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Impulsive observer design for a class of switched nonlinear systems with unknown inputs[☆]

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Abstract

This paper investigates hybrid observer design of a class of unknown input switched nonlinear systems. The distinguishing feature of the proposed method is that the stability of all subsystems of the error switched systems is not necessarily required. First, an output derivative-based method and time-varying coordinate transformation are considered to eliminate the unknown input. Then in order to maintain a satisfactory estimation performance, an impulsive full-order and switched reduced-order observer are developed with a pair of upper and lower dwell time bounds and constructing time-varying Lyapunov functions combined with convex combination technique. In addition, the time-varying Lyapunov functions method is also used to analyze the stability of a class of error switched nonlinear systems with stable subsystems. Finally, two examples are presented to demonstrate the effectiveness of the proposed method.

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1. Introduction

Switched systems belong to an important class of hybrid systems, which consist of a family of subsystems described by continuous or discrete-time dynamics. Such systems have gained

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considerable interests in both theoretical researches and practical applications, such as power electronics, chaos generators, traffic control and intercepting missiles [1]. In the last two decades, some fundamental theories of switched systems such as controllability, stability and stabilization have been established, see [2–5] and references therein. Meanwhile, great efforts have also been put on the state estimations or observer designs for switched systems [6–9], since full or partial information of state is unavailable in abundant practical applications. However, these switched systems do not contain unknown inputs. As is known, there are many situations in practical systems where disturbances or partial inputs are inaccessible, such as fault detection and isolation [10], chaotic secure communication [11] and the cutting forces exerted in machine tools [12], which can all be viewed as unknown inputs. Therefore, this paper will focus on state estimation of switched systems with unknown inputs.

Recently, many interesting results on unknown inputs observer design for switched systems have been reported in literature [13–16]. We note that all of the above results are based on the assumption that the output of the system under consideration is measured continuously. However, the outputs of many control systems are only available at discrete sampled instants because of the applications of digital sensors. In this case, proposing an impulsive observer is needed, which cannot only be updated in an impulsive fashion, but also save the bandwidth of networks and communication cost when the measured information from the system transports to the observer at discrete time instants in comparison with continuous-time observers. Then, an impulsive error dynamic system is constructed, which has been investigated extensively [17–19]. In [20], a full-order impulsive observer was designed firstly for nonlinear continuous-time systems. Then some different impulsive observers, such as adaptive impulsive observers and impulsive functional observers, were proposed for different systems in [21–24]. But for switched nonlinear systems, only Li et al. [25] investigated the adaptive impulsive observers, seldom did scholars consider this issue for switched systems with unknown inputs.

In addition, on the approach of decoupling unknown inputs from the dynamics of unavailable state component, the observer designs for the unknown input switched systems were studied in [26–29]. In [26], it should be pointed out that the convergence condition of the error dynamics was based on the existence of a common time-invariant Lyapunov function. As we know, it is not easy to find this common time-invariant Lyapunov function, and many switched systems can achieve asymptotic stability under some proper switching law without possessing a common time-invariant Lyapunov function. Then how to design an appropriate switching law became one of the most challenging problems in studying switched systems. Further, we find that all of results in [21,25–29] have the requirements that all or parts of error subsystems are stable. Thus, questions naturally arise: is it possible, based on the dwell-time technique, to achieve the observers design of switched nonlinear systems with unknown inputs without posing any stability requirements on error subsystems of switched systems? If possible, under what conditions can we achieve this goal and how?

To study the problems proposed above, different approaches of decoupling unknown inputs are applied firstly to design the hybrid observer for switched nonlinear systems with all unstable error subsystems. The main characteristics of the paper are summarized as follows.

- (1) Compared with the existing literature, the impulsive full-order observer is designed even if each error subsystem is unstable. Furthermore, a novel algorithm is given to avoid the limitation of the linearization method.

