Decentralized Control of Coupled Nonlinear Dynamic Systems with Application to Quadruple-Tank Process

Frank Ferrese
Naval Surface Warfare Center
Carderock Division
Philadelphia, PA19112
frank.ferrese@navy.mil

Qing Dong
Naval Surface Warfare Center
Carderock Division
Philadelphia, PA19112
qing.dong@navy.mil

Saroj Biswas
Electrical Engineering Dept.
Temple University
Philadelphia, PA19122
sbiswas@temple.edu

Jason Batcho
Naval Surface Warfare Center
Carderock Division
Philadelphia, PA19112
jason.batcho@navy.mil

***Abstract*—This paper presents a novel method of control system design for nonlinear interconnected large-scale systems. Conventional control design techniques often involve linearization of the system model with respect to an operating state, which may lead to incorrect control performance for systems that are highly nonlinear and interconnected. In this paper, a decentralized control design consisting of a PI controller with constant feedback gains is considered. Then a nonlinear optimal control method based on integral minimum principle of Pontryagin is developed. In this method, the controller performance is defined using a standard quadratic cost function, and the control law is defined using the PI structure with constant feedback gain. The effectiveness of the developed control scheme shows that the controller is capable of maintaining the desired water level in the four tanks.**

Keywords*— *Quadruple-Tank process, decentralized system, optimal control, fluid-flow control, parameter optimization.

I. Introduction

A large-scale system is composed of a number of interconnected subsystems, and is often found in many engineering and control applications. Large-scale systems are traditionally characterized by a large number of state variables and nonlinear system dynamics. Researchers usually decompose a large-scale system into smaller and more manageable networked subsystems when traditional control design techniques can be applied. An electrical power system, a pump water control system and a team of robots, among others, are typical examples of large-scale systems with varying levels of complexity.

The size and complexity of a large-scale system makes it difficult to approach using tools from the classical centralized control method. A literature search indicated that decentralized control using decomposition has been a common solution for decades. Output feedback had also been applied to large-scale linear systems in [1]–[3]. An overlapping state feedback control incorporating both structural and algebraic constraints for decentralized control of a large-scale system has also been developed by [4]. Recently, new methods for large-scale system control have emerged, such as multi-agent based control and market based resource allocation approaches, where a large number of networked intelligent agents execute the control algorithm locally at an agent level to achieve the overall system performance. In some cases, the agents have to make efficient use of a scarce resource, such as computation power, to demonstrate distributed computational abilities. Reference tracking control for nonlinear interconnected systems using H_∞ decentralized fuzzy control was developed by [5].

Many large-scale systems exist in today's Navy machinery automation and control systems, such as distributed chilled water systems, distributed electrical power networks, etc., which can be better represented as decentralized zonal systems. For such systems, ordinary approaches, such as decomposition methods are not appropriate, and system reconfiguration is even more complicated due to scalability issues. A new innovative decentralized control method is needed to deal with these large-scale systems. This paper presents a decentralized control method with constant feedback gains for coupled nonlinear systems, and illustrates the method using quadruple-tank process.

One problem that arises from current approaches to the multiple tank system control is the use of passive flow interconnection. A lot of research is focused on fluid flow between tanks using gravity [6]–[9], and pressure differentials [10] [11]. If the fluid source in a passive flow system were to fail, the control of the water levels in each tank would no longer be available. At best the passive flow scheme will allow for a common fluid level in each tank, but if drainage is involved then all tanks will eventually drain out. This paper describes a decentralized optimal feedback control scheme based on a simple test bed consisting of four water tanks, valves, electrical pumps and a common reservoir as shown in Fig. 1.

In this experiment setup, the four water tanks are connected by gravity lines and flow control valves. In addition to unmeasured water leakage from the upper tanks to the lower level tanks, water flows out of the lower level tanks into a common reservoir by gravity force. The electrical pumps used to control the overall system pump water from the reservoir back into the upper level tanks. The interconnections in this quad tank system are a mixture of both passive and active, meaning that the controller will be able to dictate flow

978-1-4799-4032-5/14/$31.00 ©2014 IEEE

between tanks, as well as flow into each tank from the source. This system exhibits the nonlinear behavior of interconnected dynamics because each of the two pumps affects both of the outputs (tank levels).

