Analysis of the Fuzzy Controllability Property and Stabilization for a Class of T–S Fuzzy Models

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***Abstract*—In this paper, the controllability property for a class of Takagi–Sugeno (T–S) fuzzy models is analyzed, while a fully nonlinear stabilizer is designed in a practical way. It is shown that global fuzzy stabilizers can be constructed in a nonconservative way by means of a relatively simple approach. The existence of such controllers depends on the fuzzy controllability conditions, which are derived in a straightforward way. The main advantage of the proposed approach is that the convergence of the closed-loop system can be imposed arbitrarily. Some examples are given in order to illustrate the validity of the method. Finally, the proposed controller is applied on an underactuated system known as "pendubot" and the results are compared with an stabilizer designed on the basis of LMIs.**

***Index Terms*—Fuzzy Ackermann's formula, fuzzy controllability, fuzzy stability, Takagi–Sugeno (T–S) fuzzy models.**

I. Introduction

In recent years, some techniques have been developed in order to characterize nonlinear systems by means of linear local subsystems [19], [20]. One of this approaches is the well-known Takagi–Sugeno (T–S) fuzzy modeling. This technique allows describing the nonlinear dynamics by means of a suitable "blending" of linear subsystems. Thus, local controllers can be designed for each subsystems, obtaining the aggregate controller by the same procedure used to compute the T–S overall fuzzy system.

In [21], a practical approach that is based on numerical techniques, namely linear matrix inequalities (LMIs), is derived to design the overall fuzzy stabilizer, known as parallel distributed compensator (PDC). This approach depends on finding a common positive definite matrix P, which has to fulfill local and global Lyapunov conditions.

Unfortunately, the PDC has been proved to be a conservative approach. The relaxation of the conditions to design the fuzzy stabilizers for Takagi–Sugeno (T–S) fuzzy models has been thoroughly investigated by many authors [1], [4], [6], [8]–[10], [13], [17], [18], [22]–[24], [27]–[31].

For instance, while some works consider the inclusion of basis-dependent Lyapunov function, which can be described as the fuzzy version of the well-known Lyapunov function proposed in [22]; other authors suggest the addition of positive definite matrices in order to relax the approach that is based on the common positive definite matrix P, but knowledge of the membership function overlap has to be taken into account [17].

As expected, in many of the references mentioned previously, the improvement achieved by the proposed approaches has been notable, but the stability relaxation for T–S fuzzy models is not fully solved, in general. For that reason, the designing of nonconservative fuzzy stabilizers is addressed in the present paper.

In this study, the T–S fuzzy model is considered a particular type of linear time-varying systems, such that, very well-known linear techniques can be applied to design the fuzzy stabilizer.

It is important to mention that although linear techniques are used to design the stabilizer, a nonlinear controller is obtained. Such a stabilizer is defined on the basis of the membership functions, which are included in the T–S fuzzy model, guaranteeing the efficacy of the stabilizer along the whole approximation region of the T–S fuzzy system.

On the other hand, *a priori* determination of controllability property for T–S fuzzy models is an open problem. There exist very few works related to this issue, one of them is [3], where the authors proposed an approach that is based on the linearization of the T–S fuzzy model. Unfortunately, when the resulting linearized system is noncontrollable, the controllability property of the overall fuzzy model must be analyzed through a complex algorithm. In addition, as it is presented in Section IV, in some cases this method may fail at the moment of analyzing the overall fuzzy controllability property.

Therefore, the contributions of the present study are 1) to propose a practical approach to achieve arbitrary stabilization of a T–S fuzzy model, and 2) to analyze the controllability property for a class of T–S fuzzy systems in a practical way for all $t \geq 0$, independently of the form of the membership functions; but considering that, at most, two fuzzy rules are activated at the same instant. (Observe that this condition is fulfilled by a great variety of T–S fuzzy models).

This paper is organized as follows. A brief overview of the T–S fuzzy models is given in Section II. The overall fuzzy stabilizer is obtained in Section III. The analysis of the overall

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fuzzy controllability property is carried out in Section IV by using some simple examples. Then, in Section V, the approach is applied on the underactuated system. Finally, in Section VI some conclusions are drawn.

II. Takagi–Sugeno Fuzzy Models

Consider the nonlinear system given by

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the plant, and $u(t) \in \mathbb{R}$ is the input signal.

It is now well-known that a good approximation, or exact representation, for nonlinear systems is provided by the so-called T–S fuzzy modeling [21]. In this way, the nonlinear behavior can be approximated, or exactly represented, by the suitable choice of a set of linear subsystems, according to rules associated with some physical knowledge and some linguistic characterization of the properties of the system. These linear subsystems properly describe, at least locally, the behavior of the nonlinear system for a predefined region of the state space.

Therefore, nonlinear dynamics defined by (1) can be approximated, or exactly described, by the following T–S fuzzy model [20], [21]:

Plant
Rule i
IF $z_1(t)$ is $M_{i,1}$ and ... and $z_p(t)$ is $M_{i,p}$, THEN

$$\dot{x}(t) = A_i x(t) + b_i u(t) \tag{2}$$

where sets $M_{i,j}$ are fuzzy sets defined on the basis of a previous knowledge of the dynamics of the system, and premise variable $z(t) = [z_1(t)\ z_2(t)\ \ldots\ z_p(t)]$ is a function of $x(t)$.

Matrices $A_i \in \mathbb{R}^{n\times n}$ and $b_i \in \mathbb{R}^{n\times 1}$ can be obtained by linearizing the nonlinear system around some suitable operation points $(x, u) = (x^i, u^i)$, i.e.,

$$A_i = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x^i,u^i)},\ b_i = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x^i,u^i)}.$$

Then, the membership functions need to be chosen, taking into account that the overall fuzzy model has to approximate the nonlinear dynamics (1), at least in the operation region.

On the other hand, one can apply the sector nonlinearity approach presented in [21, pp. 10–23], which allows us to obtain a T–S fuzzy model capable of exactly representing the original nonlinear dynamics. This method produces both, linear matrices and membership functions that are needed to conform the T–S fuzzy model.

In any case, the resulting composite system is defined by [21]

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + b_i u(t)\} \tag{3}$$

where r is the number of rules for the fuzzy plant, $x(t) \in \mathbb{R}^n$ is the state of the plant, and $u(t) \in \mathbb{R}$ is the control signal. Besides, membership functions for the fuzzy plant satisfy

$$\varpi_i(z(t)) = \prod_{j=1}^{p} M_{i,j}(z_j(t)) \tag{4}$$

$$h_i(z(t)) = \frac{\varpi_i(z(t))}{\sum_{i=1}^{r}\varpi_i(z(t))} \tag{5}$$

for all $t \geq 0$, and the term $M_{i,j}(z_j(t))$ is the membership value for $M_{i,j}$ at $z_j(t)$. In addition, since

$$\sum_{i=1}^{r}\varpi_i(z(t)) > 0$$
$$\varpi_i(z(t)) \geq 0, \qquad i = 1, \ldots, r \tag{6}$$

one has

$$\sum_{i=1}^{r} h_i(z(t)) = 1$$
$$h_i(z(t)) \geq 0, \qquad i = 1, \ldots, r \tag{7}$$

for all $t \geq 0$.