- (2) Different from [26–28], when there is no the assumption that the original switched subsystems are strong detectable, the reduced-order observer can still be designed successfully.
- (3) By virtue of time-varying Lyapunov function that sufficiently captures the hybrid characteristic of switched impulsive systems, sufficient conditions for ensuring the stability of the error switched systems are constructed in the framework of the convex combination technique.

This paper is organized as follows. In Section 2, system description and necessary preliminaries are given. The main results are presented in Section 3. An illustrative example along with numerical and simulation results is provided in Section 4.

2. Preliminaries

We shall introduce, in this section, some notations, definitions and lemmas, which will be used in the later sections.

Let $X \geq 0$ (or, $X > 0$) denote the symmetric matrix X that is semi-positive definite (or, positive definite). I and 0 represent, respectively, the identity matrix and zero matrix with appropriate dimensions. \mathbb{R} and \mathbb{R}^+ stand for the real numbers set and positive real numbers set. \mathbb{R}^n and $\mathbb{R}^{m \times n}$ denote, respectively, the n -dimensional Euclidean space, and the set of all $m \times n$ real matrices. For any matrix X , X^T denotes its transpose, X^{-1} means its inverse, X^+ is the Moore–Penrose inverse. If X has full column rank, then $X^+ = (X^T X)^{-1} X^T$; if X has full row rank, then $X^+ = X^T (X X^T)^{-1}$. The superscript ‘ T ’ denotes the transpose of a matrix, the symbol $*$ is the transposed elements in the symmetric position of a matrix, $He(X)$ represents $X + X^T$. $eig(X)$ represents the eigenvalues of matrix X .

Consider a class of nonlinear switched systems with unknown inputs described by

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + D_{\sigma(t)}w(t) + f_{\sigma(t)}(t, x(t)), \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is state, $u(t) \in \mathbb{R}^m$ is control input, $w(t) \in \mathbb{R}^p$ is unknown input vectors, and $y(t) \in \mathbb{R}^q$ is measurement output. $f_{\sigma(t)}(t, x(t)) : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous functions in their arguments. The matrices $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$ and $D_{\sigma(t)}$ are constant matrices with appropriate dimensions. $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0^-} x(t_k - h)$. $\sigma(t) : [t_0, \infty) \rightarrow M = \{1, 2, \dots, m\}$ is the switching signal, where m is a positive integer. $j \in M$ is the active mode at time t if $\sigma(t) = j \in M$ for $t \in [t_k, t_{k+1})$ with $k = 0, 1, 2, \dots$, where t_k is the k th switching time instant. The time sequence $\{t_k, k = 0, 1, 2, \dots\}$ belongs to $\Delta(\delta_1, \delta_2) \triangleq \{t_0 < t_1 < \dots < t_k < \dots, \lim_{k \rightarrow +\infty} t_k = +\infty, \delta_1 \leq t_{k+1} - t_k \leq \delta_2\}$, where δ_1 and δ_2 are minimum and maximum dwell times, respectively. The purpose of this paper is to develop a full-order impulsive observer and a reduced-order switched observer for the switched nonlinear system (1) under the following assumptions.

Assumption 1. For any $i \in M$, there exist matrices $F_i \in \mathbb{R}^{n \times n}$ such that,

$$\|f_i(t, x_1) - f_i(t, x_2)\| \leq \|F_i(x_1 - x_2)\|.$$

Assumption 2. $\text{rank}(C_i D_i) = \text{rank}(D_i) = p, \forall i \in M$.