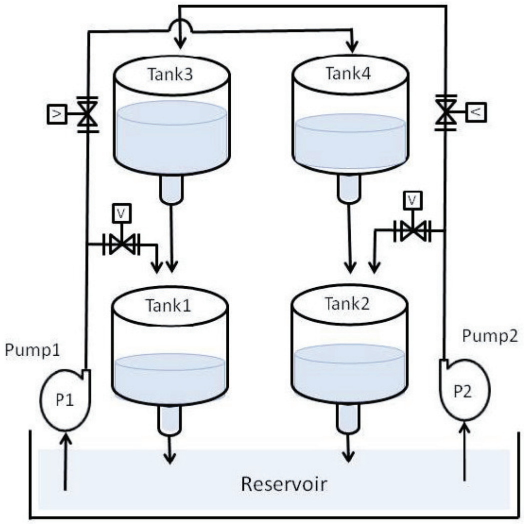

Fig. 1. Continuous Time Quad-Tank Test Bed

Research pertaining to designing controllers and predicting the output behavior for various large-scale systems, specifically multiple tank fluid systems, is readily available. Reference [10] describes controlling a two-tank fluid system to test a method of diagnosis that uses future predictions of the normal and faulty model of the system. In [12] a two-tank test facility is described that is designed to test control algorithms and steady-state optimization studies. Also, research has been conducted in systems that contain more than two tanks [7] [8]. Most of the current research, such as [6]–[9], [11] is based on transfer function of the system through linearization of the nonlinear dynamic model; furthermore the frequency domain method is favored over the time domain method for controller design.

In this paper, a decentralized control method is used for the quadruple-tank process by considering it as two coupled subsystems, such as tanks 1 and 3 forming the first subsystem, and tanks 2 and 4 for the second. A proportion-integral (PI) controller using local feedback and constant gains is then designed for each subsystem. Since the system is nonlinear, standard PI design methods cannot be used; instead we use the integral mnimum principle of Pontryagin to minimize a standard quadratic cost function. A simple proof of integral minimum principle and a numerical method for computation of the gains are presented. The effectiveness of the developed control is illustrated by numerical simulation which shows that the controller is capable of maintaining the desired water level in the four tanks.

The rest of the paper is organized as follows: A nonlinear dynamic model of the quadruple-tank process is presented in Section III along with the formulation of the decentralized PI control problem. Section IV provides the solution of the parameter optimization problem for nonlinear systems and a method of numerical solution. Simulation results are presented in Section V which is followed by concluding remarks in Section VI.

II. System Model

Water networks are generally composed of a large number of interconnected pipes, reservoir, pumps, valves and other hydraulic elements to deliver water from the reservoir to different locations. In this section a mathematical model of a quad tank system is presented which can be derived from mass conservation principle and Bernoulli's law of incompressible fluids. Water is pumped to the upper tanks by two independent pumps, and the control objective is to maintain a desired water level in the two lower tanks. The control inputs of the system are the voltages v_1 and v_2 applied to the pumps. The system model is given by

$$\begin{aligned}\dot{h}_1 &= -\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{a_3}{A_1}\sqrt{2gh_3}+\frac{\gamma_1 k_1}{A_1}v_1\\ \dot{h}_2 &= -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_4}{A_2}\sqrt{2gh_4}+\frac{\gamma_2 k_2}{A_2}v_2\\ \dot{h}_3 &= -\frac{a_3}{A_3}\sqrt{2gh_3}+\frac{(1-\gamma_2)k_2}{A_3}v_2\\ \dot{h}_4 &= -\frac{a_4}{A_4}\sqrt{2gh_4}+\frac{(1-\gamma_1)k_1}{A_4}v_1 \qquad (1)\end{aligned}$$

where
- h_i is the water level in tank i
- a_i is the cross-section area of the drainage hole
- A_i is the cross-section area of tank i
- g is the acceleration due to gravity
- γ_i is the valve setting that controls water sharing between tanks
- k_i is the pump parameter
- $k_i v_i$ is the flow rate of the pump i

The flow to tank 1 is $\gamma_1 k_1 v_1$ and to tank 4 is $(1-\gamma_1)k_1 v_1$, and flow to tanks 2 and 3 are defined in the same way. The valve parameters $\gamma_1, \gamma_2 \in (0,1)$ are preset parameters that regulate the sharing of water flow in the corresponding supplied tanks.