The T–S fuzzy models are nonlinear systems and the membership functions are responsible to reproduce such a nonlinear behavior. However, the T–S fuzzy models can be also viewed as the weighted summation of linear systems, where the weights are instantaneously defined by the membership functions which in turn depend on the states of the system.

At this point, system (3) can be rewritten as

$$\dot{x}(t) = A(t)x(t) + b(t)u(t) \tag{8}$$

with

$$A(t) = \sum_{i=1}^{r} h_i(z(t))A_i$$
$$b(t) = \sum_{i=1}^{r} h_i(z(t))b_i.$$

The rationale behind this representation is that the membership functions depend on the states of the plant $x(t)$ which in turn depend on time. Therefore, the membership functions ultimately depend on time t. However, it must to be kept in mind that (8) describes a nonlinear system.

Thus, the system (8) is actually the T–S fuzzy model to be analyzed and stabilized. See [5], [14], and [16], where the stability of linear time-varying systems is deeply studied.

In the following section, an approach that is based on Ackermann's formula is proposed to obtain a nonlinear gain $K(t)$ by considering the T–S fuzzy model as a special case of linear time-varying systems. As consequence a controller of the form $u(t) = -K(t)x(t)$ will be easily obtained. See [11], [26], and references therein, where the authors have shown how the Ackermann's formula can be used to stabilize linear time-varying systems.

III. Nonlinear Fuzzy Stabilizer Through Ackermann's Formula

Consider again the fuzzy model defined by (3). Thus, from the previous section follows that the design of the overall fuzzy stabilizer can be preformed from a linear point of view [7].

In this section, such a controller is obtained by means of Ackermann's formula, which allows us to place eigenvalues, of linear time-varying system (8), in desired locations when the aforementioned system is controllable for any t.

Before proceeding, Ackermann's formula is briefly recalled in the following algorithm:

Algorithm 3.1 (Ackermann's formula):

1) Considering the linear system $\dot{x}(t) = Ax(t) + bu(t)$. Assume that desired locations of eigenvalues are defined by $s_1, s_2, \ldots,$ and s_n, where $s_1, s_2, \ldots,$ and s_n are valid coordinates of the complex plane. Then, the resulting characteristic equation of the closed-loop system is

$$\begin{aligned} P(s) &= (s - s_1)(s - s_2) \cdots (s - s_n) \\ &= s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n. \end{aligned} \tag{9}$$

2) By Cayley–Hamilton theorem, one can obtain the matrix polynomial

$$P(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I \tag{10}$$

where I is an identity matrix of appropriate dimension.

3) Compute the controllability matrix as

$$\mathcal{C} = [\, b \quad Ab \quad A^2 b \quad \cdots \quad A^{n-1} b \,]. \tag{11}$$

4) Obtain the gain vector

$$K = [\, k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_n \,]$$

by

$$K = [\, 0 \quad \cdots \quad 0 \quad 1 \,] \, \mathcal{C}^{-1} P(A). \tag{12}$$

5) Finally, the linear stabilizer is

$$u(t) = -Kx(t) \tag{13}$$

i.e., the closed-loop system is $\dot{x}(t) = Ax(t) - bKx(t)$.

As can be clearly observed, the eigenvalues can be placed in the desired location if and only if, $\mathcal{C}$ is nonsingular, namely, the linear system defined by (A, b) is fully controllable [7].

In the following section, it is shown that the controllability property of the overall T–S fuzzy system can be analyzed on the basis of the fuzzy controllability matrices for different interpolation regions (such matrices are also time dependent).

Thus, a global nonlinear feedback stabilizer for the T–S fuzzy model can be obtained as follows.

Algorithm 3.2 (Fuzzy Ackermann's formula):

1) Considering the T–S fuzzy system (3), which can be rewritten as (8). Assume that desired locations of the fuzzy eigenvalues are defined by s_1, $s_2, \ldots,$ and s_n, where s_1, $s_2, \ldots,$ and s_n, are valid coordinates of the complex plane. Then, the resulting characteristic equation of the closed-loop system is

$$\begin{aligned} P(s) &= (s - s_1)(s - s_2) \cdots (s - s_n) \\ &= s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} \\ &\quad + \cdots + \alpha_n. \end{aligned} \tag{14}$$

2) By Cayley–Hamilton theorem, one can obtain the matrix polynomial

$$\begin{aligned} P(A(t)) &= (A(t))^n + \alpha_1 (A(t))^{n-1} \\ &\quad + \alpha_2 (A(t))^{n-2} + \cdots + \alpha_n I \end{aligned} \tag{15}$$

where I is an identity matrix of appropriate dimension.

3) Compute the overall fuzzy controllability matrix as

$$\mathcal{C}(t) = [\, \xi_0(t) \quad \xi_1(t) \quad \xi_2(t) \quad \cdots \quad \xi_{n-1} \,] \tag{16}$$

where $\xi_0(t) = b(t)$ and $\xi_j(t) = A(t)\xi_{i-1}(t) - \dot{\xi}_{i-1}(t)$ for $i = 1, \ldots, n-1$ [2], [11], [25]. Notice that the fuzzy controllability matrix reduces to

$$\begin{aligned} \mathcal{C}(t) = [\, b(t) \quad A(t)b(t) \quad (A(t))^2 \, b(t) \quad \cdots \\ \cdots \quad (A(t))^{n-1} \, b(t) \,] \end{aligned} \tag{17}$$

when the fuzzy system can be considered as a slowly time-varying linear system [15] (see Section V).

4) Obtain the gain vector

$$K(t) = [\, k_1(t) \quad k_2(t) \quad k_3(t) \quad \cdots \quad k_n(t) \,]$$

by

$$K(t) = [\, 0 \quad \cdots \quad 0 \quad 1 \,] \, \mathcal{C}(t)^{-1} P(A(t)). \tag{18}$$

5) Finally, the nonlinear stabilizer is

$$u(t) = -K(t)x(t) \tag{19}$$

i.e., the closed-loop fuzzy system is $\dot{x}(t) = A(t)x(t) - b(t)K(t)x(t)$.

Remark 3.3: Notice that computation of fuzzy matrix (16) may be impossible to accomplish when the relative degree between the premise variable and the input is less than n, i.e., when the input signal appears in (16). Besides, if the membership functions are not smooth (triangular-shaped or trapezoidal-shaped membership functions) then fuzzy matrix (16) may be difficult to obtain. In those cases, the use of (17) is suggested, when possible. Otherwise, fuzzy Ackermann's formula cannot be used to stabilize the T–S fuzzy model.