3. Main results

3.1. Full-order impulsive observer

In this section, for observer design of system (1), it is necessary to eliminate the unknown input influences. Under Assumption 2, there exist matrices $E_i \in \mathbb{R}^{n \times q}$ such that $E_i C_i D_i = D_i$ for any $i \in M$. Then the general solution of E_i is given by

$$E_i = D_i(C_i D_i)^+ + G_i(I - (C_i D_i)(C_i D_i)^+), \tag{2}$$

where G_i is an arbitrary matrix with appropriate dimensions. Thus, we have

$$\begin{aligned} D_i w(t) &= E_i C_i D_i w(t) = E_i C_i (\dot{x}(t) - A_i x(t) - B_i u(t) - f_i(t, x(t))) \\ &= E_i \dot{y}(t) - E_i C_i A_i x(t) - E_i C_i B_i u(t) - E_i C_i f_i(t, x(t)). \end{aligned} \tag{3}$$

Substituting Eq. (3) into Eq. (1) yields

$$\dot{x}(t) = N_i A_i x(t) + N_i B_i u(t) + N_i f_i(t, x(t)) + E_i \dot{y}(t), \tag{4}$$

where $N_i = I - E_i C_i$.

Remark 3.1. In fact, for system (4), a general full-order observer is designed as

$$\dot{\hat{x}}(t) = N_i A_i \hat{x}(t) + N_i B_i u(t) + N_i f_i(t, \hat{x}(t)) + E_i \dot{y}(t) + L_i(y(t) - \hat{y}(t)),$$

where $\hat{x}(t)$ is the estimation of $x(t)$, $\hat{y}(t) = C\hat{x}(t)$ is the corresponding output. L_i are observer gain matrices. Obviously, the derivative of $y(t)$ is contained, which is difficult to measure in practical applications and may cause high frequency noise in observer design. In order to avoid this problem, a coordinate transformation is introduced, which may lead the estimated state jump in switched systems, thus an impulsive observer is needed.

The impulsive observer for switched system (1) has the form:

$$\begin{cases} \dot{z}(t) = N_i A_i \hat{x}(t) + N_i B_i u(t) + N_i f_i(t, \hat{x}(t)), & t \neq t_k, \\ z(t_k^+) = z(t_k^-) + L_i(y(t_k^-) - \hat{y}(t_k^-)), & t = t_k, \\ \hat{x}(t) = z(t) + E_i y(t), \end{cases} \tag{5}$$

where z is the observer state, \hat{x} is the estimation of x , L_i are the observer gain matrices to be designed.

In addition, it is assumed that $z(t_k) = z(t_k^+)$, i.e., the solutions of the impulsive observer are right continuous, and then $\hat{x}(t_k) = \hat{x}(t_k^+)$.

Define $e = x - \hat{x}$. From Eqs. (1) and (5), the estimation error dynamic of subsystem $i \in M$ is

$$\begin{cases} \dot{e}(t) = N_i A_i e(t) + N_i \Delta f_i(t, x, \hat{x}), & t \neq t_k, \\ e(t_k) = (I - L_i C_i)e(t_k^-), & t = t_k, \end{cases} \tag{6}$$

where $\Delta f_i(t, x, \hat{x}) = f_i(t, x(t)) - f_i(t, \hat{x}(t))$.

By virtue of Assumption 1, one has

$$\|\Delta f_i(t, x, \hat{x})\| \leq \|F_i e\|, \quad \forall i \in M. \tag{7}$$

Remark 3.2. In [29], a full-order impulsive reset observer is built with the premise that each error subsystem is stable. Compared with this situation, it is more challenging and difficult that none of the individual error subsystems is stable. Thus, our task in the next section is

to identify certain classes of switching signals satisfying (10) and in the meantime to design observer gains to ensure stability of the switched error system (6).

Case a: all error subsystems are unstable.

