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Synchronization of fractional order chaotic systems using active control method

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ABSTRACT

In this article, the active control method is used for synchronization of two different pairs of fractional order systems with Lotka–Volterra chaotic system as the master system and the other two fractional order chaotic systems, viz., Newton–Leipnik and Lorenz systems as slave systems separately. The fractional derivative is described in Caputo sense. Numerical simulation results which are carried out using Adams–Bashforth–Moulton method show that the method is easy to implement and reliable for synchronizing the two nonlinear fractional order chaotic systems while it also allows both the systems to remain in chaotic states. A salient feature of this analysis is the revelation that the time for synchronization increases when the system-pair approaches the integer order from fractional order for Lotka–Volterra and Newton–Leipnik systems while it reduces for the other concerned pair.

1. Introduction

The applications of dynamical systems include various disciplines of science and engineering. The concept of a dynamical system has its origin in Newtonian mechanics. Mathematical model of a dynamical system is expressed either in the form of a differential equation for a continuous time system or a difference equation, where time is discrete. The governing differential equation or the difference equation mathematically expresses the present states of the system in terms of the past ones for a constant set of inputs and fixed parameters of the system. For a given set of initial conditions, it is possible to acquire knowledge about all the future states through solution of the governing equation and determine the respective trajectory or orbit, i.e., the graphical description of the states with respect to time.

Previously, modeling was primarily restricted to linear systems for which analytical treatment is tractable. Recently due to the advent of powerful computers and with improved computational techniques, it is possible to tackle, to some extent, nonlinear systems. After all, nonlinearity is an important feature of any real-time dynamic system and the study and analysis of non-linear dynamics have gained immense popularity during the last few decades.

Chaotic systems are deterministic, nonlinear dynamical systems which are exponentially sensitive to initial conditions. This type of sensitivity is popularly known as the butterfly effect [1]. The chaotic dynamics of fractional order systems is an important topic of study in nonlinear dynamics. In the last few years this area of research has been growing rapidly [2,3].

An example worth considering is the dynamical relationship between predator and prey which is one of the dominant themes in ecology. It was observed from the population data that interaction between a pair of predator–prey influences the population growth of both the species. The first theoretical treatment of population dynamics was presented by Malthus [4], viz., "Essay on the principle of population". Verhulst [5] formed a mathematical model based on "principle of population", viz., "The logistic equation". The next major advancement in population dynamics was presented by Lotka [6] and Volterra [7]. They presented for the first time the differential equations of

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predator–prey type (tropic interaction model). Since then more complicated but realistic predator–prey systems have been used by ecologists and mathematicians. In 1988, Samardzija and Greller [8] proposed a two-predator, one prey generalization of the Lotka–Volterra equations. The proposed three dimensional system exhibits chaotic behavior, takes an explosive route to chaos and in various regions of parameter space evolves on a fractal torus.

In 1981, Newton and Leipnik [9] obtained the set of differential equations from Euler rigid body equations which were modified with the addition of a linear feedback. This is useful to model dynamics conflict with mediation, e.g., conflict between townspeople, university students and police. For a certain set of parameters the system exhibits chaotic behavior and displays the existence of two strange attractors. An attractor is a state point toward which the trajectories of a dynamic system would normally tend to move.

The Lorenz attractor is an example of a non linear dynamic system corresponding to the long term behavior of the Lorenz oscillation. The Lorenz oscillator is a three dimension dynamical system that exhibits lemniscates type shaped chaotic flow which shows how the state of dynamical system evolves over time in a complex, non-repeating pattern. The Lorenz equations [10] deal with the stability of fluid flows in the atmosphere. In addition to its interest in the field of non linear mathematics, the Lorenz model has important implications for climate and weather prediction. The case is also applicable for simplified models for lasers [10] and dynamos [11].

The concept of fractional derivative is used increasingly to model the behavior of real systems in various fields of science and engineering including Fluid Mechanics [12,13], Viscoelasticity [14], Material Science [15], Bioengineering [16], Medicine [17], Biological Tissues [18], Cardiac Tissue [19], etc. The attribute of fractional order systems for which they have gained popularity in the investigation of dynamical systems is that they allow a greater degree of flexibility in the model. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. For this realistic property, the usage of fractional order systems is becoming popular. It is to be noted that the present states of any real-life dynamic system are dependent upon the history of its past states. The memory element is very much relevant.

The fractional operators can deal with systems with long-range time correlations, which are related to the slow decay of the inverse power-law kernel in the fractional operators. The evolution equations are obtained by replacing the first order derivative by a fractional order $q(0 < q \leqslant 1)$. Gorenflo and Mainardi [20] in their article have shown how such evolution equations can be obtained from a more general master equations which govern so called continuous time random walk (CTRW) by a properly scaled passage to the limit of compressed waiting times and jump width. CTRW is a combination of random walk on the axis of physical time and a random walk in space. Recent survey of a most efficient model, viz., anomalous diffusion equation which is based on the concept of CTRW exhibits random waiting times between two successive jumps. Fractional time or space diffusion equations are relevant in this case for the limiting dynamics of CTRW. Gorenflo et al. [21] demonstrated that for obtaining the space–time fractional diffusion equation describing diffusion process, the essential assumption is that the probabilities for waiting times and jump widths behave asymptotically like powers with negative exponents related to the orders of the fractional derivatives. In 2001, Paradisi et al. [22] have proposed a general Fick's law using fractional derivative operator which accounts for non-local phenomena by virtue of its integral nature. In their other article [23], they have presented a relationship between flux and concentration gradient during their study of a generalized non-local Fick's law derived from the fractional diffusion equation generating the Levy–Feller statistics. Recent study on fractional derivative reveals that it can behave as a dissipative term. Fa [24] has made this observation through the analysis of simple fractional oscillator.

In the last decade, fractional order modeling has been an active field of research both from the theoretical and the applied perspectives. A wide range of problems in different branches of engineering and biology have already been studied by a number of researchers of different parts of the world to explore the potential of the fractional derivative. The usage of first order time derivative with a fractional order time derivative is not only applicable for non-Gaussian but also for non-Markovian systems. Recently Hanert et al. [25] have considered the propagation of epidemic fronts during their study of fractional order epidemic model and have shown that the use of fractional order diffusion term can estimate the acceleration of front that causes a rapid spread of epidemic much faster than predicted by classical Gaussian model. Using stability analysis on a fractional order model of HIV infection of CD4+ T-cells, Ding and Ye [26] obtained a sufficient condition on the parameters for the stability of the infected steady state. The area of fractional order dynamical systems viz., Lotka–Volterra, Newton–Leipnik, Lorenz models have already been explored to a certain extent. In 2007, Ahmed et al. [27] considered the fractional order predator-prey model and the fractional order rabies model. In 2009, Das et al. [28] have solved a fractional order Lotka–Volterra model with one prey and one predator. In 2010, Das and Gupta [29] solved the same type of more general model considering the growth rate of the prey, the efficiency of predator, the growth rate of predator and the death rate of predator as functions of time. A fractional model of two preys and one predator can be found in the book of Petras [30]. The dynamics of the fractional order Lorenz model was also studied by Grigorenko and Grigorenko [31]. Based on the qualitative theory, the existence and uniqueness of solutions for a class of fractional-order Lorenz chaotic systems have been investigated theoretically by Yu et al. [32]. Fractional order diffusion less Lorenz system has been investigated numerically by Sun and Sprott [33] and results have shown that the system has complex dynamics with interesting characteristics. In 2008, Kang et al. [34] studied the influence of parameters on the dynamics of a fractional order Newton–Leipnik system and showed that the system displays comprehensive dynamical behaviors like fixed points, periodic motion, chaotic motion, etc. Diethelm et al. [35] used predictor-corrector scheme for solving fractional order Newton–Leipnik system. Recently, Zhang et al. [36] have studied the stability analysis of the fractional order Newton–Leipnik system using fractional Routh–Hurwitz criteria.

