Optimal Linear Quadratic Regulator of Switched Systems

Guangyu Wu, Jian Sun, *Member, IEEE*, and Jie Chen, *Senior Member, IEEE*

***Abstract*—This paper considers the optimal control problem of linear switched systems with LQ cost or multiple LQ cost. By adopting an embedding transformation, the switching design problem is relaxed and transformed into a traditional optimal control problem. The bang-bang-type solutions of the embedded optimal control problems are obtained for both the positive definite LQ cost case and the multiple LQ cost case, which are the optimal solution to the original problems. The switching sequence of modes and the switching instants can be calculated by solving a closed-form optimal switching condition. The optimal state feedback control law is determined simultaneously. Finally, numerical results are provided to illustrate the effectiveness of the proposed method.**

***Index Terms*—Switched system, optimal control, bang-bang-type solution, quadratic programming, switching condition.**

I. INTRODUCTION

HYBRID systems arise from the interaction between continuous variable systems and discrete event systems. Characterized by a group of subsystems with different dynamics, a hybrid system switches from one subsystem to another due to the occurrence of discrete events. Switched systems are a particular class of hybrid systems that consist of a set of subsystems, one of which is active at each instant, and a switching policy for activating a specified subsystem. Optimal control of a switched system involves finding a mode sequence, switching times between the modes and an input for each mode, which are strongly coupled.

Previous efforts in this field mainly focused on the necessary conditions for optimality, and on the approximations of the optimal switching law or suboptimal solutions. Applying the Maximum Principle, [1] got necessary conditions for a general switched optimal control problem. In context of a linear quadratic criteria, the hybrid control was determined by solving a sequence of differential Riccati equations. [2] presented the hybrid minimum principle(HMP) necessary conditions and proposed a HMP algorithm for the solution of a class of switched optimal control problems. [3] proposed an approximate dynamic programming-based algorithm for learning the optimal cost-to-go function and proved the convergence of the algorithm. [4] proposed a hybrid adaptive dynamic programming approach to solve the Bellman's equation iteratively over time, thereby adapting and optimizing the continuous and discrete control laws subject to actual system dynamics. Approximation methods usually have high computational complexity and require iterative computation. Since the computation of optimal strategy is demanding, [5] employed a sub-optimal cost function for discrete-time switched linear systems. Relaxed dynamic programming was used to reduce the solution space and optimality was relaxed within prespecified bounds. In [6], a relaxation framework was developed to simplify the computation of the value iterations of the infinite-horizon discrete-time switched linear quadratic regulator(LQR) problem with guaranteed closed-loop stability and suboptimal performance.

A fundamental issue of the optimal control of a switched system is to find the optimal switching instants with a fixed predefined mode sequence. Nonlinear-programming algorithms computing the gradient and second-order derivatives of the cost function were developed for solving the optimal switching instants of nonlinear systems. [7] first proposed a two-stage optimization method and put forward its solving algorithm. At the first stage, this method assumed there is a fixed order of active subsystems to minimize the cost function with respect to the switching instants. At the second stage, it varied the order and the number of switchings to find the optimal switching sequence. [8] obtained the derivatives of the optimal cost function based on the solution of a two point boundary value differential algebraic equation and applied it to the general switched LQ problem. To reduce the computational complexity, [9] used efficiently computable expressions for the cost function and used the gradient to solve the switching time optimization problem. [10] presented a method for computing the derivative of the optimal value for nonlinear switched systems, which resulted in a simple expression for the desired derivative.

Despite of all these existing methods, how to obtain a closed-form optimal solution of the switching sequence and the control input is still open. Even when the control input is absent, finding an optimal switching law is still challenging [11]. Up to now, the exact solution of a switched LQR problem is not available. The embedding transformation method is promising for the reason that it converts the switched LQR problem to a classical continuous optimal control problem by embedding the sequence of modes as a control variable. [12] formulated sufficient and necessary conditions for optimality of the embedded optimal control problem of a two-switched system. When necessary conditions indicated a bang-bang type of solution, one obtained a solution to the switched

This work was supported in part by the National Natural Science Foundation of China (Nos. 61522303, U1509215, 61621063, 61720106011), Program for Changjiang Scholars and Innovative Research Team in University (IRT1208), Youth ChangJiang Scholars Program, National Outstanding Youth Talents Support Program.(Corresponding author: Jian Sun.)

The authors are with The Key Laboratory of Intelligent Control and Decision of Complex System, Beijing Institute of Technology, Beijing, China 3120140377, sunjian@bit.edu.cn

Manuscript received April 19, 2005; revised August 26, 2015.

optimal control problem. [13] expressed the switching signal as polynomials and transformed a nonlinear and non-convex optimal control problem into an equivalent problem with linear and convex structure solved by high performance numerical computing.