At this point, the main problem consists of analyzing the controllability property of the overall T–S fuzzy model prior the construction of the fuzzy stabilizer, i.e., the nonsingularity of matrix $\mathcal{C}(t)$, for all t, has to be verified before constructing the stabilizer.

To this end, in this study, T–S fuzzy models with at most two rules activated at any instant $t \geq 0$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$, $\forall t \geq 0$ are considered (for instance, see Fig. 3).

Thus, the overall fuzzy controllability property can be studied by taking pairs of local subsystems satisfying $h_i \cap h_j \neq \emptyset$, for all $i, j = 1, \ldots, r$. The overall fuzzy controllability conditions are derived in the following section, by analyzing some simple examples.

IV. Main Result: Overall Fuzzy Controllability Conditions

In this section, some examples are taken into account in order to derive the overall fuzzy controllability conditions.

A. *Example 1*

Consider the T–S fuzzy model analyzed in [23], which is defined by

Model
Rule i:
IF $x_1(t)$ is M_1^i, THEN

$$\sum_i : \{ \dot{x}(t) = A_i x(t) + b_i u(t)$$

with $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $i = 1 \dots 2$. Thus, the overall nonlinear behavior is described by

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(x_1(t)) \left[A_i x(t) + b_i u(t)\right]$$

with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The membership functions for this example are

$$h_1(x_1(t)) = \frac{1 + \sin(x_1(t))}{2}, \text{ and } h_2(x_1(t)) = \frac{1 - \sin(x_1(t))}{2}.$$

As clearly presented in [23], the fuzzy system defined in this way is asymptotically stable when $u(t) = 0$. However, it is not possible to find a common Lyapunov matrix to determine the stability property of the overall fuzzy model. Even so, the conventional PDC can be constructed for this simple system. Unfortunately, the designer is limited to the solution of the convex optimization problem involved in the LMI definition. In other words, although the resulting controller is capable of stabilizing the T–S fuzzy model, an arbitrary path to the stability cannot be trivially imposed by using the PDC.

Other approaches focused in relaxing the designing of stabilizer for continuous-time and/or discrete-time T–S fuzzy models can be found in [1], [4], [6], [17], [22], [29], [30], and [31], among many others works.

As mentioned before, the method proposed in the present study cannot be used to analyze the stability of the T–S fuzzy model without control, but it can be used to arbitrarily change the behavior of the T–S fuzzy model when possible, i.e., when the T–S fuzzy model is fuzzy controllable.

First of all, continuous time-varying matrices need to be constructed as follows:

$$A(t) = \sum_{i=1}^{r} h_i(z(t)) A_i$$
$$= \begin{bmatrix} -5\,h_1(x_1(t)) - 2\,h_2(x_1(t)) & -4\,h_1(x_1(t)) - 4\,h_2(x_1(t)) \\ -h_1(x_1(t)) + 20\,h_2(x_1(t)) & -2\,h_1(x_1(t)) - 2\,h_2(x_1(t)) \end{bmatrix}$$

$$b(t) = \sum_{i=1}^{r} h_i(z(t)) b_i$$
$$= \begin{bmatrix} 0 \\ 10\,h_1(x_1(t)) + 3\,h_2(x_1(t)) \end{bmatrix}$$

where $z(t) = x_1(t)$.

By substituting $h_1(x_1(t)) = \frac{1+\sin(x_1(t))}{2}$ and $h_2(x_1(t)) = \frac{1-\sin(x_1(t))}{2}$ into the previous continuous time-varying matrices, it results

$$A(t) = \begin{bmatrix} -\dfrac{3\,\sin(x_1(t))}{2} - \dfrac{7}{2} & -4 \\ \dfrac{19}{2} - \dfrac{21\,\sin(x_1(t))}{2} & -2 \end{bmatrix}$$

and

$$b(t) = \begin{bmatrix} 0 \\ \dfrac{7\,\sin(x_1(t))}{2} + \dfrac{13}{2} \end{bmatrix}.$$

Thus, overall fuzzy controllability matrix (16) turns out to be

$$\mathcal{C}(t) = \begin{bmatrix} 0 & -14\sin(x_1(t)) - 26 \\ \dfrac{7\sin(x_1(t))}{2} + \dfrac{13}{2} & \alpha(t) \end{bmatrix}$$

with $\alpha(t) = \frac{49x_1(t)\cos(x_1(t))}{4} - 7\sin(x_1(t)) + 14x_2(t)\cos(x_1(t)) + \frac{21x_1(t)\cos(x_1(t))\sin(x_1(t))}{4} - 13$. Notice that for this example $\dot{x}_1(t) = (-\frac{3\,\sin(x_1(t))}{2} - \frac{7}{2})x_1(t) - 4x_2(t)$.

The overall T–S fuzzy system will be fuzzy controllable if and only if $\mathcal{C}(t)$ has full rank for all t. One way to verify this property is based on the analysis of the determinant. For this case, the determinant of $\mathcal{C}(t)$ is

$$\det(\mathcal{C}(t)) = (7\sin(x_1(t)) + 13)^2. \tag{20}$$

Hence, the T–S fuzzy model is fuzzy controllable if and only if $\det(\mathcal{C}(t)) \neq 0\ \forall t$.

Clearly, the problem is reduced to find the roots $(x_1(t))$ of (20). For this example the roots of $\det(\mathcal{C}(t)) = 0$ are $x_1(t) = \frac{3\pi}{2} - 1.2302i$ and $x_1(t) = -\frac{\pi}{2} + 1.2302i$, which are not real. Therefore, it can be concluded that the overall T–S fuzzy model is fuzzy controllable for all t.

In this study, the command *solve* of MATLAB is used to obtain the roots of $\det(\mathcal{C}(t)) = 0$. However, any other mathematical software or even the graphical approach can be used to compute such roots, because only *real* values are important.

Remark 4.1: Note that this method can be applied to systems with more than two fuzzy rules, provided that there are, at most, two rules activated simultaneously. In those cases, a fuzzy controllability matrix has to be constructed for each fuzzy interpolation region.

Therefore, it is proposed to relocate the fuzzy eigenvalues at $s_1 = -5$ and $s_2 = -6$. As consequence

$$P(s) = s^2 + 11s + 30$$

and

$$P(A(t)) = (A(t))^2 + 11A(t) + 30.$$

From previous analysis, it results that matrix $\mathcal{C}(t)$ has a full rank for any real value of $x_1(t)$. Roughly speaking, the T–S fuzzy model is fuzzy controllable for all t, i.e., it is possible to place the fuzzy eigenvalues to any arbitrary location in the complex plane. Therefore, $K(t)$ can be computed by applying fuzzy Ackermann's formula 3.2 using (16).