Synchronization of chaos is a phenomenon that may occur when two or more chaotic systems are coupled or one chaotic system drives the other. The pioneering work of Pecora and Carroll [37] introduced a method about synchronization between the drive (master) and response (slave) systems of two identical or non identical systems with different initial conditions, which has attracted a great deal of interest in various fields due to its important applications in ecological systems [38], physical systems [39], chemical systems [40], modeling brain activity, system identification, pattern recognition phenomena and secure communications [41,42], etc. In recent years various synchronization schemes, such as linear and nonlinear feedback synchronization [43,44], time delay feedback approach [45], adaptive control [46], active control, [47,48], etc. have been successfully applied to chaos synchronization. However, most of the methods mentioned above have synchronized two identical chaotic systems. We, the authors, have made an endeavor to study and analyze the synchronization two different types or different pairs of fractional order systems. We started off considering the typical prey–predator equation in ecological modeling followed by a thorough investigation of work done by researchers in the area during the last decade. In 2002, Ho and Hung [47] successfully applied the active control method for synchronization of two different chaotic systems, viz., easy periodic and Rossler systems. In 2006, Park [49] investigated chaos synchronization between two different chaotic systems, viz., Genesio system and Rossler system using nonlinear control laws, taking first system as a master and second one as slave. In 2007, Yan and Li [50] presented chaos synchronization of fractional order Lorenz, Rossler and Chen systems taking one as master and second one as slave. In 2008, Vincent [51] made an excellent effort for examining chaos synchronization between two nonlinear systems using two different techniques viz., active control and back stepping control in terms of transient analysis. In 2008, Zhou and Cheng [52] synchronized between different fractional order chaotic systems viz., Rossler and Chen systems and Chua and Chen systems. Synchronization between chaotic systems is also being widely investigated by Refs. [50,53,54]. Recently, Faieghi and Delavari [55] and Sundarapandian [56] have successfully applied the active control method for synchronization of identical and different chaotic systems, respectively. But to the best of authors' knowledge the synchronization of fractional order Lotka–Volterra system with fractional order Newton–Leipnik and Lorenz systems have not yet been studied by any researcher. The theme of the study is to investigate the time required for synchronization between two different pair of chaotic systems when the time derivative changes to fractional order from the standard one.

In this article the authors have done the synchronization between two different pairs of fractional order chaotic systems namely the Lotka–Volterra and Newton–Leipnik systems and the Lotka–Volterra and Lorenz systems using active control method. Using Adams–Bashforth–Moulton method (see Appendix A), numerical simulations are carried out for different order fractional derivatives and for standard one which are depicted through graphs for different particular cases.

2. Synchronization between different fractional order chaotic systems

Fractional calculus is a generalization of integration and differentiation to a non integer order integro-differential operator ${}_aD_t^q$ is defined by

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & R(q) > 0, \\ 1, & R(q) = 0, \\ \int_a^t (d\tau)^{-q} & R(q) < 0, \end{cases} \tag{1} $$

where q is the fractional order which can be a complex number, $R(q)$ denotes the real part of q and $a < t$, a is the fixed lower terminal and t is the moving upper terminal.

There are some definitions for fractional derivative. The commonly used definition is Riemann–Liouville definition, defined by

$$ {}_aD_t^q x(t) = \frac{d^n}{dt^n} j_t^{n-q} x(t), \quad q > 0, \tag{2} $$

$n = \lceil q \rceil$ i.e., n is the first integer which is not less than q, j_t^α is the α-order Riemann–Liouville integral operator which is described as follows:

$$ j_t^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{3} $$

where $0 < \alpha \leqslant 1$ and $\Gamma(\cdot)$ is gamma function.

In our work the following definition is used:

$$ D_t^q x(t) = j_t^{n-q} x^{(n)}(t), \quad q > 0, \tag{4} $$

where $n = \lceil q \rceil$ and the operator D_t^q is the Caputo differential operator of order q. For an initial problem of Riemann–Liouville type, one would have to specify the values of certain fractional derivatives (and integrals) of unknown solution at the initial point $t = 0$. However, it is not clear what the physical meanings of fractional derivatives of x are when we deal with a concrete physical application, and hence it is also not clear how such quantities can be measured. When we deal with an initial problem of Caputo type, the situation is different. We may specify the initial values. This clearly helps to understand the physical meaning.

Again Riemann–Liouville initial value problems require homogeneous initial conditions though for Caputo initial value problem holds for both homogeneous and non-homogeneous conditions. For this reason choice derivative is better than the Riemann–Liouville derivative.

The important reason of choosing Caputo derivatives for solving initial value fractional order differential equations is that it does not introduce any problems when trying to model real world phenomena with fractional time and apace derivatives which are noticed for Riemann–Liouville derivative.

Another important difference between the Riemann–Liouville definition and Caputo definition is that the Caputo derivative of a constant is zero, whereas in the case of the Riemann–Liouville fractional derivative of the constant c is not equal to zero but ${}_0D_t^q c = c\frac{t^{-q}}{\Gamma(1+q)}$.

Let us consider the two different fractional order chaotic systems

$$\frac{d^q x}{dt^q} = Ax + F(x), \tag{5}$$

$$\frac{d^q y}{dt^q} = By + G(y), \tag{6}$$

where $0 < q \leqslant 1$ is the order of the fractional time derivative, x, $y \in R^n$ are the state vectors, $F(x), G(y) \in R^n$ are the nonlinear terms of the system (5) and system (6), and are smooth functions. A, B are constant matrices. For synchronization of the above two different systems, system (5) represents the master (drive) system and system (6) represents the slave (response) system.

Now introducing the active control parameter $u \in R^n$ in the system (6), we get.

$$\frac{d^q y}{dt^q} = By + G(y) + u. \tag{7}$$

The purpose of chaos synchronization is how to design the active controller u, which is able to synchronize the states of both the master and the slave systems. If we define the error vector as $e = y - x$, the error dynamical system becomes

$$\frac{d^q e}{dt^q} = Be + (B - A)x + G(y) - F(x) + u. \tag{8}$$

Theorem. *Suppose*

$$u = (A - B)x + F(x) - G(x) + DG(x)e + Ce, \tag{9}$$

where DG(x) is the Jacobi matrix of G(x). C is a controller gain matrix to be designed later.

If $|\arg(\lambda_i(B + C))| > 0.5\pi q,\ i = 1, 2, \ldots, n$ by choosing suit controller gain matrix control C, then the fractional-order chaotic system (5) and the fractional-order chaotic system (7) can arrive to the asymptotic synchronization. Here $\arg(\lambda_i(B + C))$ denotes the argument of the eigenvalues λ_i of $(B + C)$.

Proof. The proof of above theorem can be found in Appendix B with the help of [52]. □

2.1. The fractional-order Lotka–Volterra system

The fractional-order Lotka–Volterra system (in the article 5.13 from the book by Petras [30]) is given by

$$\begin{aligned} \frac{d^{q_1} x}{dt^{q_1}} &= ax - bxy + ex^2 - szx^2, \\ \frac{d^{q_2} y}{dt^{q_2}} &= -cy + dxy, \\ \frac{d^{q_3} z}{dt^{q_3}} &= -pz + szx^2, \end{aligned} \tag{10}$$

where parameters a, b, c, $d > 0$, 'a' represents the natural growth rate of the prey in the absence of predators, 'b' represents the effect of predation on the prey, 'c' represents the natural death rate of the predator in the absence of prey, 'd' represents the efficiency and propagation rate of the predator in the presence of prey, and e, p, s are positive constants.

During synchronization the parameters are taken as $a = 1,\ b = 1,\ c = 1,\ d = 1,\ e = 2,\ s = 2.7,\ p = 3$ and the initial condition is $[1, 1.4, 1]$, $0 < q_i \leqslant 1$ is the order of the derivative. At $q_i = 0.95$ $(i = 1, 2, 3)$, Eq. (10) represents the fractional order Lotka–Volterra chaotic system and the chaotic attractors of the system (10) are described through Fig. 1.The phase portraits in x–y–z space and x–y, y–z, z–x planes are shown through Fig. 1(a)–(d) respectively.

2.2. The fractional-order Newton–Leipnik system

The fractional-order Newton–Leipnik system [57] is given by

$$\begin{aligned} \frac{d^{q_1} x}{dt^{q_1}} &= -ax + y + 10yz, \\ \frac{d^{q_2} y}{dt^{q_2}} &= -x - 0.4y + 5xz, \\ \frac{d^{q_3} z}{dt^{q_3}} &= bz - 5xy, \end{aligned} \tag{11}$$

where 'a' and 'b' are variable parameters. Usually the parameter 'b' is taken in the interval $(0, 8.0)$. The system is ill-behaved outside this interval. As $b \to 0$, the system relatively shows uninteresting dynamics and for $b \geqslant 0.8$, the given system becomes explosive i.e., the solution diverge to infinity for any initial condition other than the critical points.