For open-loop switched systems, a closed-form optimal switching condition with LQ cost was dealt as a 0-1 integer programming in [14] and a two point boundary value problem formed by the state and co-state was solved. [15] gave optimal necessary conditions and a switching law of the open-loop switched LQ problem. The obtained switching conditions were optimal in some generic cases when the optimal control was constant. In this technical note, we extend the results in [14] and [15] by solving the switched LQR problem of closed-loop systems with cost function defined on the state trajectory and the control input. Applying the embedding transformation method, we investigate two closed-form switching conditions involved by the switching law for LQ cost and multiple LQ cost when the mode sequence and the switching instants are unspecified. The switching dependent state feedback control law can be determined simultaneously. The main contributions of this paper are summarized as follows:

1) For the switched LQR problem, we show that computing the optimal switch input leads to a quadratic programming problem. The minimization of a concave function is solved and a bang-bang type solution is obtained. Therefore, a closed-form optimal switching condition of subsystems can be developed. The mode sequence and the switching instants are determined afterwards.

2) For the multiple switched LQR problem, we prove the Hessian matrix of the Hamilton function is negative semi-definite when the weighting matrix corresponding to the input is a diagonal matrix with positive diagonal entries. Therefore, the optimal solution of the embedded optimal problem is of bang-bang type.

The paper organization is as follows. In section 2, we formulate and solve the switched LQR problem with positive definite cost. We formulate the multiple switched LQR problem and derive an optimal switching condition in section 3. In section 4, two numerical examples are given to show results of the methods. Section 5 concludes this paper.

II. Switched LQ Regulator

Consider a switched system comprising a collection of N subsystems described by linear equations together with a switching rule.

$$\dot{x}\left(t\right) = A_{\sigma(t)} x\left(t\right) + B_{\sigma(t)} u\left(t\right) \tag{1}$$

where $x(t)$ is a n-dimensional state, $u(t)$ is a m-dimensional control input. A_σ and B_σ are matrices with appropriate dimensions. Moreover, switching function $\sigma\left(t\right)$ returns the index of active subsystem at time $t \in [t_0, t_f]$, where $\sigma\left(t\right) \in \{1, 2, \ldots, N\}$, t_0 is a fixed initial time and t_f is a fixed final time.

We investigate the situation where each subsystem (A_i, B_i) is controllable. We do not make any assumptions and constraints about the number of switches nor about the mode sequence. For simplicity,

$$\sum_{i=1}^{N} w_i(t) \triangleq \sum_{i} w_i \qquad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i(t) w_j(t) \triangleq \sum_{i,j} w_i w_j$$

The switched system can be represented by a combination of N subsystems.

$$\dot{x}\left(t\right) = \sum_{i} w_i(t) \left[A_i x\left(t\right) + B_i u\left(t\right)\right] \tag{2}$$

where $w_i\left(t\right) \in \{0, 1\}$. The switch input vector $w\left(t\right) = \left[w_1\left(t\right), \ldots, w_N\left(t\right)\right]^T$ can be given by a switching sequence as:

$$\xi = \left[\left(t_0, \sigma\left(t_0\right), u_0\right), \ldots, \left(t_K, \sigma\left(t_K\right), u_K\right)\right] \tag{3}$$

where $t_0 \leq t_1 \leq \ldots \leq t_K \leq t_f$ and $t_1, \ldots, t_K$ are the switching instants and K is the number of switching.

The problem of switched linear quadratic regulator (SLQR) can be defined as determining a switch input $w(t)$ and a control input $u(t)$ associated with a general LQ cost function for evaluating the system's performance quantitatively in a finite horizon $[t_0, t_f]$.

$$\min J = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T Q x + 2u^T S x + u^T R u\right) dt \tag{4}$$

where Q is a $n \times n$ positive semi-definite matrix, R is a $m \times m$ positive definite matrix.

Define

$$\bar{B}_{ij} = B_i R^{-1} B_j^T \tag{5}$$

where $B_i = \left[b_1^i, \cdots, b_n^i\right]^T$ and $b_j^i (j = 1, \cdots n)$ is a m-dimensional row vector. The elements of $\bar{B}_{ij}$ can be obtained

$$\bar{B}_{ij}\left(s, t\right) = b_s^i R^{-1} b_t^{j\,T} \tag{6}$$

where $s, t = 1, \cdots, n$.

