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Consensus of linear multi-agent systems via event-triggered control

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This paper proposes an event-triggered control for multi-agent systems in which all agents have an identical linear dynamic mode. The asynchronous event-triggered control algorithms are proposed based on the triggering time sequences of all agents. The main contribution of this paper is to extend the event-triggered control method to investigate general linear multi-agent systems. First, by applying variable substitution method, we give the asynchronous triggering conditions. Based on the conditions, the consensus can be achieved both under fixed and switching topologies. Meanwhile, all the proposed event-triggered algorithms can exclude Zeno behaviours of the closed-loop systems. Then, the asynchronous results are applied to cope with formation control problem. Finally, numerical simulations are presented to illustrate the effectiveness of the part event-triggered protocol designs.

Keywords: multi-agent system; event-triggered control; consensus; formation control

1. Introduction

Distributed coordination of multi-agent systems has received significant attention in recent years. This is partly due to the broad applications of multi-agent systems including consensus, formation, flocking (see Lafferriere, Williams, Caughman, & Veerman, 2005; Ren & Beard, 2008; Olfati-Saber & Murray, 2004; Tanner, Jadbabaie, & Pappas, 2007; Xiao, Wang, Chen, & Gao, 2009 and references therein). Many researchers have focused on the consensus problem and got substantial results. Most of the early results concentrated upon the algorithms taking the form of first-order dynamics (Ren & Beard, 2005; Sun, Wang, and Xie, 2008). Extensions of consensus algorithms to second-order systems, linear systems and heterogeneous systems are investigated in Xie and Wang (2007), Zheng and Wang (2012), Wang, Cheng, and Hu (2008), and Tuna (2008).

Linear multi-agent systems have attracted considerable attention in the past few years. In Wang et al. (2008), the authors consider the consensus problem for linear multi-agent systems, in which all agents have an identical mode. If the linear dynamic mode is completely controllable, the consensus problem can be solved based on the neighbouring information. In Scardovi and Sepulchre (2009), the synchronisation of a group of identical linear systems has been investigated. The constructed dynamic output feedback coupling ensures that the solutions exponentially synchronise to a solution of the decoupled open-loop system under some assumptions. In Tuna (2008), linear feedback law based on algebraic Riccati equation has been shown and the synchronisation of the closed-loop linear multi-agent systems can be obtained if the graph topology has a spanning tree. Consensusability of linear time-invariant multi-agent systems has been studied in Ma and Zhang (2010). Some necessary and sufficient conditions on consensusability are given under some mild conditions. In Guan et al. (2013), the decentralised stabilisability for linear multi-agent systems under fixed and switching topologies is studied. The authors give sufficient and necessary conditions on stabilisability for fixed topology.

Event-triggered control seems to be a very useful method to deal with the consensus problems of multi-agent systems. Many research studies have been carried out on this topic. In Tabuada (2007), a simple event-triggered controller based on a feedback mechanism is investigated, and the controller can relax traditional periodic execution requirements. Recently, Heemels, Johansson, and Tabuada (2012) provide an introductory overview on part work in the event-triggered control and self-triggered control. An event-triggered implementation of the consensus protocol for multi-agent systems is considered in Dimarogonas, Frazzoli, and Johansson (2012). The authors present centralised and distributed event-triggered cooperative control such that all agents are asymptotically stabilised to their initial average. In Seyboth, Dimarogonas, and Johansson (2013), a novel event-triggered control strategy for distributed multi-agent coordination is proposed. The novel event-triggered scheduling strategy bounds each agent's measurement error by a constant or a time-dependent threshold. In Chen and

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Hao (2012), both event-triggered control and self-triggered control for multi-agent systems are investigated. The authors give a sufficient condition described by an LMI to determine the parameter matrices in the event condition, which can render the system to average consensus. However, a drawback of the existing event-triggered controllers with state-dependent thresholds is that each agent requires to be triggered at the neighbours' event time. In order to remove the drawback, the authors propose a combinational measurement approach to design events and give a new event-triggered algorithm where continuous measuring of the neighbour states has been avoided in Fan, Feng, Wang, and Song (2013). An average consensus problem based on event-triggered control algorithms for multi-agent systems over fixed and switching topologies is considered in Meng and Chen (2013). The authors introduce a new Lyapunov function, and the derivative of the Lyapunov function is made negative by choosing appropriate event conditions. In Xiao, Meng, and Chen (2012), a scheme of sampled control triggered by edge events for distributed state consensus is set-up in order to reduce the communication cost. As intriguing topics, edge events are first proposed for multi-agent systems.

With this background, we consider asynchronous event-triggered strategy for linear multi-agent systems on undirected and connected graphs. The synchronous strategy, also named centralised approach in Dimarogonas et al. (2012), is useful but conservative. In this strategy, all agents have to be aware of the global information when enforcing the triggering condition. In order to cope with the drawback of synchronous strategy, we advance asynchronous strategy for linear multi-agent systems over fixed and switching topologies. The main difference between this work and previous works (Dimarogonas et al., 2012; Fan et al., 2013; Meng & Chen, 2013; Seyboth et al., 2013), which also is the major contribution of this work is as follows. The authors consider first-order or second-order multi-agent systems in Dimarogonas et al. (2012), Seyboth et al. (2013), Fan et al. (2013), and Meng and Chen (2013). We consider more general case, where the dynamic of each agent is an nth-order linear system. The increase of the order will bring new features and difficulties when studying the consensus problem.

