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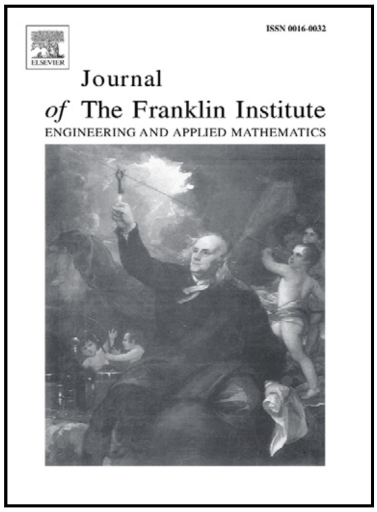

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Output Feedback Composite Nonlinear Feedback Control for Singular Systems with Input Saturation*

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Abstract

This paper addresses the output feedback composite nonlinear feedback (CNF) controller design for a tracking control problem of single-input single-output (SISO) singular linear systems with input saturation. The output feedback CNF control law is constructed based on a state feedback CNF control law for the tracking control problem and a state observer. The stability of the closed-loop system under the output feedback CNF control law is established for an output feedback CNF control law with a singular full state observer. The design procedure and the improvement of the transient performance of the closed-loop system are illustrated with an example.

Keywords: singular linear systems; input saturation; transient performance; composite nonlinear feedback; output feedback

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1 Introduction

The composite nonlinear feedback (CNF) control technique was first proposed by Lin *et al.* [1] to improve the transient performance for the tracking control problem of a second order linear system with input saturation. The CNF control law consists of a linear feedback law and a nonlinear feedback law. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for quick response. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. Chen *et al.* developed a CNF control technique to a more general class of systems with measurement feedback [2]. The CNF control technique is extended to multivariable systems by Turner *et al.* [3] and He *et al.* [4]. Moreover, Lan *et al.* extended the CNF control technique to a class of nonlinear systems [5][6]. Besides the development in theory, the CNF control is applied to design various servo systems, such as hard disk drive (HDD) servo systems [2] [7] [8] [9], helicopter flight control systems [10] [11] and position servo systems [12] [13].

In recent two decades, various control problems for singular linear systems with input saturation are investigated in the literature, such as stabilization problems [14][15][16], singular systems with time-delay [17][18][19], control of switching singular systems [20], output regulation problems [21][22], and robust control of singular systems [23][24][25], to name just a few. Most results are focused on the solvability of the investigated problems, the transient performance of the closed-loop system is not well addressed. Recently, to improve the transient performance of the closed-loop system, we introduce the CNF control technique to the tracking control problem of a single-input single-output (SISO) singular linear system with input saturation in [26], where a state feedback CNF control law is designed for the tracking control problem. Considering the fact that the state feedback is not always implementable, we will design an output feedback CNF control law for the tracking control problem in this paper. The output feedback CNF control law is constructed based on a state feedback CNF control law and a state observer. To show the stability of the closed-loop system, a full order singular observer is designed firstly. Under the assumption that the system is impulse observable, the singular observer can be reduced to a normal observer. We would like to point out that the development of the output feedback CNF control law is not trivial. Though the construction of the observer is quite routine, the proof of the stability of the closed-loop system by output feedback CNF control is more difficult than the state feedback case. In fact, some extra conditions on the nonlinear function are required to guarantee the stability of the closed-loop system. Please refer to Section 3 for the details on the construction of the output feedback CNF controller and the proof of the stability of the closed-loop system.

The remainder of the paper is organized as follows. Section 2 formulates a tracking control problem for a SISO singular linear system with input saturation, and briefly reviews the state feedback CNF controller design for the tracking control problem and the construction of state observer. In Section 3, an output feedback CNF control law is constructed to solve the tracking control problem. The design procedure of output feedback CNF control law and the improvement of the transient performance by the CNF control law are demonstrated by an illustrative example in Section 4. Finally, Section 5 concludes the paper with some concluding remarks.

2 Problem Formulation and Preliminaries

Consider the following singular linear system with input saturation

$$\begin{cases} E\dot{x} &= Ax + B\,\mathrm{sat}(u), \quad x(0) = x_0 \\ y &= Cx \end{cases} \tag{1}$$

where $x \in \Re^n$, $u \in \Re$, $y \in \Re$ are respectively the state, control input, and controlled output of the system. E, A, B, C are constant matrices with appropriate dimensions. The system (1) is said to be singular if $q = \mathrm{rank}(E) < n$. We assume that the given singular system (1) is regular, i.e., $\det(sE - A) \not\equiv 0$ for some $s \in \mathcal{C}$. And $\mathrm{sat}(u) : \Re \to \Re$ represents the actuator saturation defined as:

$$\mathrm{sat}(u) = \mathrm{sgn}(u) \min\{u_{\max}, |u|\} \tag{2}$$

with $u_{\max}$ being the saturation level of the input. We will address the following tracking control problem.
Tracking Control Problem by Output Feedback CNF Control. Consider the system (1) and a step tracking target of amplitude r, design an output feedback CNF control law

$$\begin{aligned} E_v\dot{x}_v &= \phi(x_v, y, u) \\ u &= Fx_v + Gr + u_N(x_v, y, r) \end{aligned}$$

where $x_v \in \Re^{n_v}$, and find a compact set $\mathbf{X} \subset \Re^{n+n_v}$ such that the state trajectory of the following closed-loop system

$$\begin{aligned} E\dot{x} &= Ax + B\,\mathrm{sat}(u) \\ E_v\dot{x}_v &= \phi(x_v, y, u) \\ u &= Fx_v + Gr + u_N(x_v, y, r) \\ y &= Cx \end{aligned} \tag{3}$$

is bounded for $t > 0$ and $(x(0), x_v(0)) \in \mathbf{X}$, and the controlled output y of the closed-loop system tracks the step tracking target of amplitude r asymptotically.

To establish the solvability conditions for the above tracking control problem, we list the following assumptions on the system (1).

A1: $\mathrm{rank}[sE - A \quad B] = n$ for all $s \in \mathcal{C}$, s finite;

A2: $\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \mathrm{rank}E$;

A3: The linear singular system (E, A, B, C) is invertible[1], and has no zeros at $s = 0$.

A4: $\mathrm{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n$ for all $s \in \mathcal{C}$, s finite;

A5: $\mathrm{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \mathrm{rank}E$;

Remark 2.1 The tracking control problem by state feedback CNF control for the system (1) is investigated in [26]. It is shown in [26] that we can construct a state feedback CNF controller which solves the tracking control problem if the system (1) satisfies assumptions A1-A3. Specifically, the state feedback CNF controller is given by

$$u = Fx + Gr + \rho(r, y)B^{\mathrm{T}}ME(x - x_e) \tag{4}$$

where F is such that

P1: $\sigma(E, A + BF) \subset \mathcal{C}^-$;

P2: $\mathrm{degdet}(sE - (A + BF)) = \mathrm{rank}E$;

where $\sigma(E, A+BF) = \{s \mid s \in \mathcal{C}, \det(sE - A - BF) = 0\}$, and $\mathcal{C}^- = \{s \mid s \in \mathcal{C}, Re(s) < 0\}$. Such an F exists because of assumptions A1 and A2. After F is selected, G is given by

$$G = -\left(C(A + BF)^{-1}B\right)^{-1}$$

which can be calculated because of assumption A3. $M > 0$ is the solution of the following generalized Lyapunov equation

$$(A + BF)^{\mathrm{T}}ME + E^{\mathrm{T}}M(A + BF) + E^{\mathrm{T}}WE = 0 \tag{5}$$

for some $W > 0$, and $x_e = -(A + BF)^{-1}BGr$. In this paper, we will consider the design of the output feedback CNF control law for the system (1). To this end, we need to design a state observer for the system (1). ■

Remark 2.2 Consider the matrix pencil $(E, A + BF)$, where F is such that the properties P1 and P2 are satisfied. It is shown in [27] that there exist nonsingular matrices P and Q such that

$$QEP = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad Q(A + BF)P = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix} \tag{6}$$

where $\sigma(I_q, A_1) = \sigma(E, A + BF)$. Moreover, given $W > 0$, and partition $Q^{-T}WQ^{-1}$ accordingly

$$Q^{-T}WQ^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{7}$$

[1] (E, A, B, C) is said to be invertible if the determinant of $G(s) = C(sE - A)^{-1}B$ does not vanish identically.

with $W_{11} \in \Re^{q\times q}$ and $W_{22} \in \Re^{(n-q)\times(n-q)}$. A positive definite solution $M > 0$ of the generalized Lyapunov equation (5) is given by

$$M = Q^T \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} Q \tag{8}$$

where $M_1 \in \Re^{q\times q}$ is the positive definite solution of the following Lyapunov equation

$$A_1^T M_1 + M_1 A_1 + W_{11} = 0 \tag{9}$$

for $W_{11} > 0$, and $M_2 > 0$ is any positive definite matrix. Moreover,

$$E^T M E = P^{-T} \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

for any $M_2 > 0$. ■

Remark 2.3 Consider the system (1) without input saturation, i.e.,

$$\begin{cases} E\dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \tag{10}$$