By adopting the embedding transformation method, we embed the switched system into a larger family of systems by allowing $w_i(t)$ to vary continuously in the range $[0, 1]$. The SLQR problem can be transformed into the embedded switched LQR problem(ESLQR) as follows.

$$\begin{aligned} ESLQR: \min J &= \frac{1}{2}\int_{t_0}^{t_f} \left(x^T Q x + 2u^T S x + u^T R u\right) dt \\ s.t. \quad \dot{x}\left(t\right) &= \sum_{i} w_i(t)\left[A_i x\left(t\right) + B_i u\left(t\right)\right] \end{aligned} \tag{7}$$

The time-varying vector $w(t)$ belongs to a convex set W.

$$W = \left\{ w \in R^N : \sum_{i} w_i = 1, w_i \geq 0 \right\} \tag{8}$$

The set of trajectories of the embedded system (7) contains the trajectories of the switched system [12]. If a bang-bang-type solution of ESLQR is optimal, that is $w_i\left(t\right) \in \{0, 1\}$, then this type of solution is the solution of SLQR. This is discussed in the proof of the following theorem.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2872204, IEEE Transactions on Automatic Control

Theorem 1. *The switching condition of system (1) that minimize the cost functional (4) is*

$$i(t) = \arg\min_{i=1,\cdots N} \lambda^T(t)\left[(A_i - B_i R^{-1} S)x(t) - \frac{1}{2}\bar{B}_{ii}\lambda(t)\right] \tag{9}$$

and the optimal control input is

$$u(t) = -R^{-1}\left[Sx(t) + B_{i(t)}{}^T\lambda(t)\right] \tag{10}$$

where $\lambda(t) = [\lambda_1, \ldots, \lambda_n]^T$ *is the solution of*

$$\dot{\lambda}(t) = -\left[Qx(t) + S^T u(t) + A_{i(t)}^T \lambda(t)\right] \tag{11}$$

with the boundary condition $\lambda(t_f) = 0$.

Proof. The Hamilton function is defined as

$$\begin{aligned} &H[x,u,w,\lambda] \\ =&\frac{1}{2}\left[x(t)^T Q x(t) + 2u(t)^T S x(t) + u(t)^T R u(t)\right] \\ &+ \lambda^T(t)\sum_i w_i(t)\left[A_i x(t) + B_i u(t)\right] \end{aligned} \tag{12}$$

From (11), it is clear

$$\dot{\lambda}(t) = -\left[Qx(t) + S^T u(t) + \sum_i w_i A_i^T \lambda(t)\right] \tag{13}$$

By the coupled equation, it can be obtained

$$u(t) = -R^{-1}\left[Sx(t) + \sum_i w_i B_i{}^T \lambda(t)\right] \tag{14}$$

Substituting (14) into (12) yields

$$\begin{aligned} &H[x,w,\lambda] \\ =&\frac{1}{2}x^T(t)(Q - S^T R^{-1} S)x(t) + \lambda^T(t)\sum_i w_i A_i x(t) \\ &- \frac{1}{2}\lambda^T(t)\sum_{i,j} w_i w_j B_i R^{-1} B_j{}^T \lambda(t) \\ &- \lambda^T(t)\sum_i w_i B_i R^{-1} S x(t) \end{aligned} \tag{15}$$

Minimizing H with respect to $w(t)$ can be simplified to minimize

$$\begin{aligned} \bar{H}[x,w,\lambda] = &-\frac{1}{2}\lambda^T(t)\sum_{i,j} w_i w_j B_i R^{-1} B_j{}^T \lambda(t) \\ &+ \lambda^T(t)\sum_i w_i (A_i - B_i R^{-1} S)x(t) \end{aligned} \tag{16}$$

Minimizing $\bar{H}$ with respect to $w(t)$ can be viewed as a quadratic programming problem.

$$\begin{aligned} \min \quad & -\frac{1}{2}w(t)^T G(t) w(t) + q(t)^T w(t) \\ s.t. \quad & w(t) \in W \end{aligned} \tag{17}$$

where $q(t) = [q_1, \cdots, q_N]^T$ and

$$G(i,j) = \lambda(t)^T \bar{B}_{ij} \lambda(t) \tag{18}$$

$$q_i = \lambda(t)^T (A_i - B_i R^{-1} S)x(t) \tag{19}$$

Due to

$$\bar{B}_{ji} = B_j R^{-1} B_i{}^T = \bar{B}_{ij}^T \tag{20}$$

$$G(j,i) = \lambda(t)^T \bar{B}_{ji} \lambda(t) = G(i,j) \tag{21}$$

matrix $G(t)$ is symmetric.

To clearly express $G(t)$, we construct a new matrix M_{st}. The s-th row and t-th column element of $\bar{B}_{ij}$ are used as the i-th row and j-th column element of matrix M_{st}, i.e.,

$$M_{st}(i,j) = \bar{B}_{ij}(s,t) = b_s^i R^{-1} b_t^{j\,T} \tag{22}$$

Therefore,

$$M_{st} = \begin{bmatrix} b_s^1 R^{-1} b_t^{1\,T} & \cdots & b_s^1 R^{-1} b_t^{N\,T} \\ \vdots & & \vdots \\ b_s^N R^{-1} b_t^{1\,T} & \cdots & b_s^N R^{-1} b_t^{N\,T} \end{bmatrix} = N_s R^{-1} N_t{}^T \tag{23}$$

where $i,j = 1, \cdots N$ and $s,t = 1, \cdots n$.