In the applications of linear multi-agent systems in many areas including unmanned air vehicles, autonomous underwater vehicles, attitude alignment of clusters of satellites, the implementation of the controller is realised on a digital platform (Olfati-Saber & Murray, 2004; Ren & Beard, 2008). Specifically, each agent is equipped with an embedded microprocessor, which collects information from neighbouring agents and coordinates the controller updates in a discrete-time manner. In any computer-controlled system, energy consumption is correlated with the rate at which microprocessors are computing control inputs and actuator signals are being updated. The conventional time-triggered control is costly and might lead to inefficient implementation. More importantly, actuator updates periodically is often unnecessary (Dimarogonas et al., 2012; Heemels et al., 2012). Event-triggered control may be better alternative in order to reduce the number of the actuator updates and hence reduce waste of communication resources. Meanwhile, when the limited resources of embedded processors are considered, an event-triggered control approach also seems more favourable for the application. We prove that the event-triggered control can be applied to deal with the formation problem. The previous works, such as Dimarogonas et al. (2012), Seyboth et al. (2013), Fan et al. (2013), and Meng and Chen (2013), do not involve the application of the event-triggered control. The theoretical results are illustrated through simulations.

The paper is organised as follows. In Section 2, some basic concepts and preliminaries are introduced. The asynchronous event-triggered controllers are considered to solve the consensus problem in Section 3. In Section 4, the results are applied to solve the formation control problem. The simulation results are presented to show the effectiveness of our results in Section 5. Section 6 offers a summary of the results of this paper and its possible extensions.

Notation: The following notations will be used throughout this paper. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ dimensional real matrices. I_n and $\mathrm{diag}\{a_1, \dots, a_n\}$ represent the $n \times n$ identity and diagonal matrices, respectively. Let $\mathbf{1}_n$ denote the all-1 vector with dimension n. Matrix $P > 0 (\geq 0, < 0, \leq 0)$ means P is positive definite(positive semi-definite, negative definite, or negative semi-definite). For a given vector or matrix Q, Q^T denotes its transpose. $\otimes$ denotes the Kronecker product. $|\cdot|$ denotes the Euclidean norm both for vectors and matrices.

2. Preliminaries and problem statement

2.1 *Graph preliminaries*

In this section, some useful concepts about algebraic graph theory are briefly reviewed, since algebraic graph theory is a useful tool to solve the coordination problems of multi-agent systems. A weighted undirected graph is denoted by $\mathbb{G} = (V, E, \mathcal{A})$, where $V = \{1, 2, \dots, N\}$ represents the index set of N agents with i representing the ith agent and $E \subseteq V \times V$ represents the edge set of paired agents; $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix, with the property that $(i, j) \in E$ if and only if $a_{ij} > 0$. Note that for an undirected graph $\mathbb{G}$, the corresponding adjacency matrix $\mathcal{A}$ is symmetric. The existence of an edge $(i, j), i \neq j$ means that there exists an available information channel connecting agent i with agent j. In this paper, we assume that there are no self-loops, i.e. $e_{ii} \notin E$ for any $i \in V$. The set of neighbours of agent i is denoted by $N_i = \{j \in V | (j, i) \in E\}$. Each agent is assigned a subset $N_i \subset \{1, 2, \ldots, N\}$ of the other agents, called agent i's neighbour set, which includes the agents with which it can communicate. The degree matrix of graph $\mathbb{G}$ is represented by $D = diag\{d_1, d_2, \ldots, d_N\}$, where $d_i = \Sigma_{j=1}^{N} a_{ij}$ is the degree of agent i. Correspondingly, the Laplacian matrix of $\mathbb{G}$ is defined as $L = D - \mathcal{A}$. A path from agent i to agent j is a sequence of distinct agents, such that each pair of consecutive agents is adjacent. The graph is said to be connected if there exists a path between any two distinct agents.

2.2 *Problem statement*

The multi-agent system studied in this technical note consists of N agents, labelled 1 through N, and they follow the linear dynamics, described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), i = 1, 2, \ldots, N, \tag{1}$$

where the state $x_i(t) \in \mathbb{R}^n$ denotes the state of agent i and $u_i(t) \in \mathbb{R}^n$ its control input or protocol. Matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices.

Assumption 2.1:

$$\text{rank}(AB) = \text{rank}(A). \tag{2}$$

Remark 1: Assumption 2.1 is not a very strong restriction on the system, because the first-order multi-agent systems and second-order multi-agent systems can be regarded as special cases of system (1) under the assumption. Also, it is a weaker condition since we can easily get (2) if $B = I_n$. If $A = 0$ and $B = I_n$, the dynamics of the ith agent are described as follows:

$$\dot{x}_i(t) = u_i(t), i = 1, 2, \ldots, N,$$

which are first-order multi-agent systems studied in Ren and Beard (2005) and Sun et al. (2008). Note that the second-order multi-agent systems which also have been investigated in many early works such as Xie and Wang (2007) and so on can be written as follows:

$$\dot{\zeta}_i(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \zeta_i(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i(t),$$

where $\zeta_i(t) = [x_i^T(t), v_i^T(t)]^T$. Furthermore, we consider the generalised second-order multi-agent systems, which are described as

$$\dot{x}_i(t) = A_{veh}x_i(t) + B_{veh}u_i(t), i = 1, 2, \ldots, N, x_i \in \mathbb{R}^{2n},$$

where

$$A_{evh} = \left(\begin{pmatrix} 0 & 1 \\ 0 & a_{22}^1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 0 & a_{22}^n \end{pmatrix} \right), B_{evh} = I_n \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The model has been studied in Lafferriere et al. (2005) and Ma and Zhang (2012). As special cases of system (1), it can be concluded that all the three typical multi-agent systems satisfy Assumption 2.1.