During synchronization the parameters are taken as $a = 0.4$, $b = 0.175$, initial condition = $[0.19\ \ 0\ \ -0.18]$ and $0 < q_i \leqslant 1$ is order of derivative. At $q_i = 0.95$ $(i = 1,\ 2,\ 3)$, Eq. (11) becomes the fractional order Newton–Leipnik chaotic equation, and the chaotic attractors of fractional order system (11) are described in Fig. 2. The phase portraits in x–y–z space and x–y, y–z, z–x planes are shown through Fig. 2(a)–(d) respectively.

2.3. The fractional-order Lorenz system

The fractional-order Lorenz system [31,58] is given by

$$\begin{aligned} \frac{d^{q_1} x}{dt^{q_1}} &= \sigma(y - x), \\ \frac{d^{q_2} y}{dt^{q_2}} &= x(\rho - z) - y, \\ \frac{d^{q_3} z}{dt^{q_3}} &= xy - \beta z, \end{aligned} \tag{12}$$

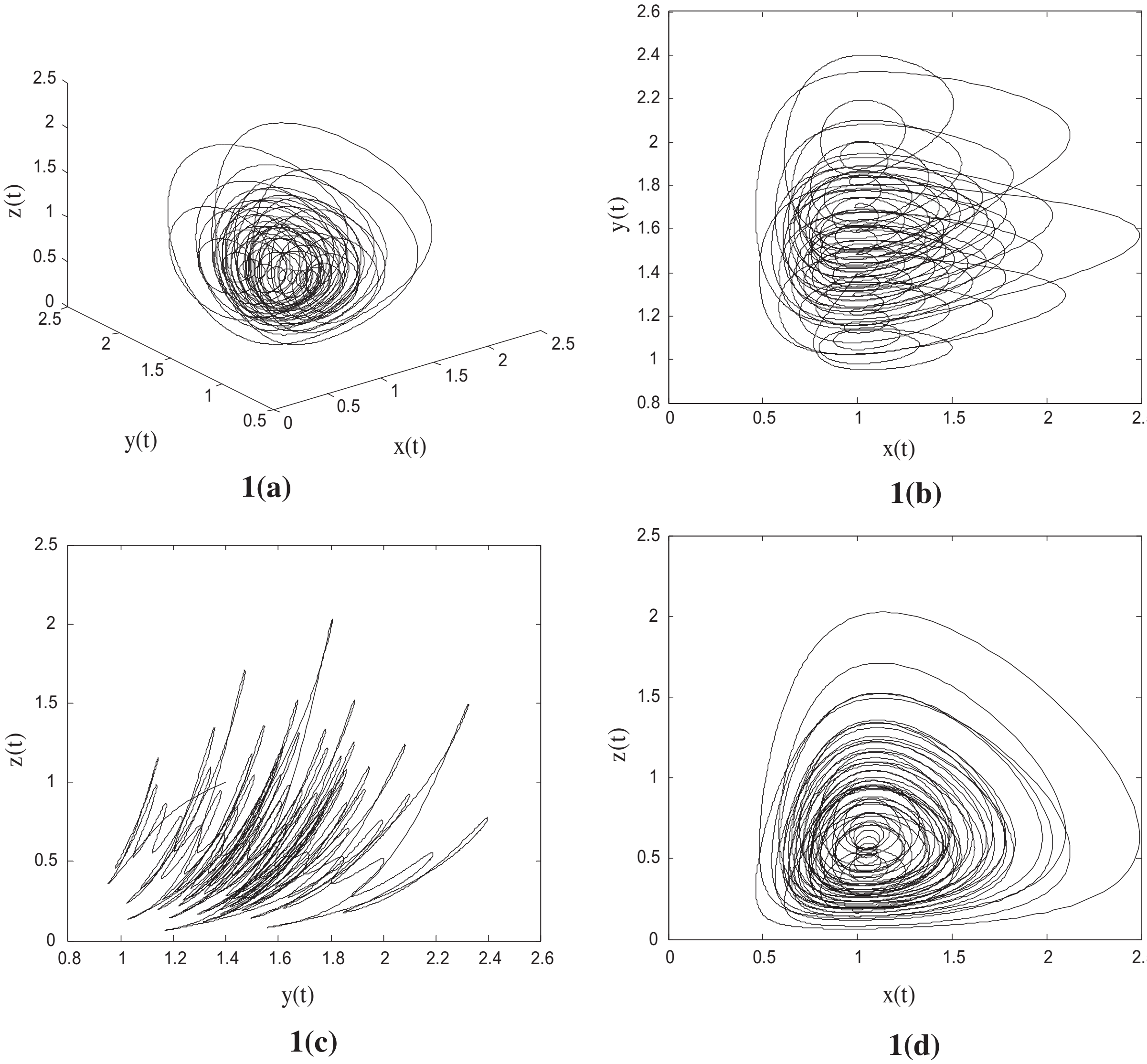


Fig. 1. Phase portraits of Lotka–Volterra system in x–y–z space and x–y, y–z, x–z planes.

where σ is the Prandtl number, ρ is the Rayleigh number and μ is the size of the region approximated by the system.

During synchronization the parameters are taken as $\sigma = 10$, $\rho = 28$, $\mu = 8/3$, the initial conditions = [0.1, 0.1, 0.1], and $0 < q_i \leqslant 1$ is order of derivative. When $q_i = 0.993$ ($i = 1,2,3$), Eq. (12) represents the fractional order Lorenz chaotic equation and the chaotic attractors of fractional order system (12) are described in Fig. 3. The phase portraits in x–y–z space and x–y, y–z, z–x planes are shown through Fig. 3(a)–(d) respectively.

2.4. Synchronization of fractional order Lotka–Volterra and Newton–Leipnik systems via active control method

In this section the synchronization behavior between two different fractional orders Lotka–Volterra and Newton–Leipnik is made. We assume that Lotka–Volterra system drives the Newton–Leipnik system. Therefore, we define the Lotka–Volterra as a master system and Newton–Leipnik as a slave system as follows:

The master system is described by (10) as

$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}} = a_1x_1 - b_1x_1y_1 + Ex_1^2 - sz_1x_1^2, \\ \frac{d^{q_2}y_1}{dt^{q_2}} = -c_1y_1 + d_1x_1y_1, \\ \frac{d^{q_3}z_1}{dt^{q_3}} = -pz_1 + sz_1x_1^2. \end{cases} \tag{13}$$

The slave system is described by (11) as

$$\begin{cases} \frac{d^{q_1}x_2}{dt^{q_1}} = -a_2x_2 + y_2 + 10y_2z_2 + u_1(t), \\ \frac{d^{q_2}y_2}{dt^{q_2}} = -x_2 - 0.4y_2 + 5x_2z_2 + u_2(t), \\ \frac{d^{q_3}z_2}{dt^{q_3}} = b_2z_2 - 5x_2y_2 + u_3(t), \end{cases} \tag{14}$$

where three active control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ are introduced in Eq. (14). Our goal is to investigate the synchronization of system (13) and (14). We define the er-