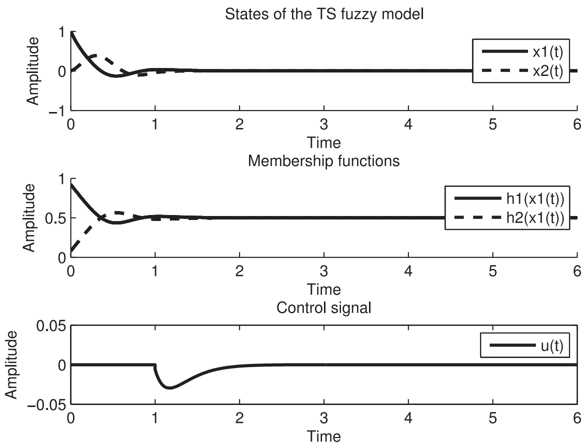

Fig. 1. States, membership functions, and control signal for the T–S fuzzy model.

These steps can be easily performed by means of mathematical software like MATLAB, Mathematica, or Maple, and the result is

$$K(t)=\left[-\frac{\alpha(t)}{7\sin(x_1(t))+13}-\frac{18}{7} \quad \frac{116}{7\,(7\sin(x_1(t))+13)}-\frac{3}{7}\right]$$

where $\alpha(t)=\frac{9\sin(x_1(t))^2}{8}-\frac{2831}{56}$.

Therefore, the control signal $u(t)=-K(t)x(t)$ is applied on the fuzzy system after 1s of free evolution, with initial conditions $x(0)=[1 \quad 0]^T$.

From Fig. 1, it can be observed how the T–S fuzzy system is asymptotically stable with decaying oscillations. However, when the control signal is activated at $t=1$ s, the behavior changes, becoming exponentially stable. This is consequence of fuzzy eigenvalues relocation. Besides, from this graphic, one can observe the membership functions and the control signal, as well.

B. Example 2

Now, a different two-rule T–S fuzzy model defined by

$$A_1=\begin{bmatrix}1 & 10\\ 3 & 1\end{bmatrix},\ b_1=\begin{bmatrix}0\\ 10\end{bmatrix},\ A_2=\begin{bmatrix}1 & -20\\ 4 & 1\end{bmatrix},\ b_2=\begin{bmatrix}0\\ 3\end{bmatrix}$$

is considered. For this case, the membership functions are the same to those given in previous example.

As a result, one has

$$A(t)=\begin{bmatrix} h_1(x_1(t))+h_2(x_1(t)) & 10\,h_1(x_1(t))-20\,h_2(x_1(t))\\ 3\,h_1(x_1(t))+4\,h_2(x_1(t)) & h_1(x_1(t))+h_2(x_1(t))\end{bmatrix}$$

$$b(t)=\begin{bmatrix}0\\ 10\,h_1(x_1(t))+3\,h_2(x_1(t))\end{bmatrix}.$$

Besides, in this example the fuzzy system is assumed as a slowly time-varying linear system. Therefore, overall fuzzy controllability matrix (17) for this case is

$$\mathcal{C}(t)=\begin{bmatrix}0 & (10\,h_1(x_1(t))-20\,h_2(x_1(t)))\,\beta(t)\\ \beta(t) & (h_1(x_1(t))+h_2(x_1(t)))\,\beta(t)\end{bmatrix} \tag{21}$$

with $\beta(t)=(10\,h_1(x_1(t))+3\,h_2(x_1(t)))$. Now, $h_2(x_1(t))=1-h_1(x_1(t))$ is considered, and the determinant can be computed by means of any numerical software. For this case, it results

$$\det(\mathcal{C}(t))=-(7h_1(x_1(t))+3)^2(30h_1(x_1(t))-20)$$

and the roots of $\det(\mathcal{C}(t))=0$ are $h_1(x_1(t))=0.6667$, $h_1(x_1(t))=-0.4286$ and $h_1(x_1(t))=-0.4286$. This means that matrix $\mathcal{C}(t)$ becomes singular when $h_1(x_1(t))=0.6667$ and $h_2(x_1(t))=0.3333$. Therefore, it is not possible to place fuzzy eigenvalues arbitrarily. In other words, the overall T–S fuzzy system is not fuzzy controllable. Therefore, the stabilization of this particular system cannot be guaranteed by means of the proposed approach.

On the other hand, according to [3] the controllability matrix of the linearized T–S fuzzy system around the origin is

$$G=\begin{bmatrix}0 & -32.5\\ 6.5 & 6.5\end{bmatrix}$$

the rank of which is equal to 2 (full rank), meaning that the T–S fuzzy system is controllable.

Hence, for this example, the fuzzy controllability property cannot be correctly determined by the approach given in [3].

Now, the same problem will be solved by using (16), i.e., the fuzzy model will not be considered as a slowly time-varying system.

Therefore, after substituting the membership functions in $A(t)$ and $b(t)$, as in Example 1, the fuzzy controllability matrix (16) is

$$\mathcal{C}(t)=\begin{bmatrix}0 & \dfrac{5\,(3\sin(x_1(t))-1)\,(7\sin(x_1(t))+13)}{2}\\ \dfrac{7\sin(x_1(t))}{2}+\dfrac{13}{2} & \alpha(t)\end{bmatrix} \tag{22}$$

where $\alpha(t)=\frac{7\sin(x_1(t))}{2}-\frac{7x_1(t)\cos(x_1(t))}{2}+\frac{35x_2(t)\cos(x_1(t))}{2}-\frac{105x_2(t)\cos(x_1(t))\sin(x_1(t))}{2}+\frac{13}{2}$, and

$$\det(\mathcal{C}(t))=-\frac{5\,(3\sin(x_1(t))-1)\,(7\sin(x_1(t))+13)^2}{4}.$$

The roots of $\det(\mathcal{C}(t))=0$ are $x_1(t)=0.3398$, $x_1(t)=2.8018$, $x_1(t)=\frac{3\pi}{2}-1.2302i$, and $x_1(t)=-\frac{\pi}{2}+1.2302i$. Again, the complex ones are disregarded, but for both of the real roots, it can readily obtain that $h_1(x_1(t))=0.6667$ and $h_1(x_1(t))=0.3333$, which is consistent with the previous result.

Remark 4.2: For this example both linear subsystems are controllable, but the overall fuzzy system is not fuzzy controllable. As expected, the local controllability does not imply fuzzy controllability for T–S fuzzy models.

Remark 4.3: For T–S fuzzy systems with several interpolation regions, the "overall controllability matrix," constructed from

(8), is the one to be considered in fuzzy Ackermann‘s formula 3.2. However, the fuzzy controllability property has to be analyzed through fuzzy controllability matrices corresponding to the different fuzzy interpolation regions. For T–S fuzzy models assumed as slowly time-varying linear systems, this procedure is valid if at most two fuzzy rules are activated at the same time.

Remark 4.4: The controllability of the local subsystems can be analyzed by means of the determinant of the fuzzy controllability matrix obtained from two adjacent subsystems satisfying $h_i(z(t)) + h_j(z(t)) = 1$ for all $i, j = 1, \ldots, r$, which define a fuzzy interpolation region.