$$N_s = \begin{bmatrix} b_s^1 \\ \vdots \\ b_s^N \end{bmatrix} \tag{24}$$

Owing to

$$\begin{aligned} G(i,j) &= \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \lambda_t \bar{B}_{ij}(s,t) \\ &= \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \lambda_t M_{st}(i,j) \end{aligned} \tag{25}$$

matrix $G(t)$ can be expressed as a linear combination of M_{st}.

$$\begin{aligned} G(t) &= \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \lambda_t M_{st} \\ &= \sum_{s=1}^{n}\sum_{t=1}^{n} \lambda_s \lambda_t N_s R^{-1} N_t{}^T \\ &= T R^{-1} T^T \end{aligned} \tag{26}$$

where matrix T is a linear combination of N_s.

$$T = \sum_{s=1}^{n} \lambda_s N_s \tag{27}$$

As R is positive definite, it is clear $-G(t) \leq 0$. Therefore, problem (17) is considered as a minimization of a concave function. In this case, the global minimum point of $\bar{H}$ is always attained at the extreme point of the convex set W, i.e., the optimal solution of the ESLQR problem is of bang-bang type. Therefore,

$$\begin{aligned} \bar{H}_m &= \min \bar{H} \\ &= \min_{i=1,\cdots N} \lambda^T(t)\left[(A_i - B_i R^{-1} S)x(t) - \frac{1}{2}\bar{B}_{ii}\lambda(t)\right] \\ &= \lambda^T(t)\left[(A_k - B_k R^{-1} S)x(t) - \frac{1}{2}\bar{B}_{kk}\lambda(t)\right] \end{aligned} \tag{28}$$

where $w_k = 1$ and $w_i = 0, \forall i \neq k$. This completes the proof. □

Remark 1. *A singular case is such that there exist at least two indices i, j for which*

$$\begin{aligned}&\lambda^T(t)\left[(A_i - B_iR^{-1}S)x(t) - \frac{1}{2}\bar{B}_{ii}\lambda(t)\right]\\ =&\lambda^T(t)\left[(A_j - B_jR^{-1}S)x(t) - \frac{1}{2}\bar{B}_{jj}\lambda(t)\right]\end{aligned} \tag{29}$$

on a non zero measure time interval [16]. The second order necessary conditions are given in the literature [17]. In this article, we only consider the nonsingular case.

Remark 2. *For the infinite horizon case,*

$$\lambda(t) = P_{i(t)}x(t) \tag{30}$$

and the algebra Riccati equation

$$P_iA_i + A_i^TP_i - \left(P_iB_i + S^T\right)R^{-1}\left(B_i^TP_i + S\right) + Q = 0 \tag{31}$$

The optimal control law is

$$u(t) = -R^{-1}\left(S + {B_i}^TP_i\right)x(t) \tag{32}$$

and the closed-loop system becomes

$$\dot{x}(t) = \left(A_i - B_iR^{-1}S - \bar{B}_{ii}P_i\right)x(t) \tag{33}$$

Remark 3. *For the indefinite cost case that matrix R is indefinite, matrix $G(t)$ is also indefinite. Therefore. problem (17) becomes a linearly constrained indefinite quadratic programming problem, which is a fundamental problem in global optimization. Since the exact global optimum is difficult to obtain, a number of approaches have been proposed to find the global approximate solutions such as [18] and [19]. Since the approximations may not be bang-bang type, only suboptimal solutions of the SLQR problem can be constructed according to [12].*

III. Multiple Switched LQ Regulator

For the multi-objective LQR problem, the tradeoffs among different performance indices vary in accordance with different system status [20]. We formulate the multiple switched LQR problem, using different tradeoff with respect to each subsystem. The overall objective function J is a sum of multiple quadratic performance indices J_i when the system switches from one subsystem to another.

$$J = \sum_i w_i(t)J_i(t) \tag{34}$$

$$J_i = \frac{1}{2}\int_{t_0}^{t_f}\left(x^TQ_ix + 2u^TS_ix + u^TR_iu\right)dt \tag{35}$$

where $w_i(t) \in \{0, 1\}$. In this section, we deal with the case that the weighting matrix R_i is a diagonal matrix with positive diagonal entries, i.e.,

$$R_i = diag(\gamma_i^1, \cdots, \gamma_i^m), \quad \gamma_i^k > 0 \tag{36}$$

where $k = 1, \cdots, m$. The optimal solution of the multi-objective LQR problem can be obtained by solving its embedded problem described as follows.

$$\begin{aligned}\min \quad & J\\ s.t. \quad & \dot{x}(t) = \sum_i w_i(t)\left[A_ix(t) + B_iu(t)\right]\\ & w(t) \in W\end{aligned} \tag{37}$$

Theorem 2. *The switching condition of system (1) that minimize the multiple LQ cost functional (34) is*

$$i(t) = \underset{i\in\{1,2,\cdots N\}}{\arg\min} -\frac{1}{2}\sum_{k=1}^{m}\frac{(f_i^k)^2}{{\gamma_i}^k} + q_i \tag{38}$$

with the optimal input

$$u(t) = -R_{i(t)}^{-1}\left[S_{i(t)}x(t) + {B_{i(t)}}^T\lambda(t)\right] \tag{39}$$

where $\lambda(t)$ is the solution of

$$\dot{\lambda}(t) = -\left[Q_{i(t)}x(t) + {S_{i(t)}}^Tu(t) + A_{i(t)}^T\lambda(t)\right] \tag{40}$$

with the boundary condition $\lambda(t_f) = 0$.