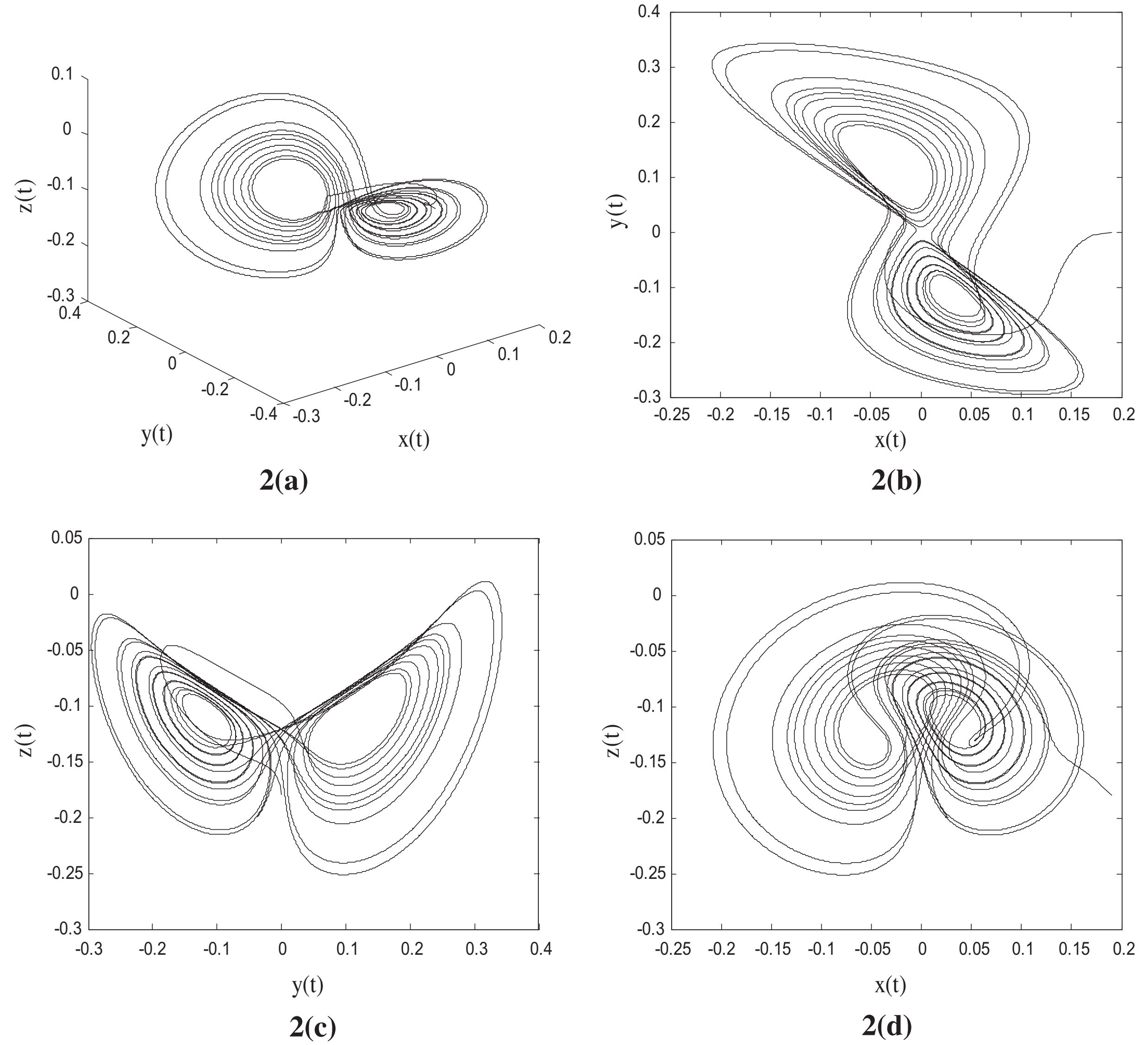


Fig. 2. Phase portraits of Newton–Leipnik system in x–y–z space and x–y, y–z, x–z planes,

ror states $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$. The corresponding error dynamics can be obtained by subtraction of Eq. (13) from Eq. (14)

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = -a_2 e_1 - (a_1 + a_2)x_1 + e_2 + y_1 - E x_1^2 + b_1 x_1 y_1 + s z_1 x_1^2 + 10 y_2 z_2 + u_1(t), \\ \frac{d^{q_2} e_2}{dt^{q_2}} = -e_1 - 0.4 e_2 - x_1 + (c_1 - 0.4) y_1 - d_1 x_1 y_1 + 5 x_2 z_2 + u_2(t), \\ \frac{d^{q_3} e_3}{dt^{q_3}} = b_2 e_3 + (b_2 + p) z_1 - s z_1 x_1^2 - 5 x_2 y_2 + u_3(t). \end{cases} \tag{15}$$

Then we define the active control inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$ as

$$\begin{cases} u_1(t) = v_1(t) + (a_1 + a_2)x_1 - y_1 + E x_1^2 - b_1 x_1 y_1 - s z_1 x_1^2 - 10 y_2 z_2, \\ u_2(t) = v_2(t) + x_1 - (c_1 - 0.4) y_1 + d_1 x_1 y_1 - 5 x_2 z_2, \\ u_3(t) = v_3(t) - (b_2 + p) z_1 + s z_1 x_1^2 + 5 x_2 y_2, \end{cases} \tag{16}$$

which leads to

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = -a_2 e_1 + e_2 + v_1(t), \\ \frac{d^{q_2} e_2}{dt^{q_2}} = -e_1 - 0.4 e_2 + v_2(t), \\ \frac{d^{q_3} e_3}{dt^{q_3}} = b_2 e_3 + v_3(t). \end{cases} \tag{17}$$

The synchronization error system (17) is a linear system with active control inputs, $v_1(t)$, $v_2(t)$ and $v_3(t)$. Next design an appropriate feedback control which stabilizes the system so that e_1, e_2, e_3 converge to zero as time t tends to infinity, which implies that Lotka–Volterra and Newton–Leipnik system are synchronized with feedback control. There are many possible choices for the control inputs $v_1(t)$, $v_2(t)$ and $v_3(t)$. We choose

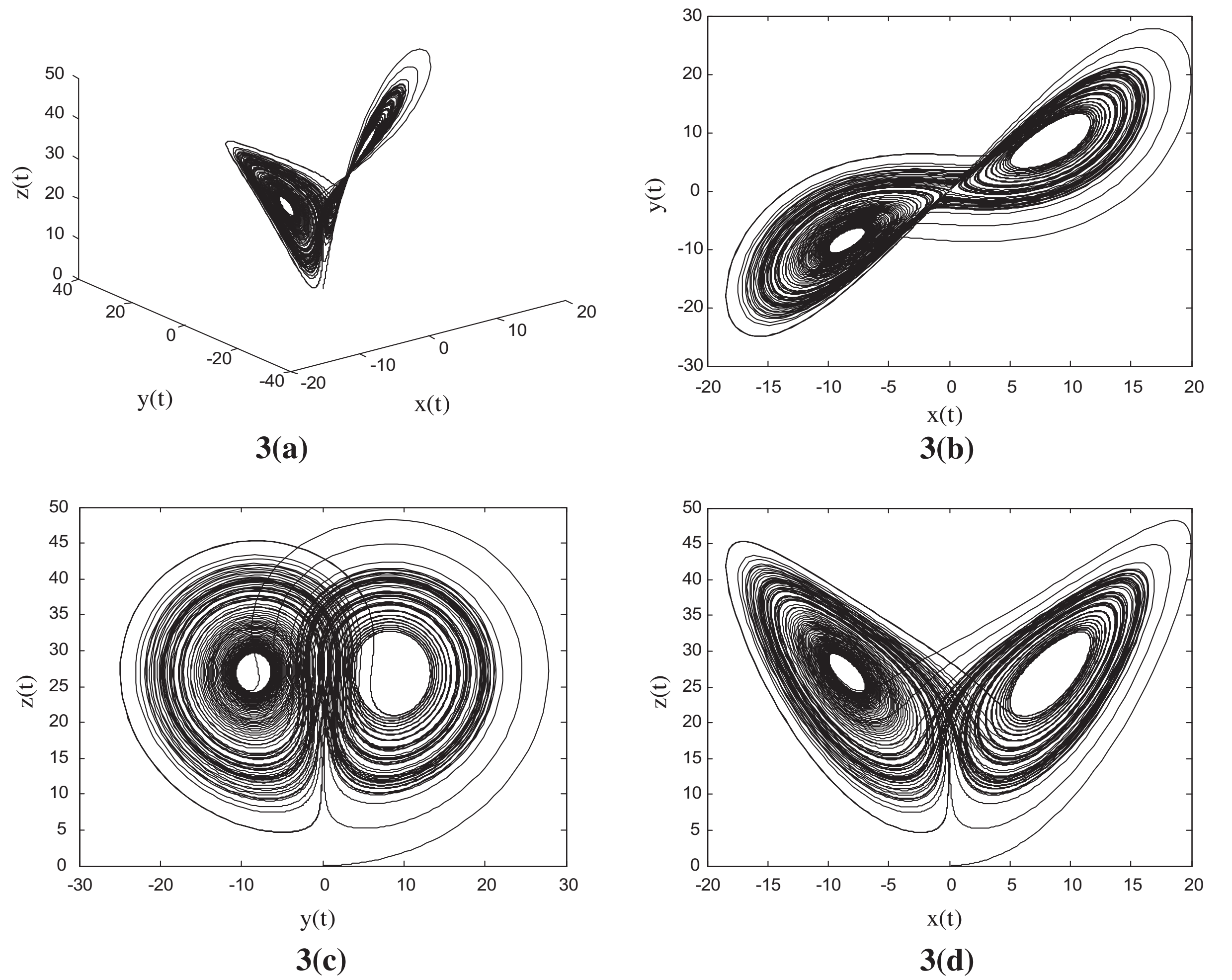


Fig. 3. Phase portraits of Lorenz system in *x–y–z* space and *x–y*, *y–z*, *x–z* planes.

