\tau(t)\big)\big)\right\|\|\theta_2\| \\
&\quad\leqslant l_1\delta_{\theta_2}\left|1-\dot{\tau}(t)\right|\|e\|\left\|x\big(t-d_1-\tau(t)\big) - x\big(t-\tau(t)\big)\right\| \leqslant l_1\delta_{\theta_2}\left|1-\dot{\tau}(t)\right|q\|e\|_1 \leqslant l_1\delta_{\theta_2}(1+\vartheta)q\|e\|_1,
\end{aligned} \tag{40}$$

$$\begin{aligned}
&e^T\big(G_2\big(y(t-d_2)\big) - G_2\big(y(t)\big)\big)\beta_2 \\
&\quad\leqslant \|e\|\left\|G_2\big(y(t-d_2)\big) - G_2\big(y(t)\big)\right\|\|\beta_2\| \leqslant l_2\delta_{\beta_2}\|e\|\left\|y(t-d_2) - y(t)\right\| \\
&\quad\leqslant l_2\delta_{\beta_2}\|e\|\left\|x(t-d_2) - x(t)\right\| + l_2\delta_{\beta_2}\|e\|\left\|e(t-d_2)\right\| + l_2\delta_{\beta_2}\|e\|^2 \\
&\quad\leqslant l_2\delta_{\beta_2}q\|e\|_1 + \left(\frac{l_2^2\delta_{\beta_2}^2}{2} + l_2\delta_{\beta_2}\right)\|e\|^2 + \frac{1}{2}\left\|e(t-d_2)\right\|^2.
\end{aligned} \tag{41}$$

Using inequalities (40)–(41) and adaptive laws (31)–(34) in (39), we further have

$$\dot{V} \leqslant l_1\delta_{\theta_2}(1+\vartheta)q\|e\|_1 + l_2\delta_{\beta_2}q\|e\|_1 + \left(\frac{l_2^2\delta_{\beta_2}^2}{2} + l_2\delta_{\beta_2} + 1\right)\|e\|^2 - k\|e\|^2 - \gamma\|e\|_1 + \sum_{j=1}^{n}\frac{\dot{k}_j}{a_j}\left(k_j - k^*\right)$$
$$+ \sum_{j=1}^{n}\frac{\dot{\gamma}_j}{b_j}\left(\gamma_j - \gamma^*\right) - \frac{1}{2}\|e\|^2. \tag{42}$$

Here, the k^* and γ^* in equality (31) are selected as

$$k^* = \frac{l_2^2\delta_{\beta_2}^2}{2} + l_2\delta_{\beta_2} + 1, \tag{43}$$

$$\gamma^* = l_1\delta_{\theta_2}(1+\vartheta)q + l_2\delta_{\beta_2}q. \tag{44}$$

If the adaptation laws for k and γ are chosen by (35) and (36) respectively, inequality (42) becomes

$$\dot{V} \leqslant -\frac{1}{2}e^T e \leqslant 0. \tag{45}$$

Based on Lyapunov-stability theory and Barbalat's lemma, the proposed controller will guarantee that error system (5) is asymptotically stable. □

Remark 4. In the control law (30) signum function (sign(e)) appears which leads to chattering phenomenon in practical applications. In order to handle this undesirable phenomenon, instead of signum function (sign(e)), high slope saturation function (sat($\frac{e}{\delta}$)) can be used where δ is a small positive constant or signum function can be replaced by tanh(εe) function (ε is a big positive constant).

Corollary 2. *If the adaptive controller and updated algorithm are selected as*

$$u = -Ke - \Lambda\,\mathrm{sign}(e) - f_2(y) + f_1\bigl(x(t-\tau)\bigr) - F_2(y)\bar{\beta}_1 + F_1\bigl(x(t-\tau)\bigr)\bar{\theta}_1 - G_2\bigl(y(t)\bigr)\bar{\beta}_2 + G_1\bigl(x(t-\tau)\bigr)\bar{\theta}_2, \tag{46}$$

and the update laws are chosen as:

$$\dot{\bar{\theta}}_1 = -k_1 F_1^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)e, \tag{47}$$

$$\dot{\bar{\theta}}_2 = -k_2 G_1^T\bigl(x\bigl(t-\tau(t)\bigr)\bigr)e, \tag{48}$$