It is quite clear from the above that the system model is nonlinear which has prompted researchers to resort to linearization of the system. Considerable research results, both simulated and experimental, can be found in the literature [13]–[15]. In this paper, we investigate control design based on the nonlinear dynamics. In particular, consider the decentralized control system shown in Fig. 2.

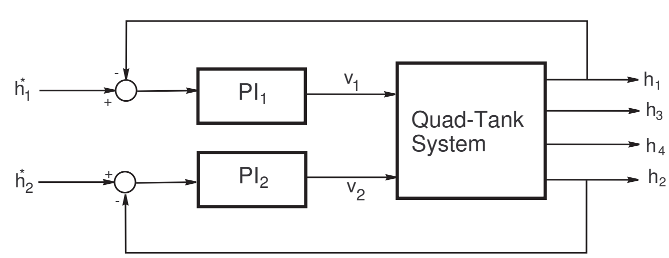

Fig. 2. Decentralized Controller for Quad Tank System

The above closed loop system utilizes two proportional-integral (PI) controllers as shown which are based on decentralized feedback given by

$$\begin{aligned} \text{PI}_1: \quad & v_1(t) = k_{p1}(h_1^* - h_1(t)) + k_{i1}\int_0^t (h_1^* - h_1(\tau))\,d\tau \\ \text{PI}_2: \quad & v_2(t) = k_{p2}(h_2^* - h_2(t)) + k_{i2}\int_0^t (h_2^* - h_2(\tau))\,d\tau \end{aligned} \tag{2}$$

where k_{p1}, k_{p2} and the proportional gains, and k_{i1}, k_{i2} are the integral gains of the two controllers. The desired level of water in the two tanks are denoted by h_1^* and h_2^* respectively. Note that the control system does not use the water level in tanks 3 and 4.

In order to express the complete closed loop system in state space form, we define two new state variables z_1, z_2 as

$$\begin{aligned} \dot{z}_1 &= h_1^* - h_1 \\ \dot{z}_2 &= h_2^* - h_2 \end{aligned} \tag{3}$$

This gives the complete closed loop system as

$$\begin{aligned} \dot{h}_1 &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \\ \dot{h}_2 &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \\ \dot{h}_3 &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 \\ \dot{h}_4 &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 \\ \dot{z}_1 &= h_1^* - h_1 \\ \dot{z}_2 &= h_2^* - h_2 \end{aligned} \tag{4}$$

and the feedback controller as

$$\begin{aligned} \text{PI}_1: \quad & v_1 = k_{p1}(h_1^* - h_1) + k_{i1}z_1 \\ \text{PI}_2: \quad & v_2 = k_{p2}(h_2^* - h_2) + k_{i2}z_2 \end{aligned} \tag{5}$$

Substitution of the feedback control law (5) into the system model (4) expresses the system in the standard form

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \alpha, t) \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{6}$$

where $\alpha = [k_{p1} \quad k_{p2} \quad k_{i1} \quad k_{i2}]$ is the vector of unknown gains of the controller. The objective of the control design is to find the optimum α so as to maintain the desired levels of water h_1^* and h_2^* in tanks 1 and 2 respectively. Note that the above system model is nonlinear so that design methods based on linear control theory cannot be used. The next section shows the design of the controller based on nonlinear optimization theory.

III. Parameter Optimization

The optimization problem defined in the above section is in the same general class commonly known as *parameter optimization* of dynamic systems. In typical optimal control problems, one seeks to find the best time varying control function that minimizes a certain objective function. In this case, the unknown controllable quantities are certain constant parameters rather than time varying functions.

Parameter optimization problem can be viewed as a generalization of the optimal control problem; the '*control*', in the case, is a constant parameter rather than a time varying function. Boltyanski [16] [17] provided the fundamental result on the necessary conditions that the optimal parameter must satisfy; the method is formally known as 'Integral Minimum Principle'. An alternate proof was later developed in [18]. Certain applications of parameter optimization problems can be found in [19]. In what follows, we follow the concepts of calculus of variations for the optimization of parameters for unconstrained problems.