Assumption A4 is equivalent to that the linear singular system (10) is R-observable (see Definition 2-3.2 in [28]). Thus, if the system (10) satisfies assumption A4, we can design a singular observer for (10),

$$E\dot{x}_v = (A + KC)x_v + Bu - Ky \tag{11}$$

such that $\lim_{t\to\infty} ||x_v(t) - x(t)|| = 0$ if $K \in \Re^{n\times 1}$ is such that $\sigma(E, A + KC) \subset \mathcal{C}^-$. Assumption A5 implies that the linear singular system (10) is impulse observable [28]. Then, if the system (10) satisfies assumptions A4 and A5, the observer gain matrix K can be designed to have the following properties,

P3: $\sigma(E, A + KC) \subset \mathcal{C}^-$;

P4: $\mathrm{degdet}(sE - (A + KC)) = \mathrm{rank}E$;

Moreover, as claimed in Remark 2.2, there exist two nonsingular matrices $P_1 \in \Re^{n\times n}$ and $Q_1 \in \Re^{n\times n}$ such that

$$Q_1 E P_1 = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_1(A + KC)P_1 = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n-q} \end{bmatrix}, \quad q = \mathrm{rank}E$$

and $\bar{A}_1 \in \Re^{q\times q}$, $\sigma(I_q, \bar{A}_1) = \sigma(E, A + KC)$. Then, the singular observer (11) can be reduced to a normal observer in the following form

$$\begin{aligned} \dot{x}_c &= \bar{A}_1 x_c + B_1 u - K_1 y \\ x_v &= P_1 \begin{bmatrix} I \\ 0 \end{bmatrix} x_c - P_1 \begin{bmatrix} 0 \\ I \end{bmatrix} B_2 u + P_1 \begin{bmatrix} 0 \\ I \end{bmatrix} K_2 y \end{aligned}$$

such that $\lim_{t\to\infty} ||x_v(t) - x(t)|| = 0$, where $x_c \in \Re^q$, and

$$Q_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \Re^{q\times 1}, \quad B_2 \in \Re^{(n-q)\times 1}$$

$$Q_1 K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad K_1 \in \Re^{q\times 1}, \quad K_2 \in \Re^{(n-q)\times 1}$$

Please refer to [28] and [29] for the details on observer design for singular linear systems. ■

3 CNF Control with Output Feedback

In this section we will construct an output feedback CNF controller for the system (1) based on a state feedback CNF controller and a state observer. First, a singular full-order observer is considered. If the system (1) satisfies assumptions A4 and A5, the singular observer can be reduced to a normal reduced-order observer as described in Remark 2.3. The design procedure of the output feedback CNF control law can be described as follows.

Step 1 Under assumptions A1–A3, design a state feedback CNF controller for the system (1) in the form of (4) as described in Remark 2.1.

Step 2 Under assumptions A4 and A5, construct a singular full-order state observer for the system (1) in the form of

$$E\dot{x}_v = (A + KC)x_v + B\,\mathrm{sat}(u) - Ky \tag{12}$$

where K is such that P3 and P4 are satisfied.

Step 3 Combining the state feedback CNF control law (4) and the state observer (12), the output feedback CNF control law is given by

$$\begin{aligned} E\dot{x}_v &= (A + KC)x_v + B\,\mathrm{sat}(u) - Ky \\ u &= Fx_v + Gr + \rho(r, Cx_v)B^{\mathrm{T}}ME(x_v - x_e) \end{aligned} \tag{13}$$

Given a positive definite matrix $W_T \in \Re^{n\times n}$ such that

$$W_T > \frac{1}{||E||^2} F^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MBF \tag{14}$$

where $M > 0$ is the positive definite solution of the generalized Lyapunov equation (5). Let $T > 0$ be the positive definite solution of the following generalized Lyapunov equation

$$(A + KC)^{\mathrm{T}}TE + E^{\mathrm{T}}T(A + KC) + E^{\mathrm{T}}W_TE = 0 \tag{15}$$

Such a T exists because K is designed such that P3 and P4 are satisfied [27]. For any $\delta \in (0, 1)$, let c_δ be the largest scalar satisfying the following condition:

$$\left| [F \quad F] \begin{pmatrix} x \\ x_v \end{pmatrix} \right| \le u_{\max}(1 - \delta), \ \ \forall \begin{pmatrix} x \\ x_v \end{pmatrix} \in X_{F\delta} \tag{16}$$

where

$$X_{F\delta} := \left\{ \begin{pmatrix} x \\ x_v \end{pmatrix} \middle| \begin{pmatrix} x \\ x_v \end{pmatrix}^T \begin{bmatrix} M & 0 \\ 0 & T \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\} \tag{17}$$

The stability of the closed-loop system consisting of (1) and (13) is described by the following theorem.