C. Example 3

Finally, another two-rule T–S fuzzy system is considered. The matrices defining this model are

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ b_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with

$$A(t) = \begin{bmatrix} 2h_2 & h_1 \\ h_2 - h_1 & 3h_2 \end{bmatrix}$$

$$b(t) = \begin{bmatrix} 0 \\ 10h_1 + h_2 \end{bmatrix}$$

where $h_1 \equiv h_1(x_1(t))$ and $h_2 \equiv h_2(x_1(t))$.

Again, consider the same membership functions used in previous examples. This time, the T–S fuzzy model is not assumed as a slowly time-varying linear system; therefore, the overall fuzzy controllability matrix is obtained from (16), resulting

$$\mathcal{C}(t) = \begin{bmatrix} 0 & \frac{(\sin(x_1(t))+1)(9\sin(x_1(t))+11)}{4} \\ \frac{9\sin(x_1(t))}{2} + \frac{11}{2} & \beta(t) \end{bmatrix}$$

with

$$\beta(t) = \frac{\alpha(t)}{2} - \left(\frac{3\sin(x_1(t))}{2} - \frac{3}{2}\right)\left(\frac{9\sin(x_1(t))}{2} + \frac{11}{2}\right)$$

where

$$\alpha(t) = 9\cos(x_1(t)) \times \left(x_1(t)(\sin(x_1(t))-1) - x_2(t)\left(\frac{\sin(x_1(t))}{2} + \frac{1}{2}\right)\right)$$

and

$$\det(\mathcal{C}(t)) = -\frac{(\sin(x_1(t)) + 1)(9\sin(x_1(t)) + 11)^2}{8}.$$

As before, fuzzy controllability matrix (16) is obtained after substituting the membership functions in the continuous time-varying matrices $A(t)$ and $b(t)$.

The roots of $\det(\mathcal{C}(t)) = 0$ are $x_1(t) = \frac{3\pi}{2} - 0.6549i$, $x_1(t) = -\frac{\pi}{2} + 0.6549i$, which are not real and $x_1(t) = -\frac{\pi}{2}$ which is real. Therefore, the first two roots can be disregarded because they are complex numbers, but $h_1(x_1(t)) = 0$ with the third value of $x_1(t)$. Consequently, $\mathcal{C}(t)$ lost rank when $h_1(x_1(t)) = 0$ and $h_2(x_1(t)) = 1$. In other words, the overall T–S fuzzy model is not fuzzy controllable because the second subsystem is not controllable.

From [3], the controllability matrix of the linearized T–S fuzzy system around the origin is

$$G = \begin{bmatrix} 0.5 & 8.75 \\ 5.5 & 8.25 \end{bmatrix}$$

which has full rank. Again, the approach given in [3] fails at the moment of analyzing the fuzzy controllability property of the T–S fuzzy system.

From all this, the following theorem can be deduced.

Theorem 4.5: A T–S fuzzy model of the form

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\,\{A_i x(t) + b_i u(t)\} \tag{23}$$

where r is the number of rules in the fuzzy model, $x(t) \in \mathbb{R}^n$ is the state of the plant, $u(t) \in \mathbb{R}$ is the control signal, and premise variable $z(t) = [z_1(t)\ z_2(t)\ \ldots\ z_p(t)]$ is a function of $x(t)$, with relative degree between $z(t)$ and $u(t)$ equal to n, which can be represented as

$$\dot{x}(t) = A(t)x(t) + b(t)u(t) \tag{24}$$

with

$$A(t) = \sum_{i=1}^{r} h_i(z(t))A_i$$

$$b(t) = \sum_{i=1}^{r} h_i(z(t))b_i$$

with, at most, two rules activated at any instant $t \geq 0$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$, is fully controllable, for all t, if the $r - 1$ determinants of fuzzy controllability matrices (16) corresponding to the fuzzy interpolation regions are different from zero, for any real value of the premise variable $z(t)$ at any instant $t \geq 0$, and the stabilizer can be designed by applying fuzzy Ackermann's formula 3.2.

Proof: Follows directly from previous discussion. □

Corollary 4.6: If the T–S fuzzy model (23) can be considered as a slowly time-varying linear system and, at most, two rules are activated at any instant $t \geq 0$ and $\sum_{i=1}^{r} h_i(z(t)) = 1$, then (23) is fully controllable, for all t, if the $r - 1$ determinants of the fuzzy controllability matrices (17) corresponding to the fuzzy interpolation regions are different from zero, for any value of the membership functions $h_i(z(t)) \in [0, 1]$ at any instant $t \geq 0$, and the stabilizer can be designed by applying fuzzy Ackermann's formula 3.2.

In the following section, a more real example is used to illustrate the validity of the proposed approach.

V. Stabilization of an Underactuated Nonlinear System

The schematic of the pendubot (Pendulum Robot) is shown in Fig. 2. Basically, it can be described as an electromechanical system composed by two rigid links. The first link (Link 1) is directly under the influence of a dc motor, which is the only

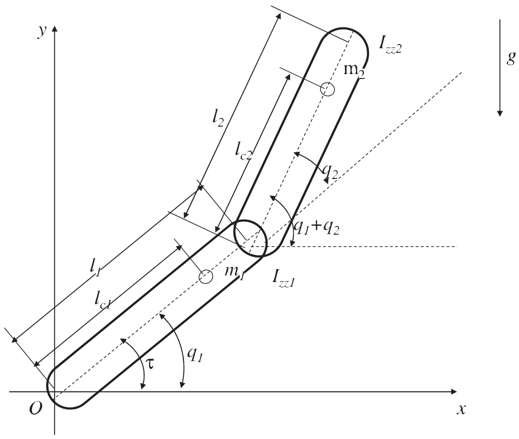

Fig. 2. Schematic of the pendubot.

actuator within the system. The second link (Link 2) is underactuated; therefore, its behavior is similar to that of the inverted pendulum on a car. Therefore, the interesting problem is to control the link 2. To do this, the system has two outputs given by two encoders placed in each one of the joints. In that way, the dynamics of the resulting system are much more richer than those given by the inverted pendulum.

The value of the parameters are [12]: length of link 1: $l_1 = 0.2032$ m, length of link 2: $l_2 = 0.3817$ m, distance of the center of mass for link 1: $l_{c1} = 0.1551$ m, distance of the center of mass for link 2: $l_{c2} = 0.1635$ m, mass of link 1: $m_1 = 0.8293$ kg, mass of link 2: $m_2 = .3402$ kg, moment of inertia for link 1, referred to its center of mass: $I_{zz1} = 59 \times 10^{-3}$ kg $\cdot$ m^2, moment of inertia for link 2, referred to its center of mass: $I_{zz2} = 43 \times 10^{-3}$ kg $\cdot$ m^2, friction constant for link 1: $\mu_1 = 0.00545$, friction constant for link 2: $\mu_2 = 0.00047$, gravitational constant: $g = 9.81$ m/s^2, angular position of link 1 (referred to x-axis): $q_1(t)$ in rad, angular position of link 2 (referred to link 1): $q_2(t)$ in rad, and torque applied by the actuator to link 1: $\tau(t)$ in N $\cdot$ m.