Consider the above N agents, the communication relationships among them are described by a graph $\mathbb{G}$. First of all, an assumption of the communication graph is given.

Assumption 2.2: *The communication graph $\mathbb{G}$ is undirected and connected.*

Now we introduce the event-triggered control scheme. The agent i monitors its own state $x_i(t)$ continuously and decides when to broadcast its current state over the network based on the neighbour's information. The latest broadcast state of agent i is given by

$$\hat{x}_i(t) = x_i(t_k^i), t \in [t_k^i, t_{k+1}^i), \tag{3}$$

where $t_0^i, t_1^i, \ldots$ is the triggering time sequence of agent i. In this work, agent i uses its last broadcast value $\hat{x}_i(t)$ and the neighbours' last broadcast value $\hat{x}_j(t)$, $j \in N_i$.

Now, we present the following protocol:

$$u_i(t) = -K \sum_{j\in N_i} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)), \tag{4}$$

where $K \in \mathbb{R}^{m\times n}$ is a parameter matrix being designed. Let $x(t)$ denote the concatenations of vectors $x_1(t), \ldots, x_N(t)$. The protocol can be written in stack vector form

$$u(t) = -(L \otimes K)\hat{x}(t). \tag{5}$$

Then the closed-loop system of (1) with the protocol (4) can be written in matrix form as

$$\dot{x}(t) = (I_N \otimes A)x(t) - (L \otimes BK)\hat{x}(t). \tag{6}$$

For each $i \in V$ and $t \geq 0$, the state measurement error is defined by

$$e_i(t) = \hat{x}_i(t) - x_i(t). \tag{7}$$

Denote the vector $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$, and the closed-loop system is then given by

$$\dot{x}(t) = (I_N \otimes A - L \otimes BK)x(t) - (L \otimes BK)e(t). \tag{8}$$

Remark 2: Under Assumption 2.2, the corresponding Laplacian matrix L is symmetric and positive semi-definite; hence, the eigenvalues of the matrix L are real and can be labelled as

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N.$$

This is a general assumption made in the early papers and works.

3. Asynchronous updates

3.1 *Fixed topology*

In order to reduce the communication between neighbouring agents and the number of control updates, we give an asynchronous updates result. First, the multi-agent networks with fixed topology are considered.

The event triggering condition for agent i has the following form:

$$|e_i(t)| = \kappa_i |z_i(t)|, i = 1, 2, \ldots, N, \tag{9}$$

where $\kappa_i > 0$ will be determined later, $z_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j)$.

Let

$$y_i(t) \triangleq e^{-A(t-t_0)} x_i(t), \; w_i(t) \triangleq e^{-A(t-t_0)} e_i(t), \tag{10}$$

and then

$$\begin{aligned}\dot{y}_i(t) &= -Ae^{-A(t-t_0)} x_i(t) + e^{-A(t-t_0)} A x_i(t) \\ &\quad - e^{-A(t-t_0)} BK \sum_{j=1}^{N} a_{ij}(x_i - x_j + e_i - e_j) \\ &= -e^{-A(t-t_0)} BK \sum_{j=1}^{N} a_{ij}(x_i - x_j + e_i - e_j).\end{aligned} \tag{11}$$

Under Assumption 2.1, there always exists $K \in \mathbb{R}^{m\times n}$ such that $ABK = A$ (Serre, 2002), then the system (11) can be written as

$$\dot{y}_i(t) = -\sum_{j=1}^{N} a_{ij}(y_i - y_j) - \sum_{j=1}^{N} a_{ij}(w_i - w_j). \tag{12}$$

Let $y(t)$, $w(t)$ be the stacking of the N vectors, i.e. $y(t) = [y_1^T, y_2^T, \ldots, y_N^T]^T$, $w(t) = [w_1^T, w_2^T, \ldots, w_N^T]^T$. Then the systems (12) can be written further in the form of vectors:

$$\dot{y}(t) = -(L \otimes I_n) y(t) - (L \otimes I_n) w(t). \tag{13}$$

Now we consider the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2} y^T(t)(L \otimes I_n) y(t). \tag{14}$$

Taking the derivative of $V(t)$ along the trajectory (13) yields

$$\begin{aligned}\dot{V}(t) &= y^T(t)(L \otimes I_n)((L \otimes I_n) y(t) - ((L \otimes I_n) w(t))) \\ &= -y^T(t)(L \otimes I_n)(L \otimes I_n) y(t) \\ &\quad - y^T(t)(L \otimes I_n)(L \otimes I_n) w(t)).\end{aligned} \tag{15}$$

Using the inequality based on matrix L is non-negative.