Theorem 3.1 *Consider the given system (1) with assumptions A1–A5 and the output feedback CNF control law (13). There exists a scalar $\rho^* > 0$, such that for any nonpositive function $\rho(r, y)$, locally Lipschitz in y and $|\rho(r, y)| \leq \rho^*$, the output feedback CNF control law (13) solves the tracking control problem described in Section 2, provided that x_0 , $x_v(0) = x_{v0}$ and r satisfy*

$$\begin{pmatrix} x_0 - x_e \\ x_{v0} - x_0 \end{pmatrix} \in X_{F\delta}, \quad |Hr| \leq \delta u_{\max}, \tag{18}$$

where

$$H = [1 - F(A + BF)^{-1}B]G$$

Proof: Let

$$\begin{aligned} \tilde{x} &= x - x_e \\ \tilde{x}_v &= x_v - x \end{aligned} \tag{19}$$

the closed-loop system consisting of the given plant (1) and the output feedback CNF control law (13) can be expressed as

$$\begin{pmatrix} E\dot{\tilde{x}} \\ E\dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{20}$$

where

$$w = \text{sat}\left[[F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho [B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right] - [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr$$

Define a Lyapunov function candidate as

$$V(z) = V(E\tilde{x}, E\tilde{x}_v) = \begin{pmatrix} E\tilde{x} \\ E\tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} M & 0 \\ 0 & T \end{bmatrix} \begin{pmatrix} E\tilde{x} \\ E\tilde{x}_v \end{pmatrix}, \quad z = \begin{bmatrix} E\tilde{x} \\ E\tilde{x}_v \end{bmatrix} \tag{21}$$

The derivative of V along the trajectories of the closed-loop system (20) is given by

$$\begin{aligned} \dot{V} &= \left(\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A + BF & BF \\ 0 & A + KC \end{bmatrix}^{\mathrm{T}} + w^{\mathrm{T}} \begin{bmatrix} B \\ 0 \end{bmatrix}^{\mathrm{T}} \right) \begin{bmatrix} M & 0 \\ 0 & T \end{bmatrix} \begin{pmatrix} E\tilde{x} \\ E\tilde{x}_v \end{pmatrix} \\ &\quad + \begin{pmatrix} E\tilde{x} \\ E\tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} M & 0 \\ 0 & T \end{bmatrix} \left(\begin{bmatrix} A + BF & BF \\ 0 & A + KC \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \right) \\ &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} (A + BF)^{\mathrm{T}}ME & 0 \\ F^{\mathrm{T}}B^{\mathrm{T}}ME & (A + KC)^{\mathrm{T}}TE \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + w^{\mathrm{T}}B^{\mathrm{T}}ME\tilde{x} \\ &\quad + \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} E^{\mathrm{T}}M(A + BF) & E^{\mathrm{T}}MBF \\ 0 & E^{\mathrm{T}}T(A + KC) \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBw \end{aligned}$$

Note that $\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBw$ is a scalar, and using (5) and (15), we have

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -E^{\mathrm{T}}WE & E^{\mathrm{T}}MBF \\ F^{\mathrm{T}}B^{\mathrm{T}}ME & -E^{\mathrm{T}}W_TE \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + 2\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBw \tag{22}$$

Case1

$$|u| = \left| [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho [B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \leq u_{\max}$$

In this case,

$$w = \rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}$$

Then

$$\begin{aligned} 2\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBw &= 2\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MB\rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \\ &= 2\rho\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBB^{\mathrm{T}}ME\tilde{x} + 2\rho\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBB^{\mathrm{T}}ME\tilde{x}_v \\ &= 2\rho\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBB^{\mathrm{T}}ME\tilde{x} \\ &\quad + \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}}\begin{bmatrix} 0 & \rho E^{\mathrm{T}}MBB^{\mathrm{T}}ME \\ \rho(B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}ME & 0 \end{bmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \end{aligned} \tag{23}$$

which yields

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}}\begin{bmatrix} -E^{\mathrm{T}}WE & E^{\mathrm{T}}MB(F+\rho B^{\mathrm{T}}ME) \\ (F+\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}ME & -E^{\mathrm{T}}W_TE \end{bmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + 2\rho\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBB^{\mathrm{T}}ME\tilde{x} \\ &\leq \begin{pmatrix} E\tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}}\begin{bmatrix} -W & MB(F+\rho B^{\mathrm{T}}ME) \\ (F+\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}M & -E^{\mathrm{T}}W_TE \end{bmatrix}\begin{pmatrix} E\tilde{x} \\ \tilde{x}_v \end{pmatrix} \\ &\leq \begin{pmatrix} E\hat{x} \\ E\tilde{x}_v \end{pmatrix}^{\mathrm{T}}\begin{bmatrix} -W & 0 \\ 0 & -W_T + \frac{1}{||E||^2}(F+\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MB(F+\rho B^{\mathrm{T}}ME) \end{bmatrix}\begin{pmatrix} E\hat{x} \\ E\tilde{x}_v \end{pmatrix} \\ &= \begin{pmatrix} E\hat{x} \\ E\tilde{x}_v \end{pmatrix}^{\mathrm{T}}\begin{bmatrix} -W & 0 \\ 0 & -\tilde{W}_T \end{bmatrix}\begin{pmatrix} E\hat{x} \\ E\tilde{x}_v \end{pmatrix} \end{aligned}$$

where

$$E\hat{x} = E\tilde{x} - W^{-1}MB(F+\rho B^{\mathrm{T}}ME)E\tilde{x}_v$$

and

$$\tilde{W}_T = W_T - \frac{1}{||E||^2}(F+\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MB(F+\rho B^{\mathrm{T}}ME)$$