Consider the system (6) where $\mathbf{x}(t) \in R^n$ is the state vector, and $\alpha \in R^m$ is a *constant* parameter vector, and the cost function is defined as

$$J(\alpha) = \Phi(\mathbf{x}(t_f)) + \int_0^{t_f} L(\mathbf{x}, \alpha, t)\,dt. \tag{7}$$

It is assumed that the functions Φ and L are positive, convex and differentiable. The objective is to find the optimal parameter vector α so that the cost is minimum. We assume that the space of admissible parameters is unconstrained.

Suppose α is the optimal parameter, and $\mathbf{x}$ is the corresponding trajectory. Also assume that $x + \delta x$ is a perturbed trajectory corresponding to a non-optimal parameter vector $\alpha + \delta\alpha$.

Defining the augmented cost function

$$\widetilde{J}(\alpha) = \Phi(\mathbf{x}(t_f)) + \int_0^{t_f} L(\mathbf{x}, \alpha, t)\,dt + \int_0^{t_f} \langle \psi(t), f(\mathbf{x}, \alpha, t)\rangle\,dt \tag{8}$$

and taking the first variation, we obtain

$$\begin{aligned} \delta\tilde{J} = \langle \frac{\partial \Phi(\mathbf{x}(t_f))}{\partial \mathbf{x}} - \psi(t_f), \delta x(t_f)\rangle + \int_{t_0}^{t_f} \langle \frac{\partial H}{\partial \mathbf{x}}(t) - \dot{\psi}, \delta x(t)\rangle\,dt \\ + \langle \int_0^{t_f} \frac{\partial H}{\partial \alpha}(t)\,dt, \delta\alpha\rangle \end{aligned} \tag{9}$$

where H is the Hamiltonian defined by

$$H(\mathbf{x}, \psi, \alpha, t) = L(\mathbf{x}, \alpha, t) + \langle \psi(t), f(\mathbf{x}, \alpha, t)\rangle. \tag{10}$$

Then since the variations are arbitrary, we obtain the necessary conditions of optimality as

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases} \tag{11}$$

$$\begin{cases} \dot{\psi} = -\dfrac{\partial H}{\partial \mathbf{x}} = -\dfrac{\partial L}{\partial \mathbf{x}} - \left[\dfrac{\partial f}{\partial \mathbf{x}}\right]' \psi \\ \psi(t_f) = \dfrac{\partial \Phi(\mathbf{x}(t_f))}{\partial \mathbf{x}}, \end{cases} \tag{12}$$

and

$$\left\{ 0 = \int_0^{t_f} \frac{\partial H}{\partial \alpha}\,dt = \int_0^{t_f} \left\{ \frac{\partial L}{\partial \alpha} + \left[\frac{\partial f}{\partial \alpha}\right]' \psi \right\} dt. \right. \tag{13}$$

Before we proceed further, a few comments are in order:

- The above result represents a set of necessary conditions for the optimal control and the corresponding state trajectory to satisfy. It is a set of $2n$ differential equations coupled with m algebraic equations (equation (13)), where n is the dimension of the state vector, and m is the dimension of the control vector.

- These necessary conditions form a complex two-point-boundary-value problem, (TPBVP), where the initial condition for the state equation is known, while the final condition $\psi(t_f)$ is known in terms of the state vector at the final time t_f, which is unknown. As such, it is not possible to obtain an analytical solution of the above set of equations, nevertheless, these equations can be solved numerically so as to obtain the optimal control v and the corresponding optimal trajectory.

A. Numerical method

One of the commonly used methods to solve the TPBVP arising from optimal control problems is the iterative gradient method that minimizes the cost by updating the control at every iteration.