$$\begin{aligned}& y^T(t)(L \otimes I_n)(L \otimes I_n) w(t)) \\ &\le \frac{1}{2} y^T(t)(L \otimes I_n)(L \otimes I_n) y(t)) \\ &\quad + \frac{1}{2} w^T(t)(L \otimes I_n)(L \otimes I_n) w(t)).\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{V}(t) &\le -\frac{1}{2} y^T(t)(L \otimes I_n)(L \otimes I_n) y(t)) \\ &\quad + \frac{1}{2} w^T(t)(L \otimes I_n)(L \otimes I_n) w(t)).\end{aligned} \tag{16}$$

Then based on the event-triggering condition (9), we can have that

$$e^T(t) e(t) \le \kappa_{\max}^2 x^T(t)(L \otimes I_n)(L \otimes I_n) x(t)),$$

where $\kappa_{\max} = \max\{\kappa_1, \kappa_2, \ldots, \kappa_N\}$. Combining the transformation (10), we get that

$$w^T(t) w(t) \le \kappa_{\max}^2 y^T(t)(L \otimes I_n)(L \otimes I_n) y(t)).$$

It follows

$$\begin{aligned}& w^T(t)(L \otimes I_n)(L \otimes I_n) w(t) \le \lambda_N^2 w^T(t) w(t) \\ &\quad \le \kappa_{\max}^2 \lambda_N^2 y^T(t)(L \otimes I_n)(L \otimes I_n) y(t)).\end{aligned}$$

Thereby,

$$\dot{V}(t) \le -\frac{1}{2}(1 - \kappa_{\max}^2 \lambda_N^2) y^T(t)(L \otimes I_n)(L \otimes I_n) y(t)), \tag{17}$$

which means $V(t)$ is non-increasing if $\kappa_{\max} < \frac{1}{\lambda_N}$. Similarly to Dimarogonas et al. (2012), we can have that $\lim_{t\to\infty} y_i(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t_0) = \frac{1}{N}\sum_{i=1}^{N} x_i(t_0) = \bar{x}$. That is to say $\lim_{t\to\infty} |e^{-A(t-t_0)} x_i - \bar{x}| = 0$, which implies that $\lim_{t\to\infty} x_i = \bar{x} e^{A(t-t_0)} = x_c(t)$ if all the eigenvalues of A belong to the imaginary axis. Thus all the states $x_i(t)$, $i = 1, \ldots, N$, asymptotically converge to $x_c(t)$. The following presents a conclusion from the above analysis.

Theorem 3.1: *Under Assumptions 2.1 and 2.2, consider the multi-agent system* (1) *with the control law* (4)*. Suppose the event condition is given by* (9)*. If* $\kappa_{max} < \frac{1}{\lambda_N}$*, then the consensus problem can be solved asymptotically for any initial conditions. Furthermore, there exists at least one agent* k *for which for any initial condition the inter-event times of agent* k *are lower bounded by* $\tau_k > 0$.

Proof: The consensus result can be obtained from the above analysis. Now we determine the lower bound on the inter-event times for all agents. Assume that at initial time t_0 all the state measurement errors are zero. Let

$k \triangleq \arg\max_i |z_i|$, combined with

$$|e_k(t)| \leq |e(t)|,$$

we can get that

$$\frac{|e_k(t)|}{N|z_k(t)|} \leq \frac{|e(t)|}{|z(t)|}$$

so that

$$\frac{|e_k(t)|}{|z_k(t)|} \leq N\frac{|e(t)|}{|z(t)|}.$$

Similar to Dimarogonas et al. (2012), the next inter-event interval of agent k is bounded by τ_k, which satisfies

$$\frac{\beta - \alpha}{\beta e^{(\alpha-\beta)\tau_k} - \alpha} - 1 = \frac{\kappa_k}{N}$$

and

$$\tau_k = \frac{1}{\alpha - \beta}\ln\left[\frac{\alpha\kappa_k + N\alpha}{\beta\kappa_k + N\alpha}\right].$$

The proof is completed. □

According to the protocol (4), the sampled values need to be determined. Compared to time-triggered control, it is more difficult to determine the sampled values since the values are sampled aperiodically. The following algorithm is given to determine the sampled values used in the event-triggered control.

Algorithm 1: Determination of sampled values $\hat{x}_i(t)$

At all times t, for agent i
Initialisation:
  $t_k^i = t_0$;
1: **while** $t > t_k^i$
2:   **if** $|e_i(t)| < \kappa_i |z_i(t)|$ holds **then**
3:     **if** there exists no agent $j \in N_i$ sending new sampled value $\hat{x}_j$ **then**
4:       the controller keeps its current state
5:     **else**
6:       update the control protocol (4)
7:     **end if**
8:   **else**
9:     sample the state value; $t_{k+1}^i = t$; update the control protocol (4)
10:   **end if**
11: **end while**
12: **return** sampled value $\hat{x}_i(t)$

Zeno behaviour is an important problem and should be excluded in event-triggered control. In the above result, we show that there exists at least one agent for which its inter-event interval is strictly positive. In the following part, we will give an improved result that ensures that the inter-event intervals for all agents are strictly positive. The events are also designed based on the local information. The improvements lie in that the events of agent i are computed based on the last sampled information transmitted by agent i and received from its neighbours. We give the following event condition:

$$|e_i(t)| = \kappa_i |\hat{z}_i(t)|, i = 1, 2, \ldots, N, \tag{18}$$

where $\kappa_i > 0$ will be determined later, $\hat{z}_i = \sum_{j=1}^N a_{ij}(\hat{x}_i - \hat{x}_j)$.