Since

$$W_T - \frac{1}{||E||^2}F^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MBF > 0$$

and $\rho(r,h)$ is locally Lipschitz, there exists a $\rho_1^* > 0$ such that $\tilde{W}_T > 0$ for any $|\rho(r,h)| \leq \rho_1^*$. Thus, $\dot{V} < 0$.

Case 2

$$u = [F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} > u_{\max}.$$

For this case,

$$\rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} > u_{\max} - [F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr = w. \tag{24}$$

Noting that $|Hr| \leq \delta u_{\max}$, and

$$\left|[F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}\right| \leq (1-\delta)u_{\max}, \quad \forall \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in X_{F\delta},$$

we have

$$\left|[F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr\right| \le u_{\max} \tag{25}$$

Using (24) and (25), we can conclude that

$$0 \le u_{\max} - \left|[F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr\right| \le w < \rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}$$

i.e.,

$$0 \le w < \rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}.$$

Thus, w can be expressed as

$$w = q\rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \tag{26}$$

where q is an appropriate positive piecewise continuous function $q(t)$, bounded by 1 for all t. In this case, the derivative of V becomes

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -E^{\mathrm{T}}WE & E^{\mathrm{T}}MB(F+q\rho B^{\mathrm{T}}ME) \\ (F+q\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}ME & -E^{\mathrm{T}}W_TE \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \\ &\quad +2q\rho\tilde{x}^{\mathrm{T}}E^{\mathrm{T}}MBB^{\mathrm{T}}ME\tilde{x} \\ &\le \begin{pmatrix} E\tilde{x} \\ \tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -W & MB(F+\rho B^{\mathrm{T}}ME) \\ (F+\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}M & -E^{\mathrm{T}}W_TE \end{bmatrix} \begin{pmatrix} E\tilde{x} \\ \tilde{x}_v \end{pmatrix} \end{aligned}$$

Similar to Case 1, we can show that

$$\dot{V} \le \begin{pmatrix} E\hat{x}_+ \\ E\tilde{x}_v \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -W & 0 \\ 0 & -\tilde{W}_{T+} \end{bmatrix} \begin{pmatrix} E\hat{x}_+ \\ E\tilde{x}_v \end{pmatrix}$$

where

$$E\hat{x}_+ = E\tilde{x} - W^{-1}MB(F+q\rho B^{\mathrm{T}}ME)E\tilde{x}_v$$

and

$$\tilde{W}_{T+} = W_T - \frac{1}{||E||^2}(F+q\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MB(F+q\rho B^{\mathrm{T}}ME)$$

Again, there exists a $\rho_2^* > 0$ such that $\tilde{W}_{T+} > 0$ for any $|\rho(r,y)| \le \rho_2^*$. Thus, we also have $\dot{V} < 0$.

Case 3 When

$$u = [F \quad F]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho[B^{\mathrm{T}}ME \quad B^{\mathrm{T}}ME]\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} < -u_{\max},$$

following the lines of Case 2, it is not difficult to show that there exists a $\rho_3^* > 0$ such that for any scalar function satisfying

$$|\rho(r,y)| \le \rho_3^* \tag{27}$$

and we have $\dot{V} < 0$.

Finally, let $\rho^* = \min\{\rho_1^*, \rho_2^*, \rho_3^*\}$. Then, for any non-positive scalar function $\rho(r,y)$ satisfying $|\rho(r,y)| \le \rho^*$,

$$\dot{V}(z) < 0, \quad \forall z = \begin{bmatrix} E\tilde{x} \\ E\tilde{x}_v \end{bmatrix} \ \text{with} \ \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in X_{F\delta} \tag{28}$$

Thus, $z(t)$ is bounded and

$$\lim_{t\to\infty} z(t) = 0 \tag{29}$$

Noting the impulsive free property P2 and P4, we have,

$$\lim_{t\to\infty} \tilde{x}_v(t) = 0, \quad \lim_{t\to\infty} x(t) = x_e, \tag{30}$$

for the initial state of the given system x_0, the initial state of the controller x_{v0}, and step command input r that satisfy (18). Hence