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = C \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{18}$$

where C is a 3×3 constant matrix. In order to make the closed loop system stable, the matrix C should be selected in such a way that the feedback system has eigenvalues λ_i of C satisfies the control $|\arg(\lambda_i)| > 0.5\pi q,\ i = 1, 2, 3$.There is not a unique choice for such matrix C, a good choice can be as follows:

$$C = \begin{pmatrix} a_2 - 1 & -1 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & -b_2 - 1 \end{pmatrix}. \tag{19}$$

Then the error system is changed to

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = -e_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} = -e_1 - e_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} = -e_3. \end{cases} \tag{20}$$

All the three eigenvalues of the system are -1, hence the conditions that all $q_i \leqslant 1$ are satisfied and we get the required synchronization.

2.5. Simulation and results

In numerical simulations, the parameters of the Newton–Leipnik system are taken as $a_2 = 0.4$ and $b_2 = 0.175$. Parameters of the Lotka–Volterra system are taken as $a_1 = 1, b_1 = 1, c_1 = 1, d_1 = 1, E = 2, p = 3, s = 2.9851$. Time step size was taken as 0.005. The initial values of the master system (Lotka–Volterra system) and the slave system (Newton–Leipnik system) are taken as [1,1.4,1] and [0.349,0,-0.160] respectively. Thus, the initial errors are $[-0.651, -1.4, -1.160]$.

Figs. 4–6 demonstrate that systems are synchronized after small duration of time for the considered fractional order time derivatives $q_i = 0.80$, $q_i = 0.95$ and also for standard case $q_i = 1, i = 1, 2, 3$. It is also seen from the figures that the time taken for synchronization of two systems increases with the increase of fractional orders approaching towards standard order system.

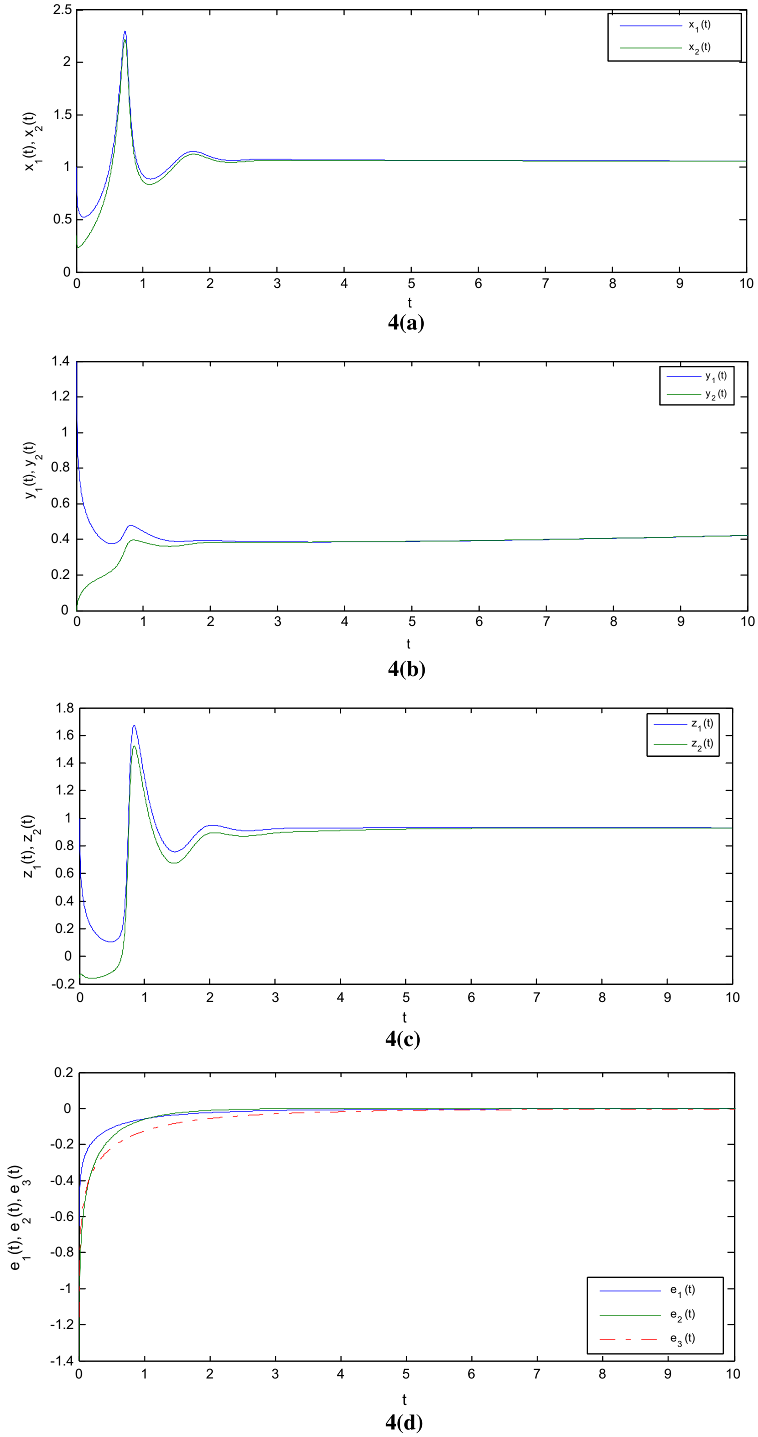


Fig. 4. State trajectories of drive system (13) and response system (14) for the fractional order $q_i = 0.80$ ($i = 1,2,3$): (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

2.6. Synchronization of fractional order Lotka–Volterra and Lorenz systems via active control method

In this section the synchronization behavior in two different fractional order Lotka–Volterra and Lorenz systems is made. We assume that Lotka–Volterra system drives the Lorenz system. Therefore, we define the Lotka–Volterra as a master system and Lorenz as a slave system.

The master system is given by (10)