Corollary 3.2: *Under Assumptions 2.1 and 2.2, consider the multi-agent system* (1) *with the control law* (4). *Suppose the event condition is given by* (18). *If $\kappa_{max} < \frac{1}{\lambda_N}$, then the consensus problem can be solved asymptotically for any initial conditions. Furthermore, the inter-event intervals of each agent are strictly positive.*

Proof: The change of variable is first given as follows:

$$\tilde{y}_i(t) \triangleq e^{-A(t-t_0)}\hat{x}_i(t). \tag{19}$$

Then, according to the similar proof of Theorem 3.1, we can safely get the conclusion that protocol (4) will solve the consensus problem under the event condition (18). Moreover, we can prove that all the agents will not exhibit Zeno behaviour; that is, the lower bounds on the inter-event times for all agents are strictly positive. Assume that agent i triggers at time $t_{k_i} \geq 0$, thus $e_i(t_{k_i}) = 0$. Between two events the evolution of the $e_i(t)$ over the interval $t \in [t_{k_i}, t_{k_i+1})$ is given by $\dot{e}_i(t) = -\dot{x}_i(t)$. Observe that

$$\begin{aligned}
\frac{d}{dt}|e_i(t)| &\leq |\dot{x}_i(t)| \\
&= \left|Ax_i(t) - \sum_{j\in N_i} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t))\right| \\
&= \left|A(\hat{x}_i(t) - e_i(t)) - \sum_{j\in N_i} a_{ij}(\hat{x}_i(t) - \hat{x}_j(t))\right| \\
&\leq |A\hat{x}_i(t) - \hat{z}_i(t)| + |A||e_i(t)|,
\end{aligned}$$

where $\hat{z}_i(t) = \sum_{j=1}^N a_{ij}((\hat{x}_i(t) - \hat{x}_j(t))$. And consider the following differential equation:

$$\dot{\xi}_i(t) = |A|\xi_i(t) + |A\hat{x}_i(t) - \hat{z}_i(t)|, \tag{20}$$

where the initial condition $\xi_i(t_{k_i}) = 0$. By some operations we can have that

$$|e_i(t)| \leq \xi_i(t) = \int_{t_{k_i}}^{t} |A\hat{x}_i(t) - \hat{z}_i(\tau)|e^{|A|(t-\tau)}d\tau. \tag{21}$$

Combined with the event condition (18), when $\hat{x}_i(t) - \hat{x}_j(t) \neq 0$, the next inter-event interval τ_{k_i} is bounded by the interval it takes for $\xi_i(t)$ to evolve from 0 to

$\kappa_i|\sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j)|$. Thus, the interval τ_{k_i} is a strictly positive value, which indicates that agent i will not exhibit Zeno behaviours. □

3.2 *Switching topology*

In the following part, we extend the event-triggered control protocol for multi-agent networks with switching topology. First, we give the collection of undirected and connected graphs with the same vertex set. Let $\Gamma = \{\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_M\}$ and $\mathbb{M} = \{1, 2, \ldots, M\}$ be the index set. The switching networks can be modelled using a dynamic graph $\mathbb{G}_{s(t)}$ parameterised with a switching signal $s(t) : [0, +\infty) \to \mathbb{M}$. Consider an infinite sequence of non-empty and continuous time intervals $[T_k, T_{k+1})$, $k = 0, 1, 2, \ldots$, where $T_0 = 0$. Then the event-triggered protocol on the switching networks can be given as follows:

$$u(t) = -(L_{s(t)} \otimes K)\hat{x}(t). \tag{22}$$

After the similar variable substitution in Section 3.1, we can get the systems:

$$\dot{y}(t) = -(L_{s(t)} \otimes I_n)y(t) - (L_{s(t)} \otimes I_n)w(t). \tag{23}$$

Based on the protocol (22), we first state the following theorem.

Theorem 3.3: *Consider the multi-agent system* (1) *with switching topology graphs* $\mathbb{G}_{s(t)}$. *At every time interval, the switching topology graphs* $\mathbb{G}_{s(t)}$ *are connected. Suppose the event condition is given by* (9). *If* $\kappa_{max} < \frac{1}{\lambda_N}$, *where* $\lambda_N = max\{\lambda_N(\mathbb{G}_1), \lambda_N(\mathbb{G}_2), \ldots, \lambda_N(\mathbb{G}_M)\}$, *then the multi-agent dynamical system* (1) *under the control law* (22) *can reach consensus asymptotically. Furthermore, there exists at least one agent* k *for which for any initial condition the inter-execution times of agent* k *are lower bounded by* $\tau_k > 0$.