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Cx(t) = Cx_e = r. \tag{31}$$

This completes the proof of Theorem 3.1. ∎

Remark 3.1 Theorem 3.1 establishes a sufficient solvability condition for the tracking control problem by output feedback CNF control. With state transformation (19), the solvability of the tracking control problem is guaranteed by the stability of closed-loop system (20). In the proof of Theorem 3.1, the stability of the closed-loop system (20) is shown under the assumptions that $|\rho(r,y)| \le \rho^*$ and (18). We would like to point out that the assumption on the function $\rho(r,y)$ is not a serious assumption. In fact, the purpose of the restriction $|\rho(r,y)| < \rho^*$ is to guarantee that

$$\tilde{W}_T = W_T - \frac{1}{||E||^2}(F + q\rho B^{\mathrm{T}}ME)^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MB(F + q\rho B^{\mathrm{T}}ME) > 0, \forall q \in [-1,1]$$

Such a ρ^* exists because

$$W_T - \frac{1}{||E||^2}F^{\mathrm{T}}B^{\mathrm{T}}MW^{-1}MBF > 0$$

Noting that the above positive definite matrix W_T can be chosen independently and arbitrarily, the bound ρ^* of the function $\rho(r,y)$ can also be arbitrary. The condition (18) is also necessary because of the constraint of input saturation. For the system with input saturation, it is very difficult to design a global controller. Instead of global control problems, semi-global or local control problems are addressed in the literature, especially, when the linear feedback controllers are considered, see, e.g., [14], [15], [21]. The CNF controller is constructed based on a linear feedback controller. Thus, local tracking control problem is investigated in this paper, and assumption (18) specifies an attractive basin of the closed-loop system (20). Moreover, $|Hr| \le \delta u_{\max}$ describes the amplitude r of the step tracking target that can be tracked by a linear state feedback control law

$$u = Fx + Gr$$

where F is such that $\sigma(E, A+BF) = \{s \mid s \in \mathcal{C}, \det(sE - A - BF) = 0\}$, and G is given by

$$G = -\left(C(A+BF)^{-1}B\right)^{-1}$$

It is not difficult to show that

$$u = Fx + Gr = F\tilde{x} + Hr$$

where $\tilde{x} = x - x_e$ with $x_e = -(A + BF)^{-1}BGr$, and $H = [1 - F(A + BF)^{-1}B]G$. Considering the steady state case, i.e., $\tilde{x} = x - x_e = 0$, we have

$$u = Hr$$

Thus, it is impossible to track a step tracking target with amplitude r that $|Hr| > u_{\max}$ because $|u| \leq u_{\max}$ for the system with input saturation. ■

Remark 3.2 The CNF control law (13) is constructed based on a singular full-order observer. Noting that K satisfies P3 and P4, according to Remark 2.3, the singular full-order observer can be reduced to a normal reduced-order observer. Thus, it is not difficult to construct an output feedback CNF control law with a normal reduced-order observer. ■

4 Illustrative Example

Consider a singular system characterized by (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

and $u_{\max} = 5$. The controlled output y is required to track a unit step input, i.e., $r = 1$. It is easy to verify that assumptions A1–A5 are all satisfied.

Let $F = [-14, -5, 0, 0]$, then

$$\sigma(E, A + BF) = \{-1 \pm 2i\}, \text{degdet}(sE - (A + BF)) = 2 = \text{rank}E$$

that is, $(E, A + BF)$ is stable and impulsive free. Calculate G, H and x_e as follows,

$$G = -\left(C(A + BF)^{-1}B\right)^{-1} = 5$$

$$H = [1 - F(A + BF)^{-1}B]G = 1$$

$$x_e = -(A + BF)^{-1}BGr = [1 \;\; -2 \;\, 2 \;\; -2]^{\mathrm{T}}$$

Solving the GLE (5) with $W = I$, we have

$$M = \begin{bmatrix} 9.5 & 1.5 & 1.5 & -1.5 \\ 1.5 & 0.5 & 0.5 & -0.5 \\ 1.5 & 0.5 & 0.5 & -0.5 \\ -1.5 & -0.5 & -0.5 & 1.0 \end{bmatrix} > 0$$

Then, the state feedback CNF control law is given by

$$u \;\; = \;\; Fx + Gr + \rho(r, y)B^{\mathrm{T}}ME(x - x_e) \tag{32}$$

where

$$\rho(r,y) = -\beta e^{-\alpha_0 \alpha |y-r|}$$

with

$$\alpha_0 = \begin{cases} \frac{1}{|y_0 - r|}, & y_0 \neq r \\ 1, & y_0 = r \end{cases}$$

The parameters α and β can be designed by trial and error. Here, we let $\alpha = 8$ and $\beta = 30$.