$$\dot{\bar{\beta}}_1 = k_3 F_2^T\bigl(y(t)\bigr)e, \tag{49}$$

$$\dot{\bar{\beta}}_2 = k_4 G_2^T\bigl(y(t)\bigr)e, \tag{50}$$

$$\dot{k}_j = a_j e_j^2, \quad j = 1, 2, \ldots, n, \tag{51}$$

$$\dot{\gamma}_j = b_j |e_j|, \quad j = 1, 2, \ldots, n, \tag{52}$$

where k_i, $i = 1, 2, \ldots, 4$, a_j, $j = 1, 2, \ldots, n$, and b_j, $j = 1, 2, \ldots, n$, are positive constants, $K = \mathrm{diagonal}(k_1, k_2, \ldots, k_n)$ and $\Lambda = \mathrm{diagonal}(\gamma_1, \gamma_2, \ldots, \gamma_n)$, then synchronization error dynamical system (5) is asymptotically stable at the origin with a constant time-lag $\tau \geqslant 0$.

Proof. Let $\dot{\tau} = 0$, we can get Corollary 2 easily from Theorem 2. We omit it. □

4. Numerical example

In this section, two examples will be simulated to illustrate the performance of the proposed controllers in chaos lag-synchronization.

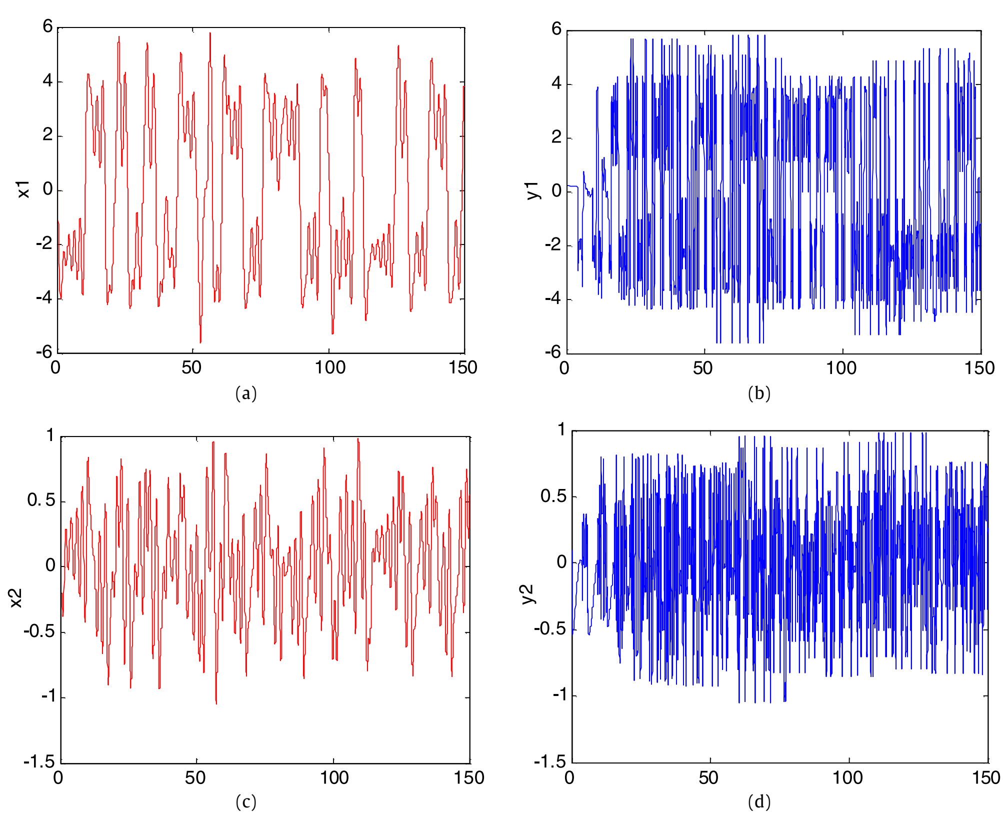


Fig. 1. State trajectories of drive, response and error systems: (a) $x_1(t)$; (b) $y_1(t)$; (c) $x_2(t)$; (d) $y_2(t)$; (e) $x_3(t)$; (f) $y_3(t)$; (g) $e_1(t)$; (h) $e_2(t)$ and (i) $e_3(t)$.