Proof: Choose the common Lyapunov function

$$V(t) = \frac{1}{2}y^T(t)y(t). \tag{24}$$

The derivative of $V(t)$ along the trajectories of system (23) is given by

$$\dot{V}(t) = y^T(t)\dot{y}(t) = -y^T(t)(L_{s(t)} \otimes I_n)y(t) - y^T(t)(L_{s(t)} \otimes I_n)w(t).$$

Then the following inequality holds

$$y^T(t)(L_{s(t)} \otimes I_n)w(t) \le \frac{1}{2}y^T(t)(L_{s(t)} \otimes I_n)y(t) + \frac{1}{2}w^T(t)(L_{s(t)} \otimes I_n)w(t).$$

Based on the event condition and similar operations in Section 4.1, we get

$$\begin{aligned}\dot{V}(t) &\le -\frac{1}{2}y^T(t)(L_{s(t)} \otimes I_n)y(t) + \frac{1}{2}w^T(t)(L_{s(t)} \otimes I_n)w(t)\\ &\le -\frac{1}{2}(1 - \kappa_{\max}^2\lambda_N^2)y^T(t)(L_{s(t)} \otimes I_n)y(t),\end{aligned} \tag{25}$$

where $\kappa_{\max}$ is defined as in Theorem 3.1, which indicates $V(t)$ is non-increasing if $\kappa_{\max} < \frac{1}{\lambda_N}$. Note that $V(t) = \frac{1}{2}y^T(t)y(t) = \frac{N}{2}\sum_{i=1}^{N}\frac{1}{N}y_i^2(t)$ and denote

$$\bar{y}(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{26}$$

It can be easily proved that $\dot{\bar{y}}(t) = 0$ based on protocol (22) and $V(t) \ge \frac{N}{2}\bar{y}^2 = V(\bar{y}\mathbf{1})$ by using Jensen's inequality. Moreover, the largest invariant set contained in the set $S = \{y \in \mathbb{R}^n | \dot{V}(t) = 0\} = \text{span}\{\mathbf{1}\}$. From Lasalle's invariance principle, $|y_i - y_j| \to 0$, that is, $|x_i - x_j| \to 0$. Furthermore, if all the eigenvalues of A belong to the imaginary axis, we can have that $\lim_{t\to\infty} x_i = \bar{x}e^{A(t-t_0)} = x_c(t)$ according to Theorem 3.1. Thus, all the states $x_i(t), i = 1, \ldots, N$, asymptotically converge to $x_c(t)$. The following presents a conclusion from the above analysis. The lower bound on the inter-execution times can be obtained straightforwardly from those of Theorem 3.1. □

Corollary 3.4: *Consider the multi-agent system* (1) *with switching topology graphs* $\mathbb{G}_{s(t)}$. *At every time interval, the switching topology graphs* $\mathbb{G}_{s(t)}$ *are connected. Suppose the event condition is given by* (18). *If* $\kappa_{max} < \frac{1}{\lambda_N}$, *where* $\lambda_N = max\{\lambda_N(\mathbb{G}_1), \lambda_N(\mathbb{G}_2), \ldots, \lambda_N(\mathbb{G}_M)\}$, *then the multi-agent dynamical system* (1) *under the control law* (22) *can reach consensus asymptotically. Furthermore, the inter-event intervals of each agent are strictly positive.*

Proof: The assertions are direct consequences of the proof of Corollary 3.2 and Theorem 3.3. The details of the proof is omitted due to page limitations. □

4. Formation control

Formation control for multi-agent systems is a charming research topic in cooperative control, due to its wide applications in unmanned aerial vehicles, mobile robot systems and so on (Lafferriere et al., 2005; Ren & Beard, 2008; Xiao et al., 2009). We will develop a formation framework for multi-agent systems by applying event-triggered controllers in this section. Define $h_i \in \mathbb{R}^n$ represents the desired formation of agent i. Let $h = [h_1^T, h_2^T, \ldots, h_N^T]^T$, and vector h defines the basic frame of the expected formation.

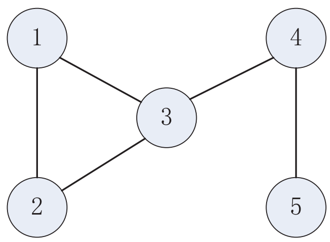

Figure 1. Communication topology.

First, for multi-agent system (1), the protocol of the agent i is described as follows:

$$u_i(t) = -K\sum_{j\in N_i} a_{ij}((\hat{x}_i(t)-h_i)-(\hat{x}_j(t)-h_j)). \quad (27)$$

The events are computed based on neighbours' information and the desired formations of their neighbours, and the event triggering condition for agent i has the following form:

$$|e_i(t)| = \mu_i\left|\sum_{j=1}^{N} a_{ij}((x_i-h_i)-(x_j-h_j))\right|, \quad (28)$$

where $\mu_i > 0$ will be determined later. Denote $\mu_{\max} = \max\{\mu_1, \mu_2, \ldots, \mu_N\}$. Then we give the following theorem to show that event-triggered strategies are very useful for the formation control of multi-agent systems.

Theorem 4.1: *Under Assumptions 2.1 and 2.2, consider the multi-agent system* (1) *with the control law* (27)*. Suppose the event condition is given by* (28)*. If $\mu_{max} < \frac{1}{\lambda_N}$ and $Ah_i = 0, i = 1, 2, \ldots, N$, then the agents converge to the formation h. Furthermore, there exists at least one agent k for which for any initial condition the inter-execution times of agent k are lower bounded by $\tau_k > 0$.*

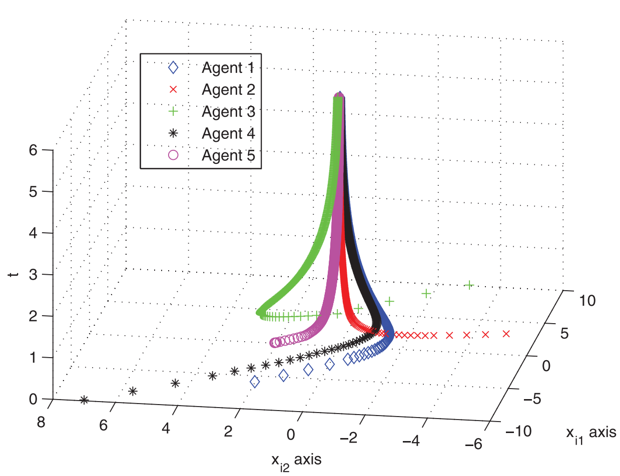

Figure 2. State trajectories of the five agents.