If the nonlinear function $\rho(r, y)$ is set to zero, then the state feedback CNF control law is reduced to a state feedback linear control law

$$u = Fx + Gr \tag{33}$$

Let

$$K = \begin{bmatrix} -6 & -43.25 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

we can verify that $\sigma(E, A + KC) = \{-1.5 \pm 6i\}$. Then, the output feedback CNF control law is given by

$$\begin{cases} E\dot{x}_v &= (A + KC)x_v - Ky + \mathrm{sat}(u) \\ u &= Fx_v + Gr + \rho(r,y)B^{\mathrm{T}}ME(x_v - x_e) \end{cases} \tag{34}$$

where the nonlinear function $\rho(r, y)$ is the same as the one in the state feedback CNF control law (32).

Assume the initial states are given by $x_0 = (0, 0.3, 0, 0)^{\mathrm{T}}$, and $x_{v0} = (0, 0, 0, 0)^{\mathrm{T}}$, then

$$\begin{aligned} \tilde{x}_0 &= x_0 - x_e = [-0.9, 2.3, -2, 2]^{\mathrm{T}} \\ \tilde{x}_{v0} &= x_{v0} - x_0 = [0, -0.2, 0, 0]^{\mathrm{T}} \end{aligned}$$

and

$$|F\tilde{x}_0 + F\tilde{x}_{v0}| + |Hr| = 5 = u_{\max}$$

The simulation results are shown in Fig. 1–Fig. 3. In Fig. 1, the controlled outputs under state feedback linear control (33), state feedback CNF control (32) and output feedback CNF control (34) are compared. The controlled outputs have quick response for all the CNF controllers and the linear controller. However, the overshoot is very large under the linear control. By introducing the nonlinear feedback part, the CNF controllers can reduce the overshoot caused by the linear feedback part. Moreover, the output response under the output feedback CNF control is quite similar to the state feedback CNF control case. The control inputs under state feedback linear control (33), state feedback CNF control (32) and output feedback CNF control (34) are shown in Fig. 2. Though the output feedback CNF control input is saturated, the stability of the closed-loop system is guaranteed by Theorem 3.1. The state response under the output feedback CNF control is shown Fig. 3. The states of the closed loop system are bounded, and converge to x_e asymptotically.

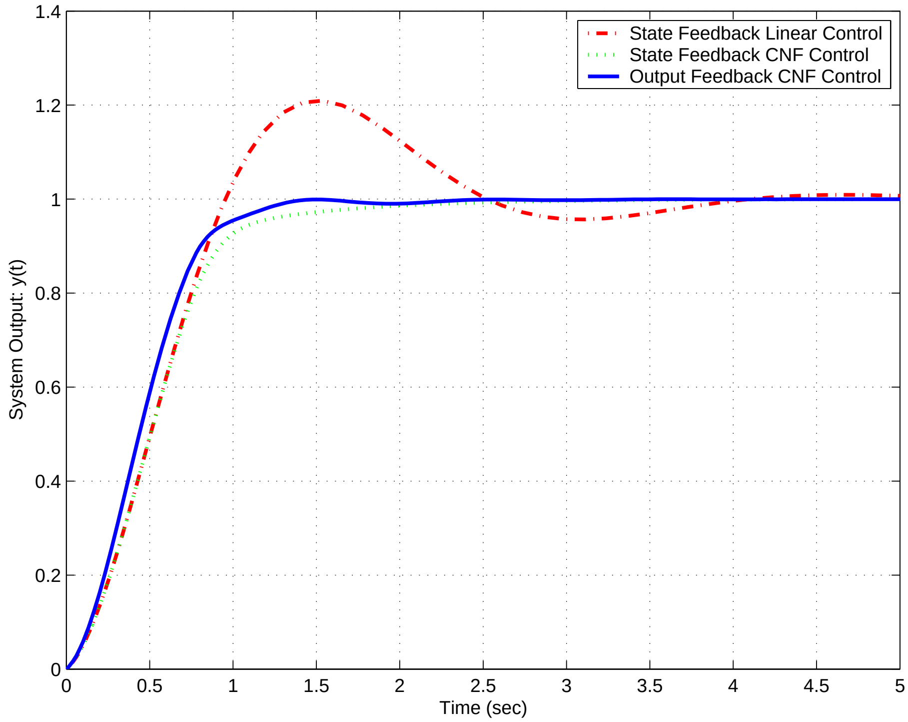

Figure 1: The trajectory of controlled output.