Proof. The Hamilton function is chosen as

$$\begin{aligned}&H[x, u, w, \lambda]\\ =&\frac{1}{2}\sum_i w_i\left[x(t)^TQ_ix(t) + 2u(t)^TS_ix(t) + u(t)^TR_iu(t)\right]\\ &+\sum_i w_i\lambda^T(t)\left[A_ix(t) + B_iu(t)\right]\end{aligned} \tag{41}$$

From (40), it is clear

$$\dot{\lambda}(t) = -\sum_i w_i\left[Q_ix(t) + {S_i}^Tu(t) + A_i^T\lambda(t)\right] \tag{42}$$

By the coupled equation, one can obtain

$$u(t) = -\left(\sum_i w_iR_i\right)^{-1}\sum_i w_i\left[S_ix(t) + {B_i}^T\lambda(t)\right] \tag{43}$$

Substituting (43) into (41) yields

$$\begin{aligned}&H[x, w, \lambda]\\ =&\frac{1}{2}\sum_i w_ix^T(t)Q_ix(t) + \sum_i w_i\lambda^T(t)A_ix(t)\\ &-\frac{1}{2}\sum_{i,j}w_iw_jx^T(t){S_i}^T\left(\sum_i w_iR_i\right)^{-1}S_jx(t)\\ &-\frac{1}{2}\sum_{i,j}w_iw_j\lambda^T(t)B_i\left(\sum_i w_iR_i\right)^{-1}{B_j}^T\lambda(t)\\ &-\sum_{i,j}w_iw_j\lambda^T(t)B_i\left(\sum_i w_iR_i\right)^{-1}S_jx(t)\end{aligned} \tag{44}$$

Simplifying the Hamilton function, we have

$$\bar{H}[x, w, \lambda] = -\frac{1}{2}\sum_{i,j}w_iw_jG(i, j) + \sum_i w_iq_i \tag{45}$$

where

$$G(i, j) = f_i(t)^T\left(\sum_i w_iR_i\right)^{-1}f_j(t) \tag{46}$$

$$f_i(t) = S_ix(t) + {B_i}^T\lambda(t) \tag{47}$$

$$q_i = \frac{1}{2}x^T(t)Q_ix(t) + \lambda^T(t)A_ix(t) \tag{48}$$

Define

$$f_i = \left[f_i^1, \cdots, f_i^m\right] \tag{49}$$

$\bar{H}$ can be rewritten as

$$\begin{aligned}\bar{H} &= -\frac{1}{2}\sum_{k=1}^{m}\frac{\sum_{i,j} w_i w_j f_i^k f_j^k}{\sum_i w_i {\gamma_i}^k} + \sum_i w_i q_i \\ &= -\frac{1}{2}\sum_{k=1}^{m}\frac{\left(\sum_i w_i f_i^k\right)^2}{\sum_i w_i {\gamma_i}^k} + \sum_i w_i q_i\end{aligned} \tag{50}$$

The second-order derivative of $\bar{H}$ with respect to w_i is

$$\frac{\partial^2 \bar{H}}{\partial w_i^2} = -\sum_{k=1}^{m}\frac{\left[h f_i^k - g\gamma_i^k\right]^2}{h^3} \leq 0 \tag{51}$$

The second-order partial derivative of $\bar{H}$ with respect to w_i and w_j is

$$\frac{\partial \bar{H}}{\partial w_i \partial w_j} = -\sum_{k=1}^{m}\frac{\left[h f_i^k - g\gamma_i^k\right]\left[h f_j^k - g\gamma_j^k\right]}{h^3} \tag{52}$$

where $\gamma^k = \left[\gamma_1^k, \ldots, \gamma_N^k\right]^T$ and

$$g = \sum_i w_j f_j^k \tag{53}$$

$$h = w^T \gamma^k \tag{54}$$

Thus, the Hessian matrix of $\bar{H}$ is

$$\begin{aligned}\frac{\partial^2 \bar{H}}{\partial w^2} &= -\frac{1}{h^3}\sum_{k=1}^{m}\begin{bmatrix} e_1^2 & e_1 e_2 & \cdots & e_1 e_N \\ e_2 e_1 & e_2^2 & \cdots & e_2 e_N \\ \vdots & \vdots & \ddots & \vdots \\ e_N e_1 & e_N e_2 & \cdots & e_N^2 \end{bmatrix} \\ &= -\frac{1}{h^3}\sum_{k=1}^{m}\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}\begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix} \leq 0\end{aligned} \tag{55}$$

where

$$e_i = h f_i^k - g\gamma_i^k \tag{56}$$

It indicates that the Hamilton function is a concave function. Similar to problem (17), the global minimum of the concave function is at one of the extreme points of the convex set W. Therefore, a closed-form optimal switching condition for the nonsingular case is obtained as

$$\min \bar{H} = \min_{i=1,\cdots N} -\frac{1}{2}\sum_{k=1}^{m}\frac{(f_i^k)^2}{{\gamma_i}^k} + q_i \tag{57}$$