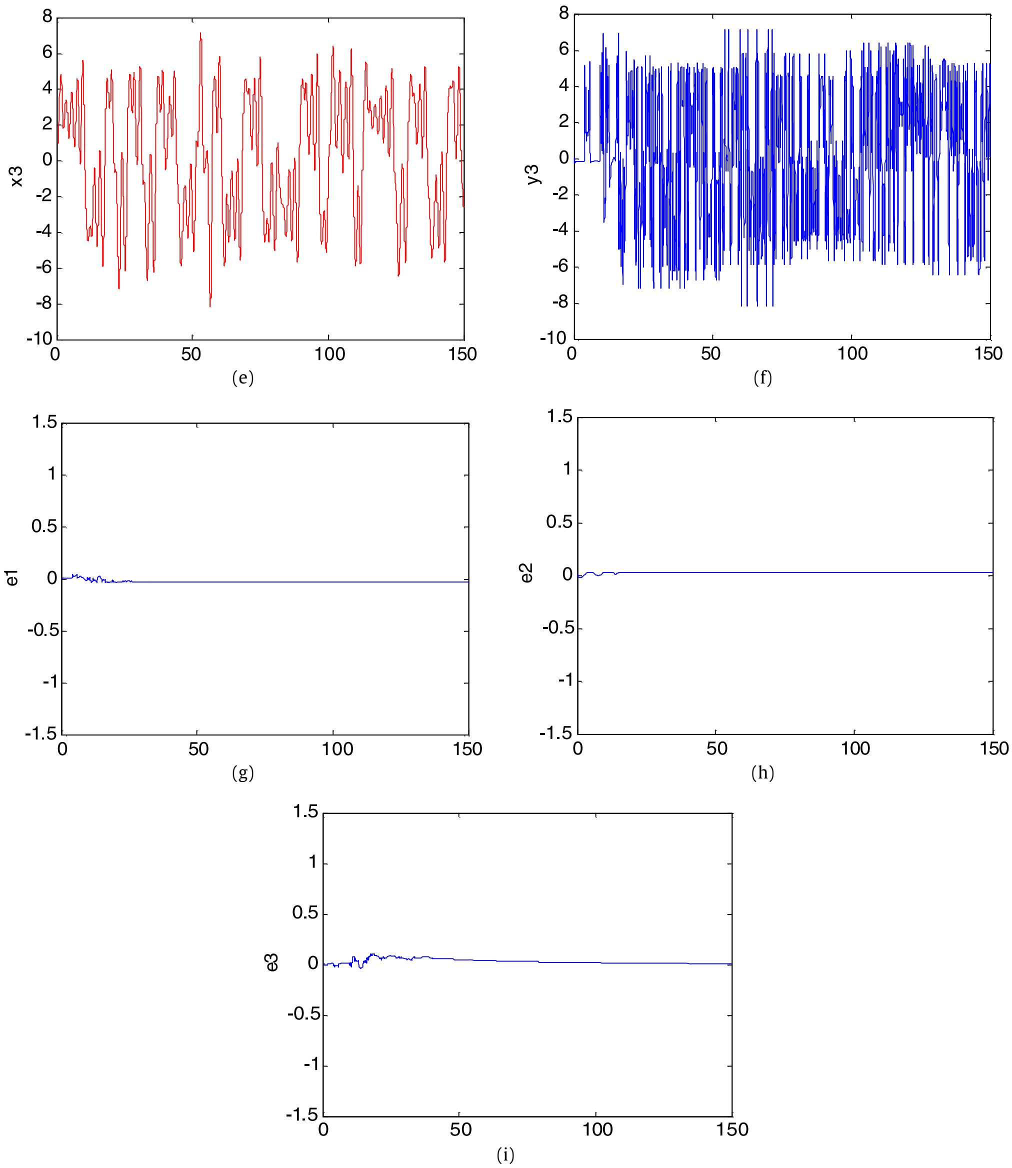


Fig. 1. (*continued*)

Example 1. Consider a time-delay Chua's circuit as the drive system and a delay Rössler dynamic system as the drive system. The time-delay Chua's circuit is described by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 - x_2 + x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 - h(x_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -x_3 & -x_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \\ \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin(\sigma(x_1(t-d_1))) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \beta\varepsilon \end{bmatrix}, \tag{53}$$

with the nonlinear characteristics $h(x_1) = bx_1 + \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|)$ and the parameters $a = -1.4325$, $b = -0.7831$, $\alpha = 10$, $\beta = 15$, $\gamma = 0.1636$, $\varepsilon = 0.2$, $\sigma = 0.5$ and the time-delay $d_1 = 1.999$. β, ε, α and γ are the unknown parameters of the drive system.