In order to obtain the state space model, the following change of variable is considered $x_1(t) = q_1(t)$, $x_2(t) = q_2(t)$, $x_3(t) = \dot{q}_1(t)$, $x_4(t) = \dot{q}_2(t)$, and $u(t) = \tau$.

Then, the dynamics of the plant can be described by a nonlinear equation of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{25}$$

where

$$f(x(t)) = \begin{bmatrix} x_3(t) \\ x_4(t) \\ f_{31}(x(t))f_{32}(x(t)) \\ f_{41}(x(t))f_{32}(x(t)) + f_{42}(x(t)) \end{bmatrix}$$

and

$$g(x(t)) = \begin{bmatrix} 0 \\ 0 \\ f_{31}(x(t)) \\ f_{41}(x(t)) \end{bmatrix}$$

with

$$\begin{aligned}
d_{11}(x(t)) &= m_1 l_{c1}^2 + m_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(x_2(t))\right) \\
&\quad + I_{zz1} + I_{zz2} \\
d_{12}(x(t)) &= m_2\left(l_2^2 + l_1 l_2 \cos(x_2(t))\right) + I_{zz2} \\
d_{22}(x(t)) &= m_2 l_{c2}^2 + I_{zz2} \\
c_1(x(t)) &= -2m_2 l_1 l_{c2} x_3(t)\sin(x_2(t))x_4(t) \\
&\quad - m_2 l_1 l_{c2} x_4^2(t)\sin(x_2(t)) \\
c_2(x(t)) &= m_2 l_1 l_{c2} x_3(t)\sin(x_2(t)) \\
g_1(x(t)) &= m_1 g l_{c1}\cos(x_1(t)) + m_2 g\,[l_1\cos(x_1(t)) \\
&\quad + l_2\cos(x_1(t) + x_2(t))] \\
g_2(x(t)) &= m_2 g l_{c2}\cos(x_1(t) + x_2(t)) \\
f_1(x(t)) &= \mu_1 x_3(t) \\
f_2(x(t)) &= \mu_2 x_4(t) \\
f_{31}(x(t)) &= \frac{d_{22}(x(t))}{d_{11}(x(t))d_{22}(x(t)) - d_{12}(x(t))^2} \\
f_{32}(x(t)) &= \frac{d_{12}(x(t))}{d_{22}(x(t))}\left[c_2(x(t)) + g_2(x(t)) + f_2(x(t))\right] \\
&\quad + c_1(x(t)) + g_1(x(t)) + f_1(x(t)) \\
f_{41}(x(t)) &= -\frac{d_{12}(x(t))}{d_{11}(x(t))d_{22}(x(t)) - d_{12}(x(t))^2} \\
f_{42}(x(t)) &= -\frac{1}{d_{22}(x(t))}\left[c_2(x(t)) + g_2(x(t))\right. \\
&\quad \left. + f_2(x(t))\right], \text{ and} \\
x(t) &= \left[x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\right]^T.
\end{aligned}$$

In this example, the latter nonlinear behavior is approximated by a three-rules T–S fuzzy model. The linear subsystems are obtained by linearizing the nonlinear plant in different operation points. For this case those points are $x^1 = [80\ 10\ 0\ 0]^T$, $x^2 = [90\ 0\ 0\ 0]^T$ and $x^3 = [100\ -10\ 0\ 0]^T$, where $x_1(t)$ and $x_2(t)$ are given in degrees, while $x_4(t)$ and $x_3(t)$ are given in degree/s. Notice that the chosen operation points fulfill the condition $x_1(t) + x_2(t) = 90°$. This is a pendubot restriction that must be satisfied during all the control process in order to keep it stable. Then, the resulting T–S fuzzy model is

Rule i

IF $x_{k,2}$ is $M_{1,i,1}$, THEN

$$\dot{x}(t) = A_i x(t) + b_i u(t) \qquad \text{for } i = 1, \ldots, 3.$$

Therefore, the overall T–S fuzzy system can be described by

$$\dot{x}(t) = \sum_{i=1}^{3} h_i(x_2(t))\{A_i x(t) + b_i u(t)\} \tag{26}$$

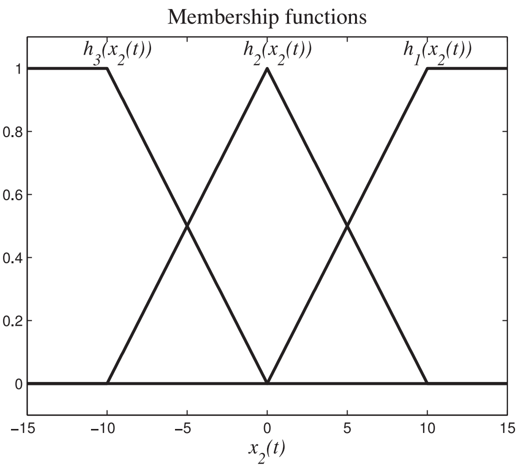

Fig. 3. Membership functions for the pendubot.

where the matrices defining the linear systems are

$$A_1 = A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 47.47 & -14.78 & -0.18 & 0.03 \\ -46.18 & 67.79 & 0.33 & -0.09 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 48.66 & -15.15 & -0.18 & 0.03 \\ -48.97 & 68.66 & 0.34 & -0.09 \end{bmatrix}$$

$$b_1 = b_3 = \begin{bmatrix} 0 \\ 0 \\ 32.58 \\ -59.65 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 0 \\ 0 \\ 32.88 \\ -60.63 \end{bmatrix}$$

and

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

with initial conditions

$$x_0 = [\,88 \quad 1 \quad 0 \quad 0\,].$$

Notice that subsystems 1 and 3 are defined by the same matrices, this is because the pendubot is symmetric around the vertical position. For this example, the membership functions representing the nonlinearity of the original systems are given in Fig. 3.

It results that the membership functions chosen for this example are not adequate to use (16); for that reason, this problem will be solved by considering the T–S fuzzy model as a slowly time-varying linear system. In other words, the fuzzy controllability matrix will be constructed from (17).

Thus, the T–S fuzzy system can be expressed in the form (8) with

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_{31}(t) & \alpha_{32}(t) & \alpha_{33}(t) & \alpha_{34}(t) \\ \alpha_{41}(t) & \alpha_{42}(t) & \alpha_{43}(t) & \alpha_{44}(t) \end{bmatrix}$$

and

$$b(t) = \begin{bmatrix} 0 \\ 0 \\ \beta_3(t) \\ \beta_4(t) \end{bmatrix}$$

where the elements $\alpha_{k,\ell}(t)$ and $\beta_k(t)$, for $k = 3, 4$ and $\ell = 1, \ldots, 4$, can be directly obtained from the fuzzy summation of the local subsystems, recalling that $A(t) = \sum_{i=1}^{3} h_i(x_2(t))A_i$, and $b(t) = \sum_{i=1}^{3} h_i(x_2(t))b_i$.