Theorem 3.1. Consider switched system (1) satisfying Assumptions 1 and 2 and the time sequence $t_k \in \Delta(\delta_1, \delta_2)$. For given $\eta > 0$, $0 < \mu < 1$ and $\rho_i > 1$, if there exist $P_{i1} > 0$, $P_{i2} > 0$, $G_i \in \mathbb{R}^{n \times q}$, $\bar{L}_i \in \mathbb{R}^{n \times q}$, $\epsilon_{imm} > 0$, $i, j \in M$, $i \neq j$, $m, n = 1, 2$, such that

$$\begin{bmatrix} \Omega_{imm} & P_{im}T_i - P_{im}G_iH_iC_i \\ * & -\epsilon_{imm}I \end{bmatrix} < 0, \tag{8}$$

$$\begin{bmatrix} -\mu P_{j1} & P_{i2} - C_i^T \bar{L}_i^T \\ * & -\rho_i P_{i2} \end{bmatrix} < 0, \tag{9}$$

$$\ln \mu + \eta \delta_2 < 0, \tag{10}$$

where

$$\Omega_{imm} = \theta_i P_{im} + He\{P_{im}(T_i - G_iH_iC_i)A_i\} + \frac{1}{\delta_n}(P_{i1} - P_{i2}) - \eta P_{im} + \epsilon_{imm}F_i^T F_i \text{ with } \theta_i = \frac{\ln \rho_i}{\delta_1},$$

$$T_i = I - D_i(C_iD_i)^+C_i, H_i = I - C_iD_i(C_iD_i)^+,$$

then the switched observer error dynamic system (6) is asymptotically stable. The observer gains are given by

$$L_i = P_{i2}^{-1}\bar{L}_i, \quad i \in M.$$

Proof. For $\forall t \in [t_k, t_{k+1})$, define $\delta_k = t_{k+1} - t_k$, $\kappa(t) = \frac{t-t_k}{\delta_k}$, $\tilde{\kappa}(t) = 1 - \kappa(t)$, then it follows that $\kappa(t) \in [0, 1]$, $\kappa(t_k) = 0$, $\kappa(t_k^-) = 1$.

Suppose $\sigma(t) = i$, $t \in [t_k, t_{k+1})$. Let $e(t)$ be a solution of system (6) and

$$V_i(e) = \psi(t)e^T P_i(t)e, \quad i \in M, \tag{11}$$

where $\psi(t) = \rho_i^{\kappa(t)-1}$, $P_i(t) = \kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2}$. Define $V_i(t) = V_i(e(t))$, then taking derivative of $V_i(t)$ along the trajectory of system (6) yields

$$\begin{aligned} \dot{V}_i(t) - \eta V_i(t) &= \dot{\psi}(t)e^T P_i(t)e + 2\psi(t)e^T P_i(t)\dot{e} + \psi(t)e^T \dot{P}_i(t)e - \eta\psi(t)e^T P_i(t)e \\ &= \psi(t)\left\{e^T \left[\frac{\ln \rho_i}{\delta_k} P_i(t) + 2P_i(t)N_iA_i + \frac{1}{\delta_k}(P_{i1} - P_{i2}) - \eta P_i(t) \right] e \right. \\ &\quad \left. + 2e^T P_i(t)N_i\Delta f_i \right\} \\ &\leq \psi(t)\left\{e^T [\theta_i P_i(t) + 2P_i(t)N_iA_i + \frac{1}{\delta_k}(P_{i1} - P_{i2}) - \eta P_i(t)]e + 2e^T P_i(t)N_i\Delta f_i \right\}, \tag{12} \end{aligned}$$

where $\Delta f_i = \Delta f_i(t, x, \hat{x})$, $\theta_i = \frac{\ln \rho_i}{\delta_1}$.

We choose a function $\nu(t) = \left(\frac{1}{\delta_k} - \frac{1}{\delta_2}\right) / \left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) \in [0, 1]$ and $\tilde{\nu}(t) = 1 - \nu(t)$, such that

$$\frac{1}{\delta_k} = \frac{1}{\delta_1}\nu(t) + \frac{1}{\delta_2}\tilde{\nu}(t). \tag{13}$$