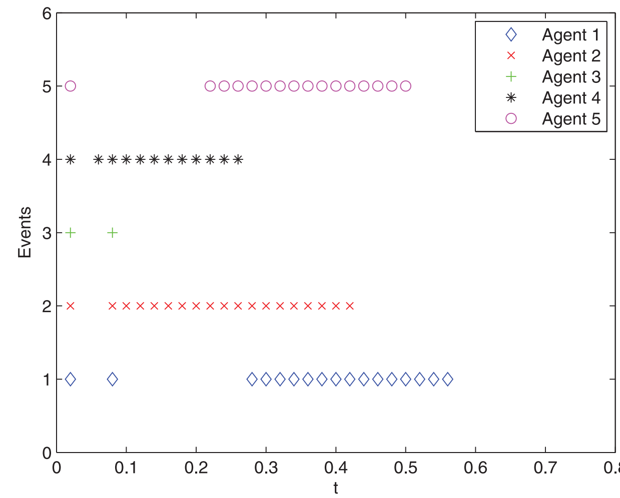

Figure 3. Event times for five agents.

Proof: Let $\tilde{x}_i = x_i - h_i$, then the system (1) can be given as $\dot{\tilde{x}}_i = A\tilde{x}_i + Ah_i + Bu_i$. Based on the condition $Ah_i = 0$, thus

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + Bu_i(t). \quad (29)$$

Define $\tilde{y}_i(t) \triangleq e^{-A(t-t_0)}\tilde{x}_i(t)$. Based on the definition and the protocol (27), the system (29) can be written as follows if $ABK = A$.

$$\dot{\tilde{y}}(t) = -(L\otimes I_n)\tilde{y}(t) - (L\otimes I_n)w(t). \quad (30)$$

The rest proof of Theorem 4.1 follows straightforwardly from the proof of Theorem 3.1. □

Remark 3: The applications to formation control in this section can be easily extended to Theorem 3.4. Due to page limitation, the corresponding result is omitted.

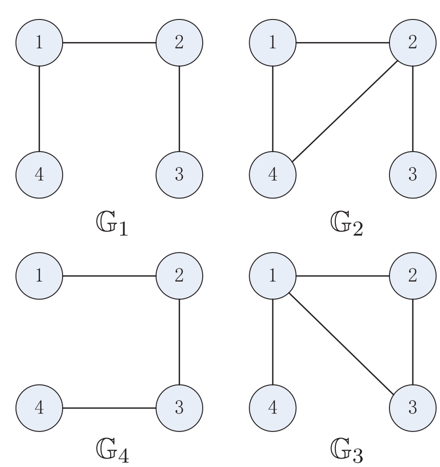

Figure 4. Switching communication topology.

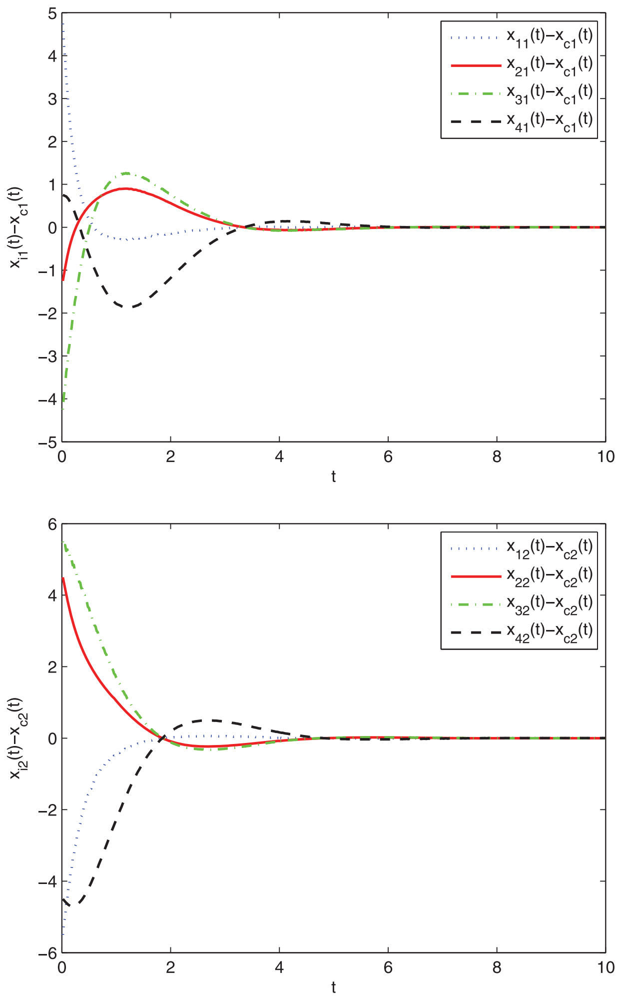

Figure 5. Errors between the states $x_i(t)$ and $x_c(t)$.