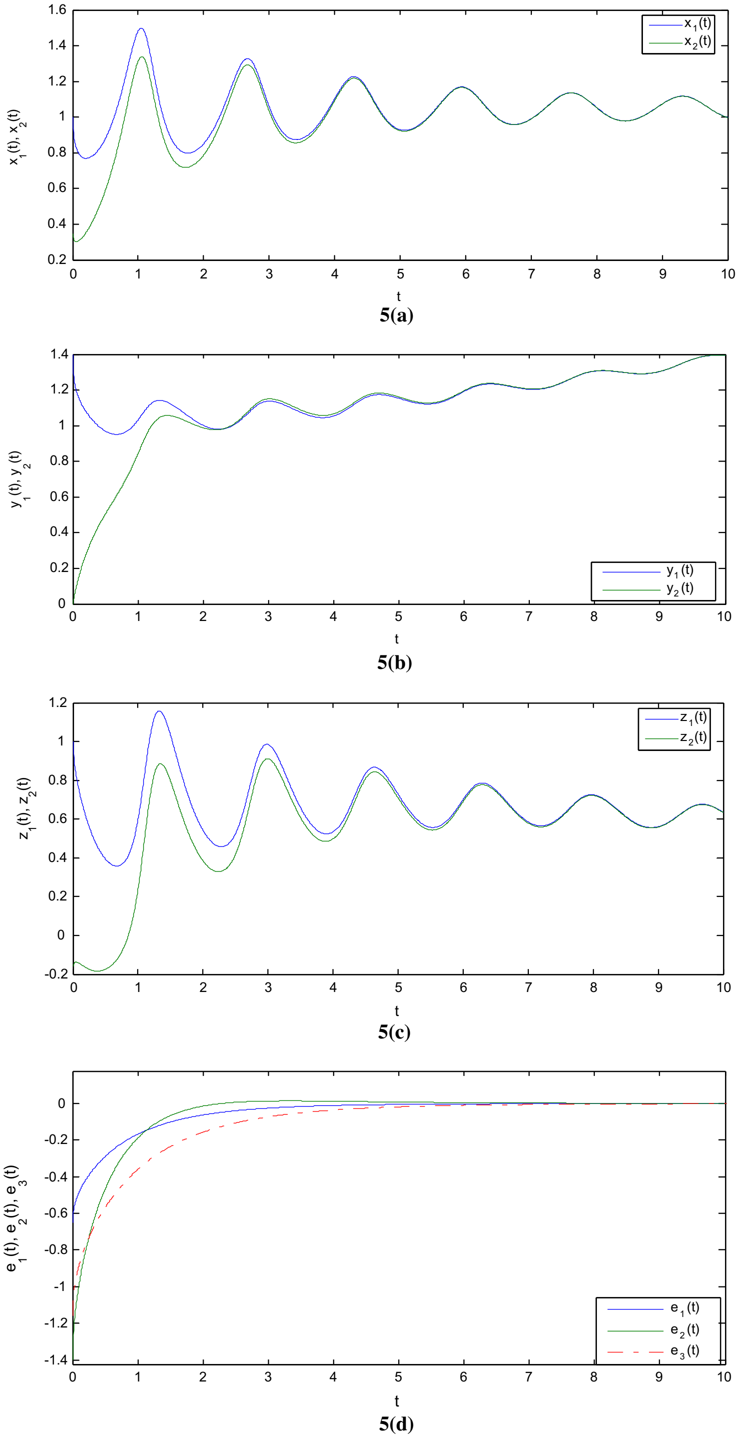
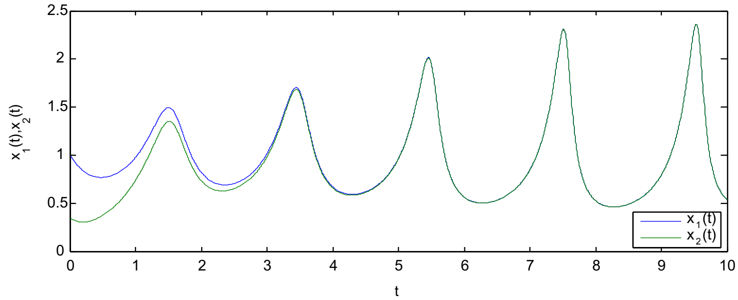


Fig. 5. State trajectories of drive system (13) and response system (14) for the fractional order $q_i = 0.95$ ($i = 1,2,3$): (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) the evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

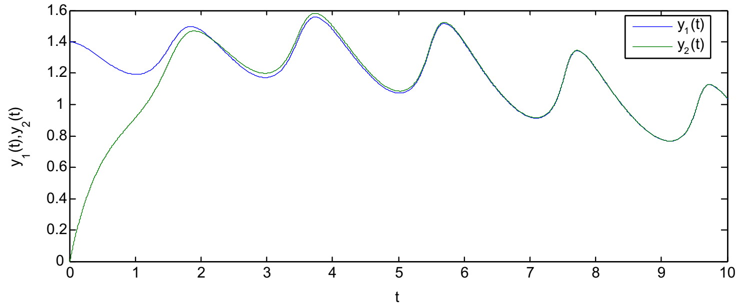
$$\begin{cases} \frac{d^{q_1}x_1}{dt^{q_1}} = ax_1 - bx_1y_1 + Ex_1^2 - sz_1x_1^2, \\ \frac{d^{q_2}y_1}{dt^{q_2}} = -cy_1 + dx_1y_1, \\ \frac{d^{q_3}z_1}{dt^{q_3}} = -pz_1 + sz_1x_1^2. \end{cases} \tag{21}$$

The slave system is described by (12)

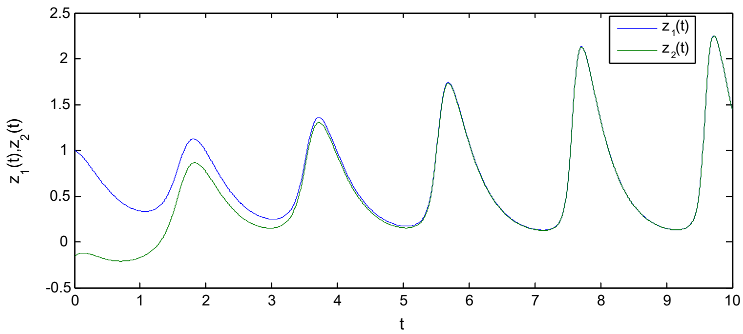
$$\begin{cases} \frac{d^{q_1}x_2}{dt^{q_1}} = \sigma(y_2 - x_2) + u_1(t), \\ \frac{d^{q_2}y_2}{dt^{q_2}} = x_2(\rho - z_2) - y_2 + u_2(t), \\ \frac{d^{q_3}z_2}{dt^{q_3}} = x_2y_2 - \beta z_2 + u_3(t). \end{cases} \tag{22}$$



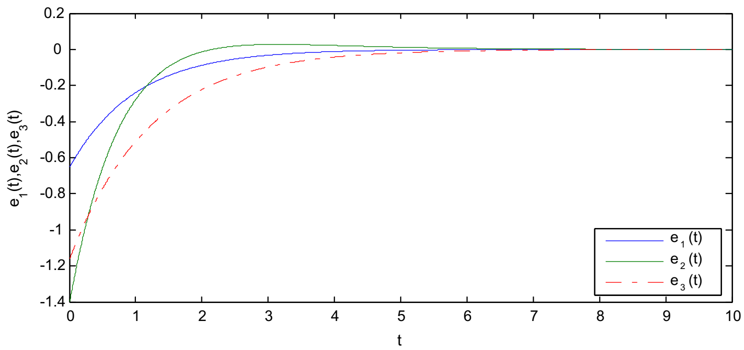
6(a)



6(d)



6(c)



6(d)

Fig. 6. State trajectories of drive system (13) and response system (14) for the standard order $q_i = 1 (i = 1,2,3)$: (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) the evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

We have introduced three active control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ in (22). Our goal is to investigate the synchronization of the systems (21) and (22). We define the error states $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$. The corresponding error dynamics can be obtained by subtraction Eq. (21) from Eq. (22) as

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = \sigma e_2 + \sigma y_1 + a e_1 - (a+\sigma) x_2 - E x_1^2 + b x_1 y_1 + s z_1 x_1^2 + u_1(t), \\ \frac{d^{q_2} e_2}{dt^{q_2}} = -c e_2 + (c-1) y_2 - d x_1 y_1 + \rho x_2 - x_2 z_2 + u_2(t), \\ \frac{d^{q_3} e_3}{dt^{q_3}} = -\beta e_3 + (p-\beta) z_1 - s z_1 x_1^2 + x_2 y_2 + u_3(t). \end{cases} \quad (23)$$

Then we define the active control inputs $u_1(t)$, $u_2(t)$, and $u_3(t)$ as

$$\begin{cases} u_1(t) = v_1(t) + (a+\sigma)x_2 - \sigma y_1 + Ex_1^2 - bx_1y_1 - sz_1x_1^2, \\ u_2(t) = v_2(t) - \rho x_2 - (c-1)y_2 + dx_1y_1 + x_2z_2, \\ u_3(t) = v_3(t) - (p-\beta)z_1 + sz_1x_1^2 - x_2y_2. \end{cases} \tag{24}$$