The overall fuzzy controllability matrix can be obtained through any mathematical software. For this case, it results

$$\mathcal{C}(t) = \left[\begin{matrix} 0 & \beta_3 & \alpha_{33}\beta_3 + \alpha_{34}\beta_4 \\ 0 & \beta_4 & \alpha_{43}\beta_3 + \beta_4(\alpha_{44}+2) \\ \beta_3 & \begin{matrix}\alpha_{33}\beta_3 \\ +\alpha_{34}\beta_4\end{matrix} & \begin{matrix}\beta_3(\alpha_{33}^2 + \alpha_{31} + \alpha_{34}\alpha_{43}) \\ +\beta_4(\alpha_{32} + \alpha_{33}\alpha_{34} + \alpha_{34}\alpha_{44})\end{matrix} \\ \beta_4 & \begin{matrix}\alpha_{43}\beta_3 \\ +\alpha_{44}\beta_4\end{matrix} & \begin{matrix}\beta_4(\alpha_{44}^2 + \alpha_{42} + \alpha_{34}\alpha_{43}) \\ +\beta_3(\alpha_{41} + \alpha_{33}\alpha_{43} + \alpha_{43}\alpha_{44})\end{matrix} \end{matrix}\right.$$
$$\left.\begin{matrix} \begin{matrix}\beta_3(\alpha_{33}^2 + \alpha_{31} + \alpha_{34}\alpha_{43}) \\ +\beta_4(\alpha_{32} + \alpha_{33}\alpha_{34} + \alpha_{34}\alpha_{44})\end{matrix} \\ \begin{matrix}\beta_4(\alpha_{44}^2 + 2\alpha_{44} + \alpha_{42} + \alpha_{34}\alpha_{43} + 4) \\ +\beta_3(\alpha_{41} + 2\alpha_{43} + \alpha_{33}\alpha_{43} + \alpha_{43}\alpha_{44})\end{matrix} \\ \begin{matrix}\beta_3(\alpha_{31}\alpha_{33} + \alpha_{32}\alpha_{43} + \alpha_{33}(\alpha_{33}^2 \\ +\alpha_{31} + \alpha_{34}\alpha_{43}) + \alpha_{34}(\alpha_{41} + \alpha_{33}\alpha_{43} \\ +\alpha_{43}\alpha_{44})) + \beta_4(\alpha_{31}\alpha_{34} + \alpha_{34}(\alpha_{44}^2 \\ +\alpha_{42} + \alpha_{34}\alpha_{43}) + \alpha_{32}(\alpha_{44}+2) \\ +\alpha_{33}(\alpha_{32} + \alpha_{33}\alpha_{34} + \alpha_{34}\alpha_{44}))\end{matrix} \\ \begin{matrix}\beta_3(\alpha_{33}\alpha_{41} + \alpha_{42}\alpha_{43} + \alpha_{43}(\alpha_{33}^2 \\ +\alpha_{31} + \alpha_{34}\alpha_{43}) + \alpha_{44}(\alpha_{41} + \alpha_{33}\alpha_{43} \\ +\alpha_{43}\alpha_{44})) + \beta_4(\alpha_{34}\alpha_{41} + \alpha_{44}(\alpha_{44}^2 \\ +\alpha_{42} + \alpha_{34}\alpha_{43}) + \alpha_{42}(\alpha_{44}+2) \\ +\alpha_{43}(\alpha_{32} + \alpha_{33}\alpha_{34} + \alpha_{34}\alpha_{44}))\end{matrix} \end{matrix}\right] \quad (27)$$

with $\alpha_{k,\ell} \equiv \alpha_{k,\ell}(t)$ and $\beta_k \equiv \beta_k(t)$, for $k = 3, 4$ and $\ell = 1, \ldots, 4$.

Clearly, the fuzzy controllability property cannot be analyzed from (27), in a practical way. However, according to Theorem 4.5 and Corollary 4.6, the overall fuzzy controllability property can be easily analyzed when the T–S fuzzy model has at most two fuzzy rules activated at the same instant, which is the case of this problem.

In order to analyze the overall fuzzy controllability property, two fuzzy controllability matrices have to be constructed by using (17), one for each interpolation region, i.e., $\mathcal{C}_{12}(t)$ for subsystems 1–2, and $\mathcal{C}_{23}(t)$ for subsystems 2–3. Notice that

$h_1(x_2(t)) = 1 - h_2(x_2(t))$ is considered for the first interpolation region, while $h_3(x_2(t)) = 1 - h_2(x_2(t))$ for the second one. As a result, the fuzzy controllability matrices for both regions are the same because of the aforementioned symmetry of the fuzzy model, i.e.,

$$\mathcal{C}_{12}(t)=\mathcal{C}_{23}(t)=\begin{bmatrix} 0 & 0.308h_2+32.6 & \begin{matrix}-0.00102h_2^2\\-0.167h_2-7.45\end{matrix} & \begin{matrix}3.15e^{-6}h_2^3\\+0.731h_2^2+90h_2\\+2433\end{matrix} \\ 0 & -0.986h_2-59.6 & \begin{matrix}0.00285h_2^2\\+0.433h_2+15.7\end{matrix} & \begin{matrix}-8.91e^{-6}h_2^3\\-1.72h_2^2-224h_2\\-5555\end{matrix} \\ 0.308h_2+32.6 & \begin{matrix}-0.00102h_2^2\\-0.167h_2-7.45\end{matrix} & \begin{matrix}3.15e^{-6}h_2^3\\+0.731h_2^2+90h_2\\+2433\end{matrix} & \begin{matrix}-9.81e^{-9}h_2^4\\-0.00437h_2^3-0.893h_2^2\\-58.2h_2-1177\end{matrix} \\ -0.986h_2-59.6 & \begin{matrix}0.00285h_2^2\\+0.433h_2+15.7\end{matrix} & \begin{matrix}-8.91e^{-6}h_2^3\\-1.72h_2^2-224h_2\\-5555\end{matrix} & \begin{matrix}2.77e^{-8}h_2^4\\+0.0113h_2^3+2.23h_2^2\\+140h_2+2688\end{matrix} \end{bmatrix}$$

where $h_2 \equiv h_2(x_2(t))$. Therefore, the determinant is

$$\begin{aligned}\det(\mathcal{C}_{12}(t)) &= \det(\mathcal{C}_{23}(t))\\ &= 4.61e^{-9}h_2^7 - 0.036h_2^6 - 2.69h_2^5 + 426h_2^4\\ &\quad + 3.14e^4h_2^3 - 1.07e^6h_2^2 - 9.02e^7h_2 - 1.29e^9\end{aligned}$$

and the roots as for $\det(\mathcal{C}_{12}(t)) = 0$ as for $\det(\mathcal{C}_{23}(t)) = 0$ are $h_2(x_2(t)) = 7811966.07$, $h_2(x_2(t)) = 78.41$, $h_2(x_2(t)) = 77.95$, $h_2(x_2(t)) = -88.04 + 0.07i$, $h_2(x_2(t)) = -88.04 - 0.07i$, $h_2(x_2(t)) = -27.64$ and $h_2(x_2(t)) = -27.37$. Therefore, it can be concluded that the overall T–S fuzzy system is fuzzy controllable.