This completes the proof. □

When the control input is absent, an algebraic switching condition can be obtained for autonomous systems.

Corollary 1. *The switching condition of the open-loop system*

$$\dot{x}\,(t) = A_{\sigma(t)} x\,(t) \tag{58}$$

that minimize the performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} x(t)^T Q_{\sigma(t)} x(t) dt \tag{59}$$

is

$$i(t) = \arg\min_{i=1,\cdots N}\left[\frac{1}{2}x(t)^T Q_i + \lambda^T(t) A_i\right] x\,(t) \tag{60}$$

where $\lambda\,(t)$ is the solution of

$$\dot{\lambda}(t) = -\left[Q_{i(t)} x(t) + A_{i(t)}^T \lambda(t)\right] \tag{61}$$

with the boundary condition $\lambda\,(t_f) = 0$.

Remark 4. *It should be noted that the switching condition (60) is the same as the algebraic condition in [15].*

IV. Illustrative Examples

A. Optimal Switched Control with positive definite LQ cost

Consider a switched system built with three controlled-systems

$$\dot{x} = \begin{cases} A_1 x + B_1 u_1, \; w = [1, 0, 0] \\ A_2 x + B_2 u_2, \; w = [0, 1, 0] \\ A_3 x + B_3 u_3, \; w = [0, 0, 1] \end{cases}$$

with

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

Choosing $Q = R = 2I$ and $S = I$ (I is an identity matrix), we obtain three positive definite solutions of (31) for three subsystems.

$$P_1 = \begin{bmatrix} 2.37 & -1.55 \\ -1.55 & 1.68 \end{bmatrix}, P_2 = \begin{bmatrix} 0.48 & -0.28 \\ -0.28 & 1.11 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.74 & -0.22 \\ -0.22 & 1.15 \end{bmatrix}$$

The initial state is $x_0 = [2, 1]^T$ and the co-state vector is $\lambda\,(t_0) = P_1 x_0 = [3.19, -1.42]^T$. Using the optimal switching condition (9), we figure out the optimal switching time, which is shown in Fig. 1. Note that the system stays in mode 2 for $0.14s \leq t < 1.31s$, in mode 3 for $t = 1.32s$, and switches frequently between modes 1 and 2 for $0.01s \leq t < 0.14s$ and $1.33s < t \leq 2$. The state trajectories under switched LQR are shown in Fig. 2. The switched feedback control input trajectories are shown in Fig. 3. The optimal value is $J = 2.01$.

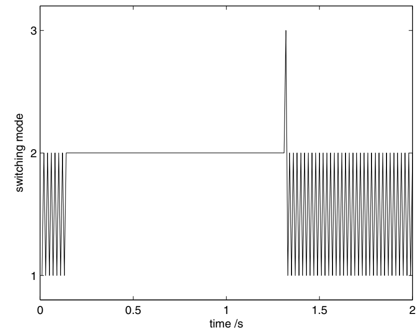

Fig. 1. Optimal switching time.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2872204, IEEE Transactions on Automatic Control

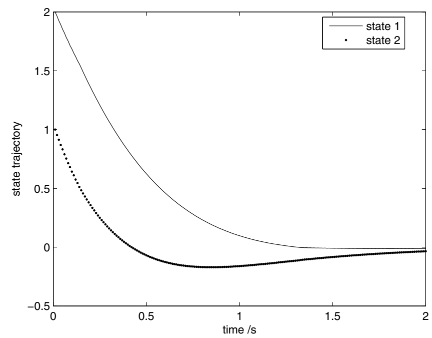

Fig. 2. State trajectories under switched LQR

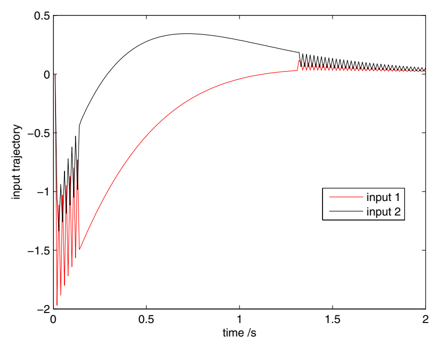

Fig. 3. Switched input trajectories

B. Application to a Power Converter

To illustrate the multiple switched LQ regulator, we take a buck-boost converter as an example [21]. The converter is described as a switched system that consists of two modes