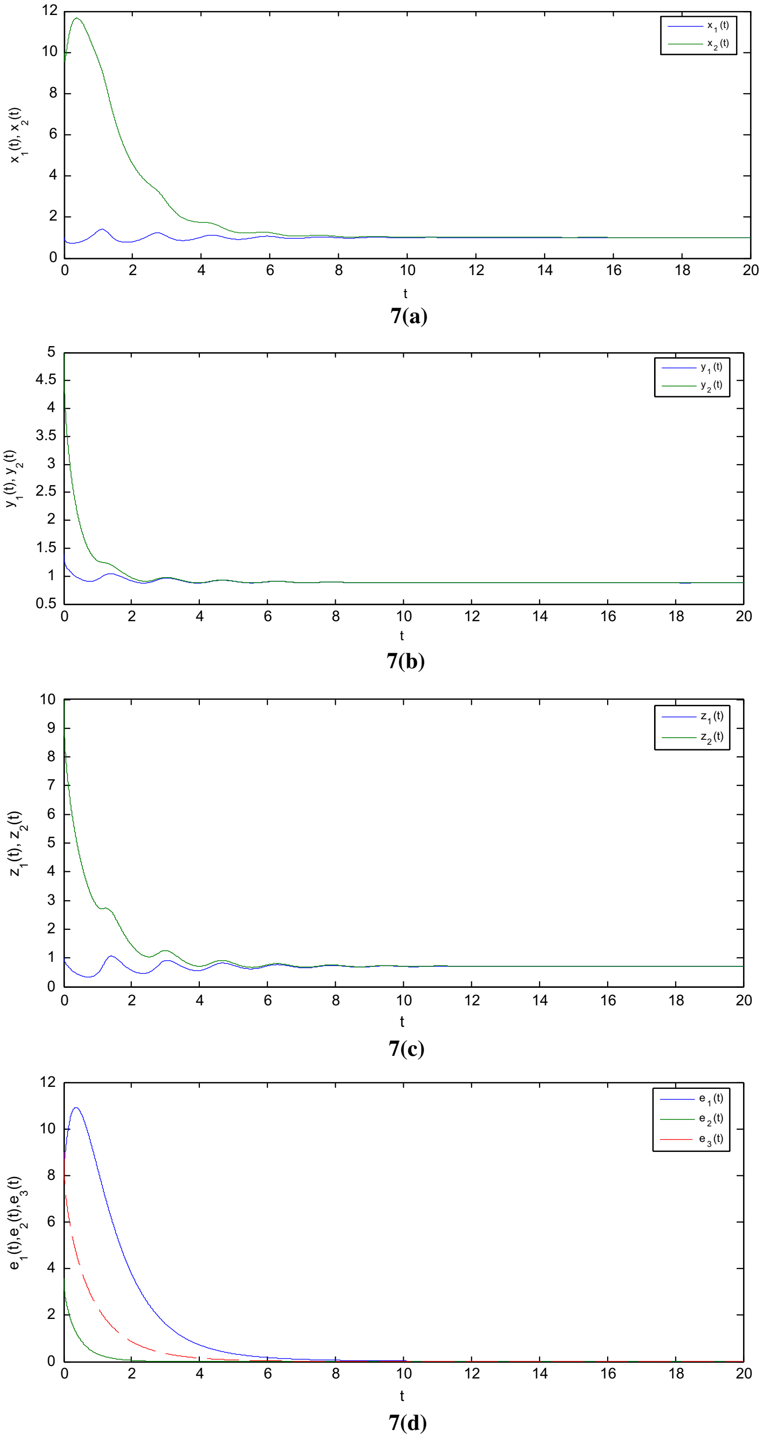


Fig. 7. State trajectories of drive system (21) and response system (22) for the fractional order $q_i = 0.95$ ($i = 1,2,3$): (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) the evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

which leads to

$$\begin{cases} \frac{d^{q_1}e_1}{dt^{q_1}} = e_1 + \sigma e_2 + v_1(t), \\ \frac{d^{q_2}e_2}{dt^{q_2}} = -ce_2 + v_2(t), \\ \frac{d^{q_3}e_3}{dt^{q_3}} = -\beta e_3 + v_3(t). \end{cases} \quad (25)$$

The synchronization error system (25) is a linear system with active control inputs, $v_1(t)$, $v_2(t)$ and $v_3(t)$. Next to design an appropriate feedback control stabilizes the system so that e_1, e_2, e_3 converge to zero as time t tends to infinity, which implies that Lotka–Volterra and Lorenz system are

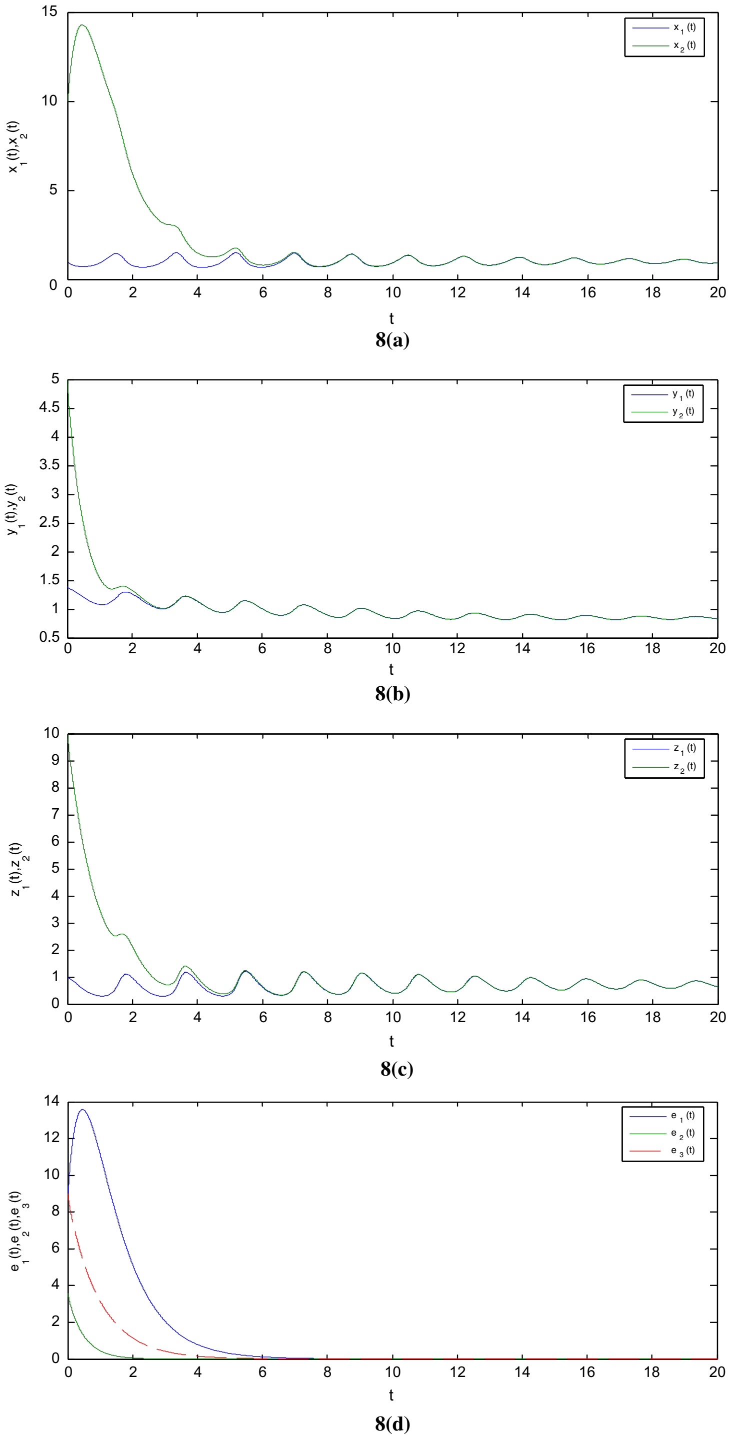


Fig. 8. State trajectories of drive system (21) and response system (22) for the fractional order $q_i = 0.993$ ($i = 1,2,3$): (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) the evolution of the error functions $e_1(t)$, $e_2(t)$ and e_3 (t).

synchronized with feedback control. As previous one, we choose

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = C \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{26}$$

where C is a 3×3 constant matrix. In order to make the closed loop system stable, the matrix C should be selected in such a way that the feedback system has eigenvalues λ_i of C satisfies the control $|\arg(\lambda_i)| > 0.5\pi q,\ i = 1, 2, 3$. Let us choose the matrix C as