Now, suppose that the pendubot has to be stabilized on its superior vertical position with decaying oscillations of 1 Hz (2π rad/s). Then, it is proposed to locate the fuzzy eigenvalues as follows: $s_1 = -0.5 + 2\pi i$, $s_2 = -0.5 - 2\pi i$, $s_3 = -10 + 2\pi i$, and $s_4 = -10 - 2\pi i$.

Thus, the fuzzy stabilizer $u(t) = -K(t)x(t)$ is computed by means of fuzzy Ackermann's formula 3.2 by considering the overall fuzzy controllability matrix (27), resulting in

$$K(t) = [\,0 \quad \cdots \quad 0 \quad 1\,]\,\mathcal{C}(t)^{-1}P(A(t))$$

where $P(A(t))$ is the matrix polynomial satisfying Cayley–Hamilton theorem.

Then, the resulting controller is applied on nonlinear plant (25). The simulations results are given in Fig. 4.

From Fig. 4, it can be observed that the fuzzy controller designed by means of Ackermann's formula is capable of stabilizing the pendubot system (25) at the vertical superior position (see Fig. 2). In order to verify that the controller induced the desired frequency on the nonlinear system, the frequency spectrum for state $x_4(t)$ is depicted in Fig. 5. As can be observed, the only frequency presented in state $x_4(t)$ correspond to 1 Hz, which is the same frequency used to design the stabilizer.

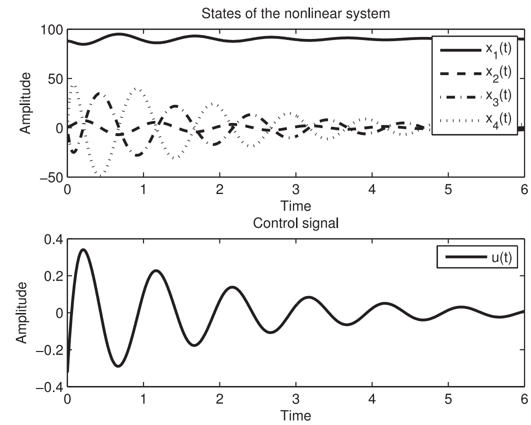

Fig. 4. States and control signal for the nonlinear model of the pendubot when the controller is obtained from fuzzy Ackermann's formula 3.2.

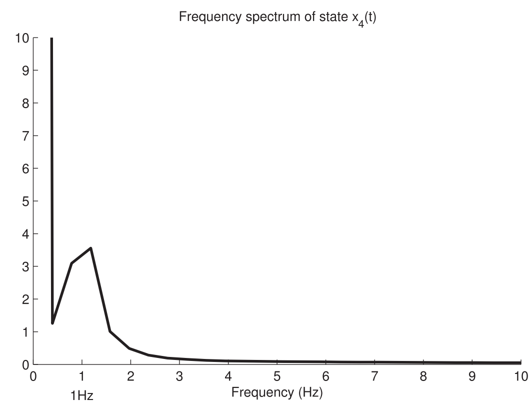

Fig. 5. Frequency spectrum for state $x_4(t)$ of the pendubot.

Notice that the results presented in Figs. 4 and 5 correspond to the application of the fuzzy controller on the "typical" nonlinear system. This means that the pendubot can be considered as a slowly time varying system, at least when the stabilization parameters are similar to those considered in this example.

Finally, a different stabilizer is computed using the simple PDC approach, obtaining $u(t) = \sum_{i=1}^{3} K_i x(t)$, with

$$\begin{aligned}K_1 = K_3 &= [\,16.64 \quad 17.83 \quad 3.17 \quad 2.26\,]\\ K_2 &= [\,16.35 \quad 17.58 \quad 3.13 \quad 2.23\,].\end{aligned}$$

The numerical results are given in Fig. 6. Notice that although the LMI design produces a simpler stabilizer, the imposing of an arbitrary behavior is not so evident.

Remark 5.1: The overall fuzzy controllability property can be analyzed in a practical way through the proposed approach.

Remark 5.2: Although the stabilizer obtained from fuzzy Ackermann's formula may be complex, the designing method is really simple. Its implementation in real-time problems can be also carried out in an easy way, because the stabilizers are completely defined by basic mathematical operations. However, in

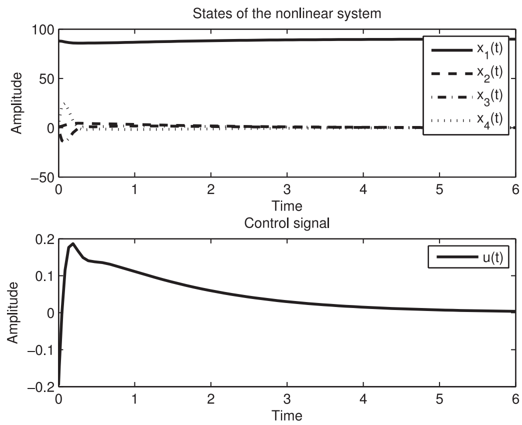

Fig. 6. States and control signal for the nonlinear model of the pendubot when the controller is obtained from the PDC approach.

some cases, a simplification process would be recommended in order to reduce the numerical errors induced by the great number of mathematical operations.

VI. Conclusion

In this study, the fuzzy controllability conditions for a class of T–S fuzzy systems have been derived. Therefore, the fuzzy controllability property can be easily analyzed and global nonlinear stabilizers can be designed in a practical way. In addition, fuzzy Ackermann's formula has been given in order to construct the fuzzy stabilizer.

Despite the fact that, stabilizers obtained by means of the proposed approach may lead to complex results, the designing method is quite simple and real-time implementations can be carried out without major inconveniences because only basic mathematical operations are involved.

Some examples in continuous-time domain were used to verify the efficacy of the proposed approach. However, it is clear that this method can be directly extended to the discrete-time domain.

It is important to mention that continuous time-varying $K(t)$ can be readily obtained by means of mathematical software like MATLAB, Mathematica, or Maple.

Finally, the proposed approach is not better or worse than other methods, it is just an alternative to design nonlinear stabilizers for T–S fuzzy models, which accomplish the features mentioned in this paper.

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