$$(A_1, B_1) = \begin{pmatrix} -R/L & 0 \\ 0 & -1/R_0C_0 \end{pmatrix}, \begin{pmatrix} 1/L \\ 0 \end{pmatrix}$$
$$(A_2, B_2) = \begin{pmatrix} -R/L & -1/L \\ 1/C_0 & -1/R_0C_0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The state $x = [x_1, x_2]^T$ are the inductor current and the capacitor voltage respectively. Simulation parameters: $R = 0.08\Omega$, $L = 500\mu H$, $C_0 = 500\mu F$, $R_0 = 2\Omega$. Two performance indexes are

$$J_1 = \frac{1}{2}\int_{t_0}^{t_f} \left[(x - x_{ref})^T Q_1 (x - x_{ref}) + u^T R_1 u\right] dt \quad (62)$$

$$J_2 = \frac{1}{2}\int_{t_0}^{t_f} (x - x_{ref})^T Q_2 (x - x_{ref}) dt \quad (63)$$

where the weighting matrices are

$$Q_1 = 400I, \quad Q_2 = 200I, \quad R_1 = 2$$

A switching strategy and its corresponding feedback control law are designed to reach the equilibrium point, while minimizing the overall performance index (34). Fig. 4 and Fig. 5 show the startup from the initial condition $x_0 = (0, 0)$ to the reference $x_{ref} = (16, 16)$. The overall performance index under optimal switching was computed to be 43.88. The

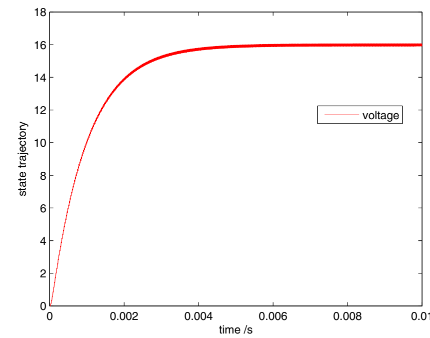

Fig. 4. State trajectories under multiple switched LQR.

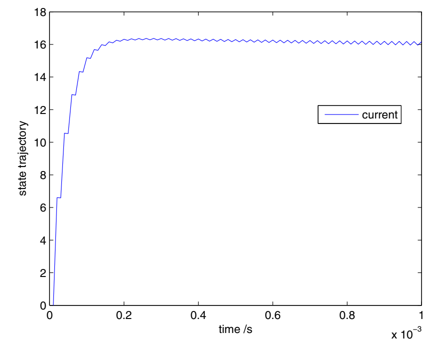

Fig. 5. State trajectories under multiple switched LQR.

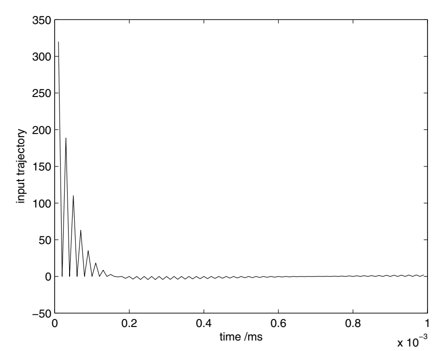

Fig. 6. Input trajectory

control input trajectory of mode 1 u_1 is shown in Fig. 6. Using the switching condition

$$i(t) = \arg\min_{i=1,\cdots N} \frac{1}{2}\tilde{x}(t)^T Q_i \tilde{x}(t) + \lambda^T(t)\left[A_i x(t) - \frac{1}{2}\bar{B}_{ii}\lambda(t)\right] \quad (64)$$

where

$$\begin{aligned} \tilde{x}(t) &= x(t) - x_{ref} \\ \dot{\lambda}(t) &= -Q_i\tilde{x}(t) - A_i^T\lambda(t) \end{aligned} \quad (65)$$

we obtain the optimal switching time in Fig. 7.

V. Conclusion

This paper has dealt with LQ regulator of switched systems, where the controlled variable is comprised of the switch

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2872204, IEEE Transactions on Automatic Control

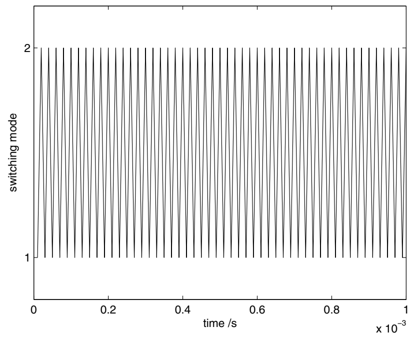

Fig. 7. Optimal switching time.

signal as well as the control input. We have investigated two optimization problems with different performance indexes and solved them by the embedding transformation method. The Hessian matrices of the Hamilton functions have been proven to be negative semi-definite, which leads to bang-bang type solutions of the optimization problems. As a result, two closed-form switching conditions are derived to obtain the optimal switching instants and optimal mode selection. Numerical examples have illustrated the efficacy of the proposed method.