Considering the first variation of the augmented cost function (9), and approximating the variations as $\Delta\mathbf{x}$, $\Delta\alpha$, and $\Delta\tilde{J}$, we have

$$\Delta\tilde{J} = \langle \frac{\partial\Phi(\mathbf{x}(t_f))}{\partial\mathbf{x}} - \psi(t_f), \Delta\mathbf{x}(t_f)\rangle + \int_0^{t_f} \langle \dot{\psi} + \frac{\partial H}{\partial\mathbf{x}}(t), \Delta\mathbf{x}(t)\rangle\, dt + \langle \int_0^{t_f} \frac{\partial H}{\partial\alpha}(t)\, dt, \Delta\alpha\rangle \tag{14}$$

Then starting with an assumed control α, it is iteratively updated so that the cost for an updated control is less than the cost in the previous iteration. Suppose at iteration i, the control parameter vector is $\alpha = \alpha_i$. Then we intend to find a new control α_{i+1} such that $J(\alpha_{i+1}) \leq J(\alpha_i)$, i.e., $\Delta J_i \leq 0$.

An iteration cycle involves solution of the state equation and the adjoint equation in a sequential manner. First the state equation (11) is solved using the control parameter α_i, Then one solves the adjoint equation (12) using the same parameter and the corresponding state trajectory. Since for this control, the state equation (11) and the adjoint equation (12) are satisfied, equation (9) simplifies to

$$\Delta\tilde{J} = \langle \int_0^{t_f} \frac{\partial H}{\partial\alpha}\, dt, \Delta\alpha\rangle.$$

Suppose the new control α_{i+1} is selected such that

$$\alpha_{i+1} = \alpha_i - \varepsilon \int_{t_0}^{t_f} \frac{\partial H}{\partial\alpha}(\mathbf{x}_i, \psi_i, \alpha_i)\, dt$$

where ε is chosen sufficiently small, then one has

$$\Delta\tilde{J} = -\varepsilon \| \int_{t_0}^{t_f} \frac{\partial H}{\partial\alpha}(\mathbf{x}_i, \psi_i, \alpha_i)\, dt \|^2.$$

Therefore, the cost is minimized for the new control. Iterations are continued until convergence.

Selection of ε is very important for success of the iteration. A large value of ε will cause the cost to increase, i.e., $J(\alpha_{i+1}) > J(\alpha_i)$ for which α_{i+1} cannot be taken as an updated parameter. On the other hand, a too small value of ε will cause the cost function to decrease too slowly requiring too many iterations to converge. Thus it is necessary that the value of ε be appropriately selected. A simple approach for this purpose is to start with a small ε, and at every successful iteration double the value of ε. If the new cost is smaller than the old cost, continue the iteration for the next update of control. On the other hand, if the new cost is larger than the old cost, take a new ε as one half of current value of ε. This method of doubling and halving of the value of ε may be repeated as necessary until convergence.

IV. Simulation Results

This section presents simulation results of the decentralized optimal control method presented above. For simulations, we choose the system parameters used in [7] given below:

A_1, A_3	28 cm^2
A_2, A_4	32 cm^2
a_1, a_3	0.071cm^2
a_2, a_4	0.057cm^2
g	981 cm/sec^2

The objective is to reach the desired water level at the final time $t_f = 100$ seconds with minimum state deviation and minimum cost of control:

$$J(u) = \frac{1}{2}s_1(h_1^* - h_1(t_f))^2 + \frac{1}{2}s_2(h_2^* - h_2(t_f))^2 + \frac{1}{2}\int_0^{t_f} \{q_1(h_1^* - h_1(t))^2 + q_2(h_2^* - h_2(t))^2 + v_1^2(t) + v_2^2(t)\}\, dt + R\{k_{p1}^2 + k_{p2}^2 + k_{i1}^2 + k_{i2}^2\} \tag{15}$$

The above cost function does not include any cost on water level in tanks 3 and 4. For simulations, we choose initial water level, $h_1(0) = 12.4$ cm, and $h_2(0) = 12.7$ cm, the pump gains, $k_1 = 3.33$ and $k_2 = 3.35$, and the valve parameters, $\gamma_1 = 0.70$ and $\gamma_2 = 0.60$. The desired water level in tanks 1 and 2 are 20 cm and 16 cm, respectively. Various weights in the cost functions are taken as: $s_1 = 10$, $s_2 = 20$, $q_1 = 90$, $q_2 = 100, R = 10$. Figures 3-5 show the results:

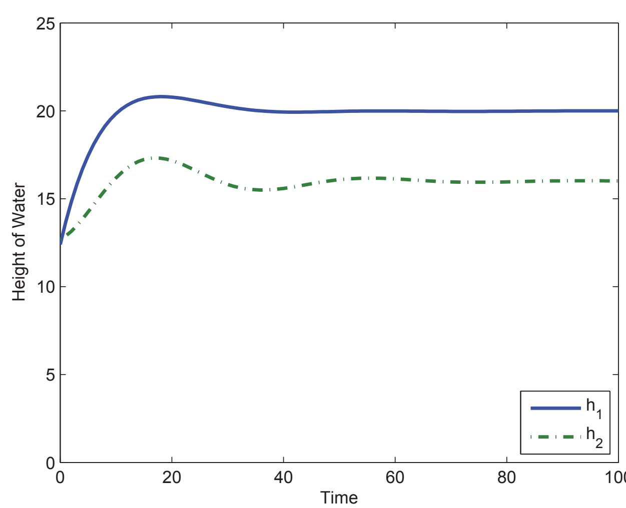

Fig. 3. Water Level in Tanks 1 and 2

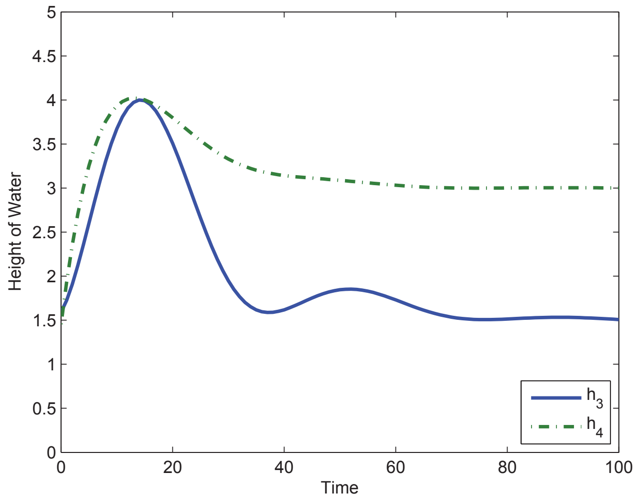

Fig. 4. Water Level in Tanks 3 and 4

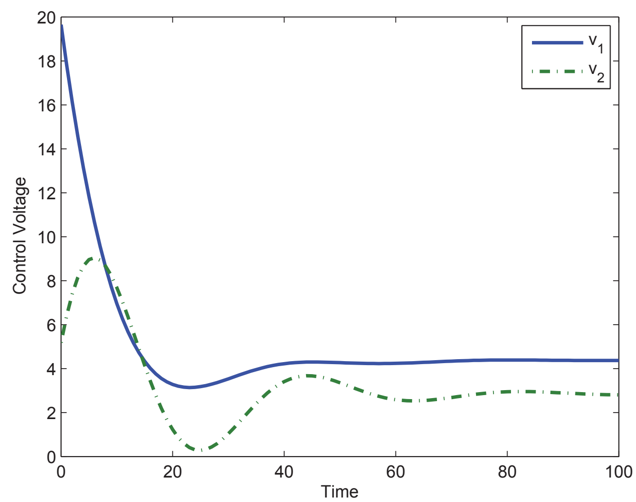

Fig. 5. Pump Control Voltage

The optimal gains for the PI controller were obtained using the iterative method described earlier. The numerical solution converged to final values of the gains as: $k_{p1} = 2.5858$, $k_{p2} = 1.5709$, $k_{i1} = 0.2190$ and $k_{i2} = 0.4714$. The water level in tanks 1 and 2 are $h_1 = 19.9978$ cm and $h_2 = 16.0126$ cm, respectively, as shown in Fig. 3 which are very close to the desired level of water in these two tanks. The corresponding water level in Tanks 3 and 4 are $h_3 = 1.5075$ cm and $h_4 = 3.0015$ cm respectively as shown in Fig. 4. The control voltages given in Fig. 5 maintain these water levels as desired.

V. CONCLUSIONS

This paper demonstrated the advantages of using decentralized optimal control for nonlinear systems for a quad tank interconnected systems. The proposed method is based on the integral minimum principle of Pontryagin which can be used for optimization of constant control parameters. The presented simulation result shows that decentralized optimal controller provides superior performance in tracking the set point with minimal error and reasonable settling time. Furthermore, the proposed nonlinear optimal control approach can be further applied in many large-scale interconnected industrial processes.