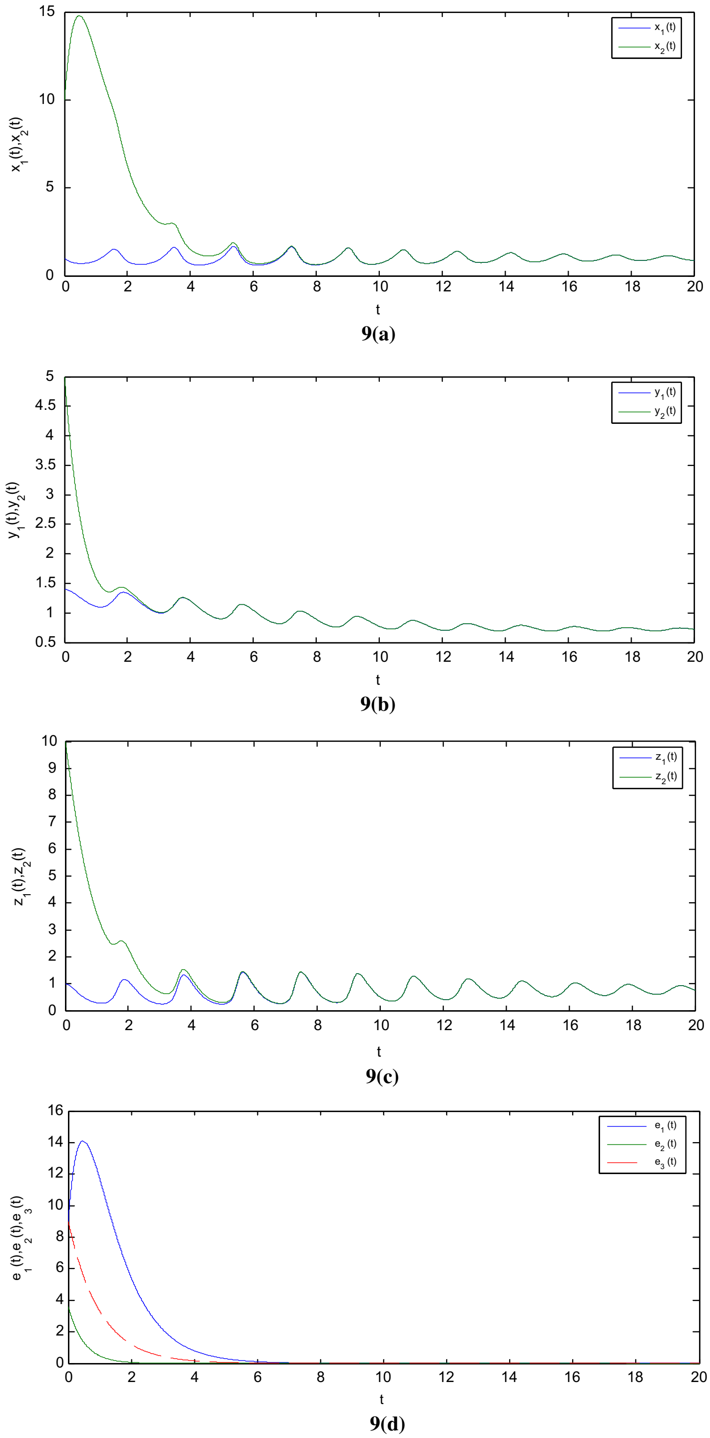


Fig. 9. State trajectories of drive system (21) and response system (22) for the standard order $q_i = 1$ ($i = 1,2,3$): (a) between x_1 and x_2, (b) between y_1 and y_2, (c) between z_1 and z_2, (d) the evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

$$C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & c-1 & 0 \\ 0 & 0 & \beta-1 \end{pmatrix}, \tag{27}$$

then the error system is changed as

$$\begin{cases} \frac{d^{q_1}e_1}{dt^{q_1}} = -2e_1 + \sigma e_2, \\ \frac{d^{q_2}e_2}{dt^{q_2}} = -e_2, \\ \frac{d^{q_3}e_3}{dt^{q_3}} = -e_3. \end{cases} \tag{28}$$

Here also all the three eigenvalues of the system are -1, hence the condition that all $q_i \leqslant 1$ is satisfied and hence we get the required synchronization.

2.7. Simulation and results

In numerical simulations, the parameters of the Lorenz system are taken as $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. Parameters of the Lotka–Volterra system are taken as $a = 1$, $b = 1$, $c = 1$, $d = 1$, $E = 2$, $p = 3$, $s = 2.9851$. Time step size is taken as 0.005. The initial values of the master system (for Lotka–Volterra system) and the slave system (Lorenz system) are taken as [1,1.4,1] and [10 5 10] respectively. Thus, the initial errors are [9,3.6,9].

Fig. 7 demonstrates that it takes higher time for synchronization of two considered chaotic systems for the fractional order $q_i = 0.95$ ($i = 1,2,3$) in comparison to the synchronization for the fractional order $q_i = 0.993$ which is displayed through Fig. 8 and also for standard order displayed through Fig. 9. This shows that for this pair of fractional order systems it takes lesser synchronization time with the increase of fractional orders while approaching towards standard order system.

3. Conclusion

There are three important goals that the authors have achieved in the present article. First one is the study of dynamical behavior of coupled chaotic systems of fractional order. Second one is employing the powerful active control method which provides us a simple way to synchronize a pair of chaotic systems. Thirdly, the observation that the synchronization time increases when the system pair approaches the standard order from fractional order with Lotka–Volterra as the master & Newton–Leipnik as the slave while it reduces when Lotka–Volterra drives the Lorenz system is a major outcome of the study. Numerical simulations are used to verify the efficiency, effectiveness and validity of the proposed method. It is worth mentioning that since synchronization of two fractional order chaotic systems assumes considerable significance in the study of nonlinear dynamics, the outcome of this research work would be appreciated and could be utilized by those researchers involved in the field of mathematical modeling of fractional order dynamical systems.

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Appendix A

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). We only derive the predictor–corrector scheme for drive–response systems. This scheme is the generalization of Adams–Bashforth–Moulton one [59,60]. We interpret the approximate solution of nonlinear fractional-order differential equations using this algorithm in the following way.

The considered differential equation

$$D_t^q y(t) = f(t, y(t)), \quad 0 \leqslant t \leqslant T, \tag{A1}$$

with

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, [q]$$

is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds. \tag{A2}$$

Set, $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$. Then (A2) can be discredited as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_h, y_h(t_j)), \tag{A3}$$

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q & \text{if } j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1} & \text{if } 0 \leqslant j \leqslant n, \\ 1 & \text{if } j = n+1, \end{cases} \tag{A4}$$

where predicted value $y_h(t_{n+1})$ is determined by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), \tag{A5}$$

$$b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q). \tag{A6}$$

The error estimate is

$$\max_{j=0,1\ldots N} |y(t_j) - y_h(t_j)| = o(h^p) \text{ in which } p = \min(2, 1+q). \tag{A7}$$

Appendix B

From Eqs. (8) and (9), we have

$$\frac{d^q e}{dt^q} = Be + G(y) - G(x) - DG(x)e + Ce. \tag{B1}$$

On the other hand, using Taylor formula in the neighborhood $U(x\|e\|)$ we derive that

$$G(y) = G(x) + DG(x)e + O(\|e\|). \tag{B2}$$

So, combining (B1) and (B2), we obtain

$$\frac{d^q e}{dt^q} = (B + C)e + O(\|e\|). \tag{B3}$$

Certainly, $e_i = y_i - x_i = 0$, $i = 1, 2, 3, \ldots, n$ is one fixed point of error dynamical system (B3). The Jacobi matrix of (B3) at this fixed point is $(B + C)$. As shown by *D*. Matignon [61] for analyzing the stability criterion for the fractional differential equation, since the argument of the eigenvalues λ_i of matrix $(B + C)$ satisfy $|\arg(\lambda_i(B + C))| > 0.5\pi q$, therefore this fixed point considered here is asymptotically stable.

Now, $\lim_{t\to\infty} e_i = 0$ $(i = 1, 2, \ldots, n)$, indicates that systems (7) and (5) can be achieved to synchronization. Thus the synchronization condition is that all the eigenvalues λ_i of matrix $(B + C)$ in (B3) satisfy $|\arg(\lambda_i(B + C))| > 0.5\pi q$ $(i = 1, 2, 3)$.

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