5. Simulation

In this section, we present three examples to demonstrate that the results of this paper are valid.

Example 5.1: Consider a group of five agents satisfying (1), where

$$A = \begin{bmatrix} -4 & 1 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}.$$

It can be easily proved that Assumption 2.1 is satisfied, and choose the control gain

$$K = \begin{bmatrix} 1/7 & -3/7 \\ 2/7 & 1/7 \end{bmatrix}.$$

The corresponding communication topology among agents is shown in Figure 1 . Note that the graph is undirected and connected.

It can be seen in Figure 2 that the five agents asymptotically reach consensus as time goes on. The time instants when the events occur for five agents are shown in Figure 3, where the time instants are determined from the triggered condition (9).

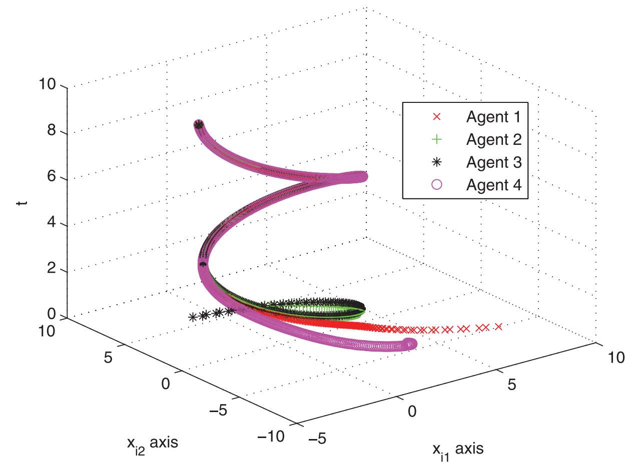

Figure 6. State trajectories of the four agents.

Example 5.2: Consider the consensus problem of system (1) with $N = 4$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $B = I_2$. The control gain can also be chosen as $K = I_2$. We consider the case of switching network topologies in Figure 4. Note that all graphs are connected. The network topology switches to another topology which is chosen randomly. Figure 5 shows the evolutions of the errors between the states $x_i(t)$ and $x_c(t)$. The evolutions of the state under the event-triggered consensus protocol are shown in Figure 6 . It can be seen that the agents reach consensus at $x_c(t)$.

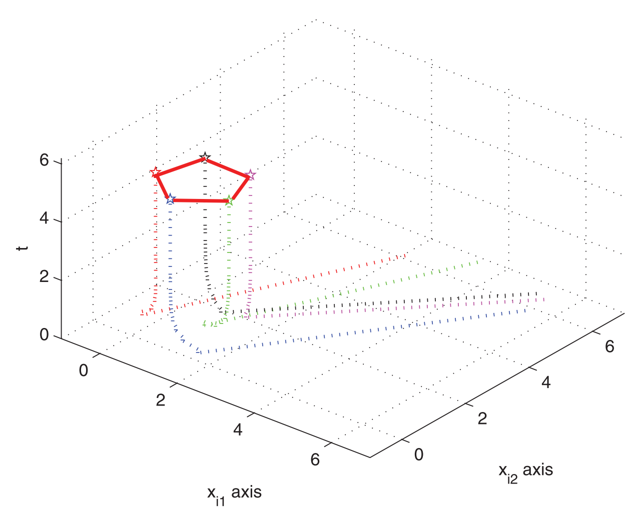

Figure 7. Five agents form a regular pentagon.

Example 5.3: Simulation results for formation control are given in this example. Consider the first-order multi-agent systems described by $\dot{x}_i(t) = u_i(t)$, $i = 1, 2, 3, 4, 5$, where the state $x_i(t) \in \mathbb{R}^2$ denotes the state of agent i and $u_i(t) \in \mathbb{R}^2$ its control input or protocol. The communication topology is the same as in Example 5.1. Let the desired formations of five agents be $h_1 = [1, 0]^T$, $h_2 = [1 - \sin(0.4\pi), 1 - \cos(0.4\pi)]^T$, $h_3 = [1 + \sin(0.4\pi), 1 - \cos(0.4\pi)]^T$, $h_4 = [1 - \sin(0.2\pi), 1 + \cos(0.2\pi)]^T$, $h_5 = [1 + \sin(0.2\pi), 1 + \cos(0.2\pi)]^T$, respectively. Figure 7 shows that the agents can converge to the desired formation according to the control protocol (27).

6. Conclusion

In this paper, event-triggered control methods have been raised to investigate the consensus problem of general linear multi-agent systems. The asynchronous event-triggered control algorithms have been proposed in order to make the systems achieve consensus on fixed and switching topologies. Zeno behaviours have not happened under the presented event-triggered control algorithms. The obtained results were applied to the formation control of multi-agent systems in the following part. Finally, the results of the paper were illustrated through simulation examples.

Future work will focus on model-based event-triggered control for linear multi-agent systems with and without communication delays. Some relevant pioneer work has been tried in Garcia and Antsaklis (2013). As an extension of event-triggered control, self-triggered control has been attracted much attention and lots of results in the field have been published (Dimarogonas et al., 2012; Heemels et al., 2012). Another future work is the study of consensus problems of linear multi-agent systems by self-triggered control.

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