The response system is assumed to be the delay Rössler chaotic system, given by

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -y_1 - y_3 \\ y_1 \\ y_1 y_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ y_2 & 0 & 0 \\ 0 & 1 & -y_3 \end{bmatrix} \begin{bmatrix} \eta \\ \mu \\ \upsilon \end{bmatrix} + \begin{bmatrix} y_1(t-d_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix}, \tag{54}$$

with the parameters $\eta = \mu = 0.2$, $\upsilon = 1.2$, $\xi = 1$ and the time-delay $d_2 = 1.999$. η, μ, υ and ξ are the unknown parameters of the response system.

We choose

$$\tau(t) = 10\big(1 + \sin(t)\big). \tag{55}$$

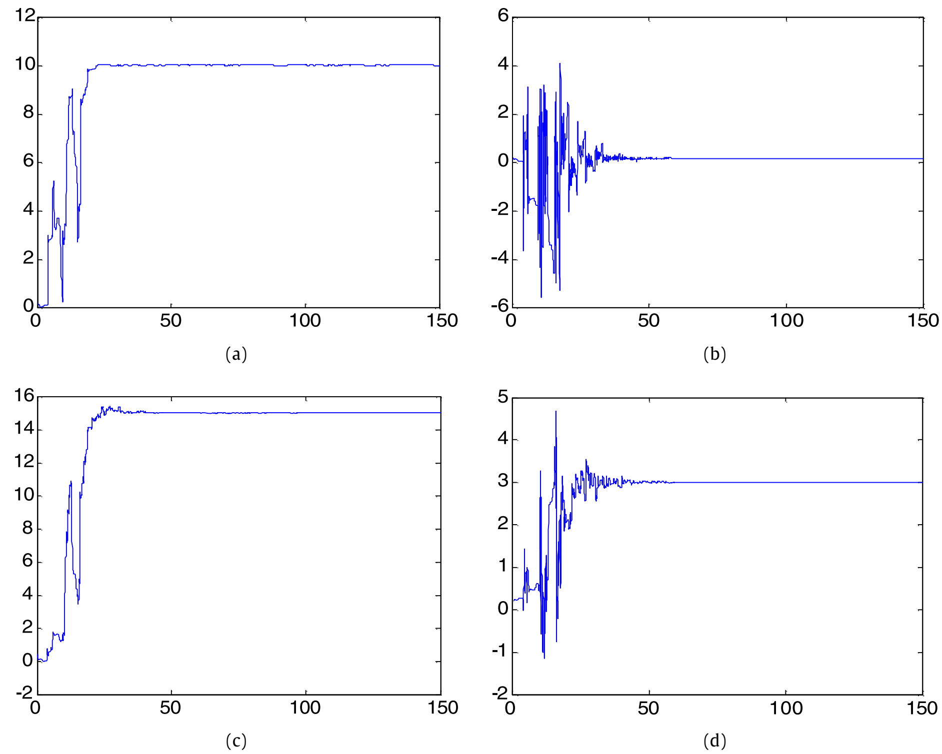


Fig. 2. Estimates of drive system parameters: (a) $\bar{\theta}_{1_1}$; (b) $\bar{\theta}_{1_2}$; (c) $\bar{\theta}_{1_3}$; (d) $\bar{\theta}_{2_3}$.