REFERENCES

[1] P. Riedinger, F. Kratz, C. Iung, and C. Zane, "Linear quadratic optimization for hybrid systems," *in IEEE Conference on Decision and Control (CDC)*, Dec. 1999, pp. 3059-3064.

[2] M. S. Shaikh, P. E. Caines, "On the Hybrid Optimal Control Problem: Theory and Algorithms,"*IEEE Transactions on Automatic Control*, vol. 52, no.9, pp. 1587-1603, 2007.

[3] A. Heydari and S. N. Balakrishnan, "Optimal Switching and Control of Nonlinear Switching Systems Using Approximate Dynamic Programming,"*IEEE Transactions on Neural Networks and Learning Systems*, vol. 25, no.6, pp.1106-1117, 2014.

[4] W. Lu, P. Zhu and S. Ferrari, "A Hybrid-Adaptive Dynamic Programming Approach for the Model-Free Control of Nonlinear Switched Systems,"*IEEE Transactions on Automatic Control*, vol. 61, no.10, pp. 3203-3208, 2016.

[5] D. Gorges, M. Izak and S. Liu, "Optimal Control and Scheduling of Switched Systems", *IEEE Transactions on Automatic Control*, vol. 56, no.1, pp. 135-140, 2011.

[6] W. Zhang, J. Hu, A. Abate, "Infinite-Horizon Switched LQR Problems in Discrete Time: A Suboptimal Algorithm With Performance Analysis", *IEEE Transactions on Automatic Control*, vol. 57, no.7, pp. 1815-1821, 2012.

[7] X. Xu and P. J. Antsaklis, "Optimal Control of Switched Systems: New Results and Open Problems",*in Proceedings of American Control Conference*, June. 2000, pp. 2683-2687.

[8] X. Xu and P. J. Antsaklis, "Optimal Control of Switched Systems Based on Parameterization of the Switching Instants", *IEEE Transactions on Automatic Control*, vol. 49, no.1, pp. 2-16, 2004.

[9] B. Stellato, S. O. Blobaum and P. J. Goulart, "Optimal Control of Switching Times in Switched Linear Systems," *in IEEE Conference on Decision and Control (CDC)*, Dec. 2016, pp. 7228-7233.

[10] M. Kamgarpour and C. Tomlin, "On Optimal control of non-autonomous switched systems with a fixed mode sequence," *Automatica*, vol. 48, no.6, pp. 1177-1181, 2012.

[11] F. Zhu and P. J. Antsaklis, " Optimal control of hybrid switched systems: A brief survey," *Discrete Event Dynamic Systems*, vol. 25, no.3, pp. 345-364, 2015.

[12] S. C. Bengea and R. A. DeCarlo, " Optimal control of switching systems," *Automatica*, vol. 41, no.1, pp. 11-27, Jan. 2005.

[13] E. M. Nava, R. Meziat, N. Quijano, A. Gauthier and N. R. Ravalontsalama, "Optimal control of switched systems: a polynomial approach," *in: IFAC Proceedings*, July. 2008, pp. 7808-7813.

[14] T. Das and R. Mukherjee, " Optimally switched linear systems," *Automatica*, vol. 44, no.5, pp. 1437-1441, May. 2008.

[15] P. Riedinger, "A Switched LQ Regulator Design in Continuous Time", *IEEE Transactions on Automatic Control*, vol. 59, no.5, pp. 1322-1328, 2014.

[16] P. Riedinger, J. Daafouz and C. Iung, "Suboptimal switched controls in context for singular arcs," *in IEEE Conference on Decision and Control (CDC)*, Dec. 2003, pp. 6254-6259.

[17] A. J. Krener, "The higher order maximal principle and its application to singular extremals," *SIAM Journal on Matrix Analysis and Applications*, vol. 15, no.2, pp. 259-293, July. 1977.

[18] T. Q. Phong, L. T. H. An, P. D. Tao, "Decomposition branch and bound method for globally solving linearly constrained indefinite quadratic minimization problems,"*Operations Research Letters*, vol. 17, no.5, pp. 215-220, 1995.

[19] Y. Yajima, T. Fujie, "A Polyhedral Approach for Nonconvex Quadratic Programming Problems with Box Constraints,"*Journal of Global Optimization*, vol. 13, no.2, pp. 151-170, 1998.

[20] L. Liao, D. Li, "Successive method for general multiple linear-quadratic control problem in discrete time", *IEEE Transactions on Automatic Control*, vol. 45, no.7, pp. 1380-1385, 2000.

[21] P. Riedinger and J. C. Vivalda, "Dynamic output feedback for switched linear systems based on a LQG design,"*Automatica*, vol. 54, pp. 235-245, April. 2015.