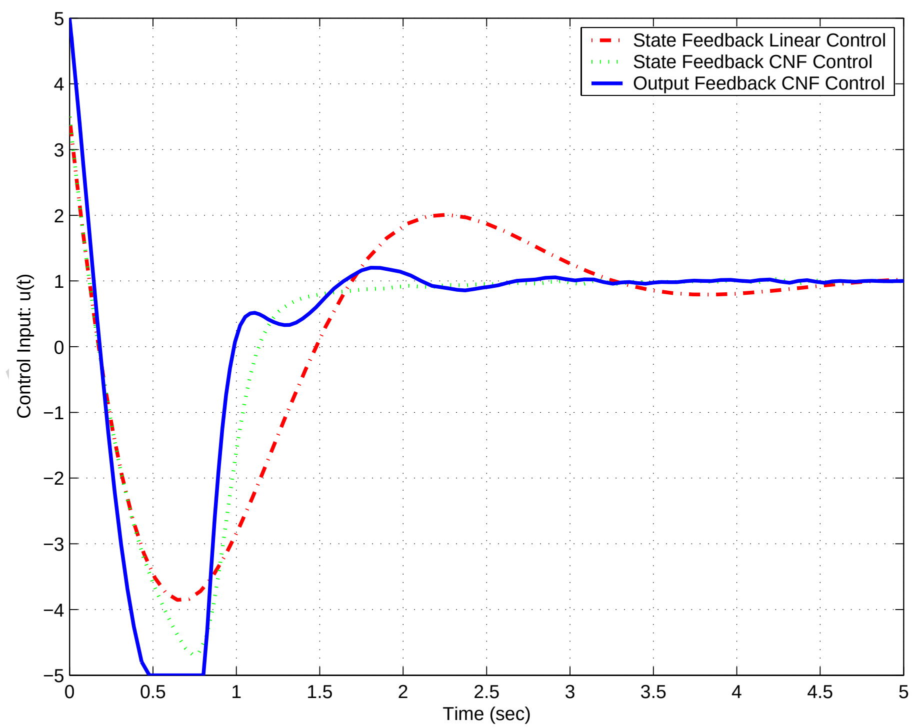

Figure 2: The Trajectory of control input.

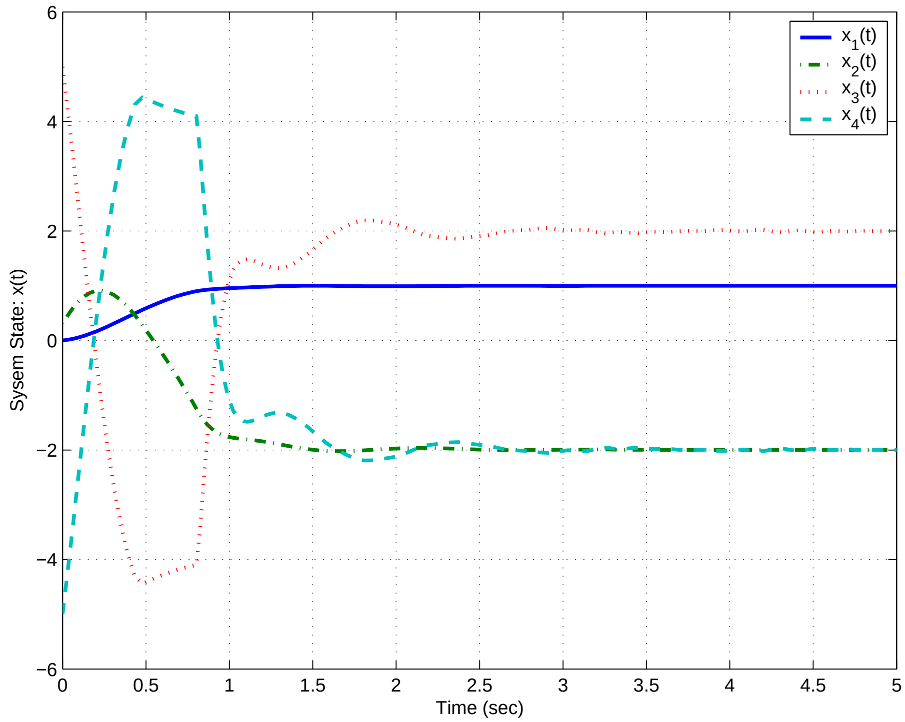

Figure 3: State response under output feedback CNF control.

5 Conclusions

The state feedback CNF control technique is extended to output feedback case for solving the tracking control problem of singular linear systems with input saturation. The output feedback CNF controller is constructed based on a state feedback CNF controller and a state observer. First, a singular full-order observer is applied to construct the output feedback CNF control law, and the stability of the closed-loop system is proved. Moreover, the singular full-order observer can be reduced to a normal reduced-order observer by some state transformation. An example is given to demonstrate the properties of the proposed CNF controller. Simulation results show that the CNF control law can significantly improve the transient performance of the closed-loop system. Only single-input single-output (SISO) systems are considered in this paper, but it is not difficult to extend the results of this paper to multi-input multi-output (MIMO) systems.

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