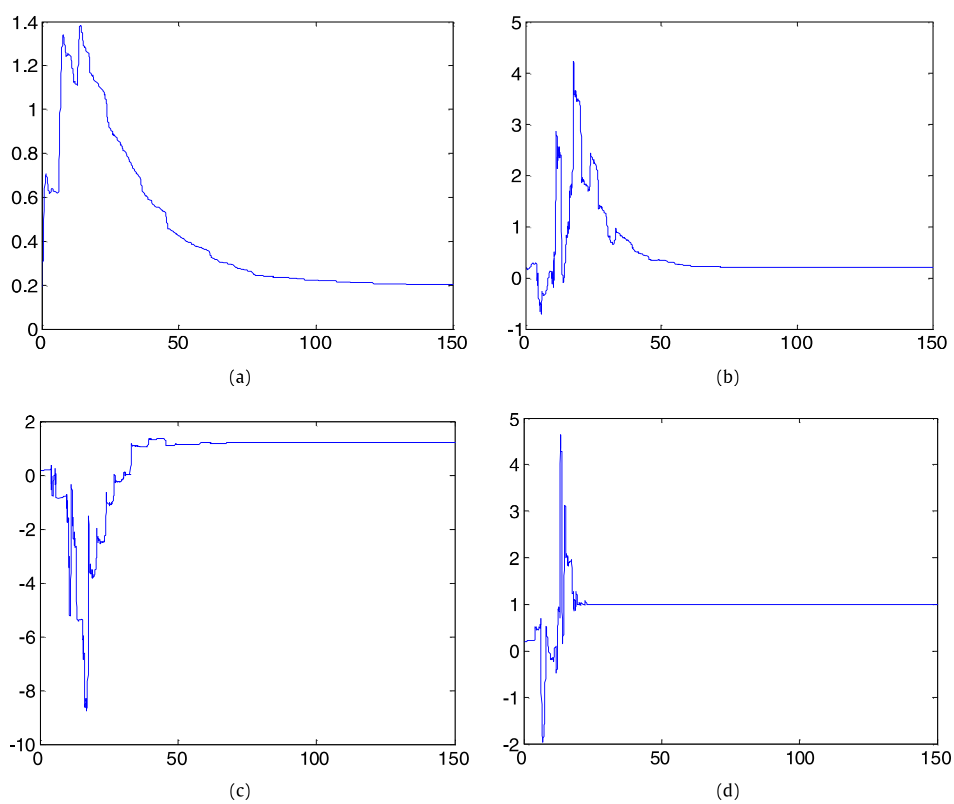


Fig. 3. Estimates of response system parameters: (a) $\bar{\beta}_{1_1}$; (b) $\bar{\beta}_{1_2}$; (c) $\bar{\beta}_{1_3}$; (d) $\bar{\beta}_{2_1}$.

The trajectories of the drive, response and error systems applying the proposed controller in Theorem 1 are shown in Fig. 1. From Fig. 1, we see that the drive system lag synchronizes with the response system at time $\tau(t)$. Fig. 2 and Fig. 3 illustrate the estimates of the drive and response system parameters. In this case as shown in Remark 3, the unknown parameters of the drive and response systems are identified completely and the estimates of the parameters converge to their actual values.

Example 2. Consider drive and response systems the same as in Example 1.

But here, it is assumed that the time-delays d_1 and d_2 are unknown. So, the controller in Theorem 2 is used for lag-synchronization of the time-delay Chua's circuit and delay Rössler chaotic system with the propagation delay $\tau(t) = 5$. Fig. 4 displays the state trajectory of the drive, response and errors system with designed controller in Eq. (24). It is indicated that lag-synchronization error vector with the proposed controller converges to the origin very fast.