REFERENCES

[1] M. Frye, C. Qian, and R. Colgren, "Decentralized control of large-scale uncertain nonlinear systems by linear output feedback," *Communications in Information and Systems*, vol. 4, no. 3, pp. 191–210, 2005.

[2] Z. Jiang, "Decentralized and adaptive nonlinear tracking of large-scale systems via output feedback," *Automatic Control, IEEE Transactions on*, vol. 45, no. 11, pp. 2122–2128, 2000.

[3] H. Khalil and A. Saberi, "Decentralized stabilization of nonlinear interconnected systems using high-gain feedback," *Automatic Control, IEEE Transactions on*, vol. 27, no. 1, pp. 265–268, 1982.

[4] D. Siljak and A. Zecevic, "Control of large-scale systems: Beyond decentralized feedback," *Annual Reviews in Control*, vol. 29, no. 2, pp. 169–179, 2005.

[5] C. Tseng and B. Chen, "H_∞ decentralized fuzzy model reference tracking control design for nonlinear interconnected systems," *Fuzzy Systems, IEEE Transactions on*, vol. 9, no. 6, pp. 795–809, 2001.

[6] J. Ramos and P. dos Santos, "Mathematical modeling, system identification, and controller design of a two tank system," in *Decision and Control, 2007 46th IEEE Conference on*. IEEE, 2007, pp. 2838–2843.

[7] K. Johansson, "The quadruple-tank process: A multivariable laboratory process with an adjustable zero," *Control Systems Technology, IEEE Transactions on*, vol. 8, no. 3, pp. 456–465, 2000.

[8] S. Alavi and M. Hayes, "Quantitative feedback design for a benchmark quadruple-tank process," in *Irish Signals and Systems Conference, 2006. IET*. IET, 2006, pp. 401–406.

[9] X. Zuo, "Liquid level control of water tank system based on improved polyclonal selection algorithm and rbf network," in *Computer Engineering and Technology (ICCET), 2010 2nd International Conference on*, vol. 2. IEEE, 2010, pp. V2–528.

[10] S. Tabatabaeipour, A. Ravn, R. Izadi-zamabadi, and T. Bak, "Active diagnosis of hybrid systems-a model predictive approach," in *Control and Automation, 2009. ICCA 2009. IEEE International Conference on*. IEEE, 2009, pp. 465–470.

[11] Y. Zhao, "Research on application of fuzzy pid controller in two-container water tank system control," in *Machine Vision and Human-Machine Interface (MVHI), 2010 Internal Conference on*. IEEE, 2010, pp. 679–682.

[12] E. Davison and H. Fathi, "Description of control laboratory multivariable experiment:"marts"," in *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, vol. 1. IEEE, 1999, pp. 813–818.

[13] E. Rusli, S. Ang, and R. D. Braatz, "A quadruple-tank process control experiment," *Comments from the Editors*, 2004.

[14] P. P. Biswas, R. Srivastava, S. Ray, and A. N. Samanta, "Sliding mode control of quadruple tank process," *Mechatronics*, vol. 19, no. 4, pp. 548–561, 2009.

[15] D. Rosinová and M. Markech, "Robust control of quadruple tank process," *ICIC Express Letters*, vol. 2, no. 3, pp. 231–238, 2008.

[16] V. Boltyanski and A. Poznyak, "Linear multi-model time-optimization," *Optimal Control Applications and Methods*, vol. 23, no. 3, pp. 141–161, 2002.

[17] A. Poznyak, T. Duncan, B. Pasik-Duncan, and V. Boltyanski, "Robust maximum principle for multi-model lq-problem," *International Journal of Control*, vol. 75, no. 15, pp. 1170–1177, 2002.

[18] R. V. Gamkrelidze, K. Makowki, and L. D. Berkovitz, *Principles of optimal control theory*. Plenum Press New York, 1978.

[19] S. Biswas and N. Ahmed, "Optimal feedback control of power systems goverened by non-linear dynamics," *Optimal Control Applications and Methods*, vol. 7, no. 3, pp. 289–303, 1986.