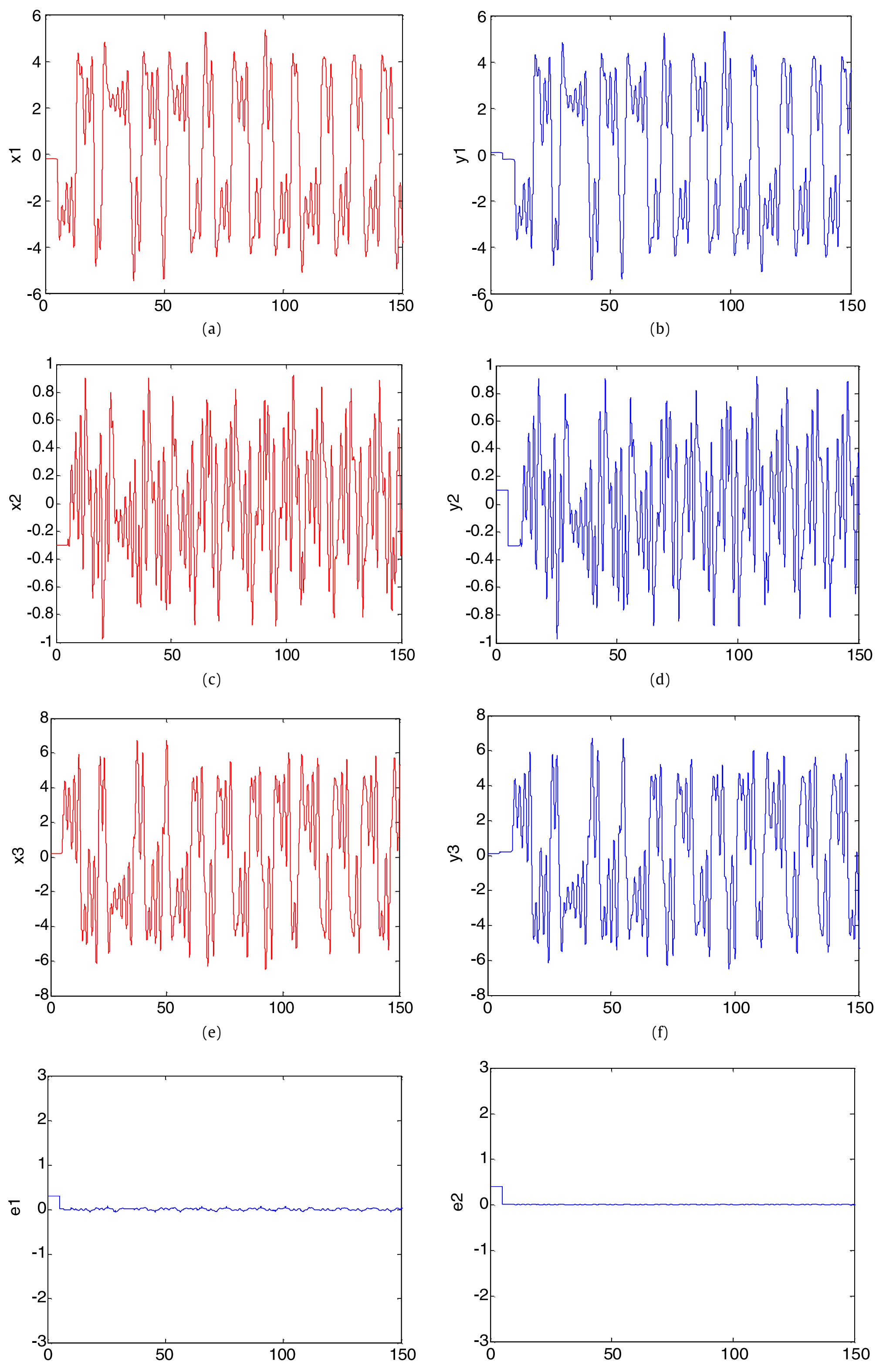
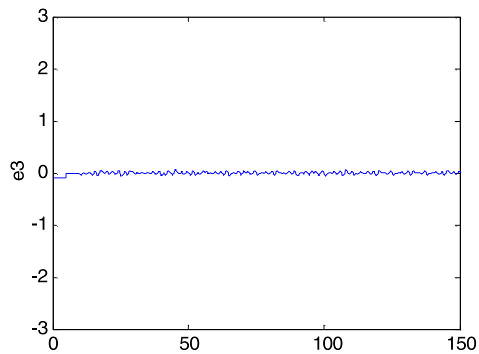


Fig. 4. State trajectories of drive, response and error systems: (a) $x_1(t)$; (b) $y_1(t)$; (c) $x_2(t)$; (d) $y_2(t)$; (e) $x_3(t)$; (f) $y_3(t)$; (g) $e_1(t)$; (h) $e_2(t)$ and (i) $e_3(t)$.



(i)

Fig. 4. (*continued*)

5. Conclusions

In this Letter, the problem of lag-synchronization of two non-identical time-delayed chaotic systems with fully unknown parameters is discussed. A simple adaptive controller is designed to achieve lag-synchronization. To make the results more practical, a controller is proposed for the case when time-delays of the drive and response systems are unknown. Numerical simulations are given to show the effectiveness and feasibility of the developed methods. One can consider the time-delays of these systems to be time-varying to improve the theoretical results in practice. This one is our future research topic.

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