From Eqs. (12) and (13), one has that

$$\begin{aligned}
 \dot{V}_i(t) - \eta V_i(t) &\leq \psi(t)\{e^T[\theta_i(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2}) + 2(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_iA_i - \eta P_i(t) \\
 &\quad + \left(\frac{1}{\delta_1}v(t) + \frac{1}{\delta_2}\tilde{v}(t)\right)(P_{i1} - P_{i2})]e + 2e^T(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_i\Delta f_i\} \\
 &= \psi(t)\{e^T[\kappa(t)(\theta_iP_{i1} + 2P_{i1}N_iA_i) + \tilde{\kappa}(t)(\theta_iP_{i2} + 2P_{i2}N_iA_i) - \eta P_i(t) \\
 &\quad + \left(\frac{1}{\delta_1}v(t) + \frac{1}{\delta_2}\tilde{v}(t)\right)(P_{i1} - P_{i2})]e + 2e^T(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_i\Delta f_i\} \\
 &= \psi(t)e^T\{\kappa(t)[v(t)(\theta_iP_{i1} + 2P_{i1}N_iA_i + \frac{1}{\delta_1}(P_{i1} - P_{i2}) - \eta P_{i1}) \\
 &\quad + \tilde{v}(t)(\theta_iP_{i1} + 2P_{i1}N_iA_i + \frac{1}{\delta_2}(P_{i1} - P_{i2}) - \eta P_{i1})] \\
 &\quad + \tilde{\kappa}(t)[v(t)(\theta_iP_{i2} + 2P_{i2}N_iA_i + \frac{1}{\delta_1}(P_{i1} - P_{i2}) - \eta P_{i2}) \\
 &\quad + \tilde{v}(t)(\theta_iP_{i2} + 2P_{i2}N_iA_i + \frac{1}{\delta_2}(P_{i1} - P_{i2}) - \eta P_{i2})]e \\
 &\quad + 2\psi(t)e^T(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_i\Delta f_i. \tag{14}
 \end{aligned}$$

According to Eq. (7), there exist $\epsilon_i(t) = \kappa(t)v(t)\epsilon_{i11} + \kappa(t)\tilde{v}(t)\epsilon_{i12} + \tilde{\kappa}(t)v(t)\epsilon_{i21} + \tilde{\kappa}(t)\tilde{v}(t)\epsilon_{i22}$ such that

$$\epsilon_i(t)(e^T F_i^T F_i e - \Delta f_i^T \Delta f_i) \geq 0. \tag{15}$$

Combining Eqs. (14) and (15), we have

$$\begin{aligned}
 \dot{V}_i(t) - \eta V_i(t) &\leq \psi(t)e^T\{\kappa(t)[v(t)(\theta_iP_{i1} + 2P_{i1}N_iA_i + \frac{1}{\delta_1}(P_{i1} - P_{i2}) - \eta P_{i1}) \\
 &\quad + \tilde{v}(t)(\theta_iP_{i1} + 2P_{i1}N_iA_i + \frac{1}{\delta_2}(P_{i1} - P_{i2}) - \eta P_{i1})] \\
 &\quad + \tilde{\kappa}(t)[v(t)(\theta_iP_{i2} + 2P_{i2}N_iA_i + \frac{1}{\delta_1}(P_{i1} - P_{i2}) - \eta P_{i2}) \\
 &\quad + \tilde{v}(t)(\theta_iP_{i2} + 2P_{i2}N_iA_i + \frac{1}{\delta_2}(P_{i1} - P_{i2}) - \eta P_{i2})]e \\
 &\quad + 2\psi(t)e^T(\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_i\Delta f_i + \epsilon_i(t)(e^T F_i^T F_i e - \Delta f_i^T \Delta f_i) \\
 &= \psi(t)\{\kappa(t)v(t)\Phi_i^{11} + \kappa(t)\tilde{v}(t)\Phi_i^{12} + \tilde{\kappa}(t)v(t)\Phi_i^{21} + \tilde{\kappa}(t)\tilde{v}(t)\Phi_i^{22}\}, \tag{16}
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi_i^{mn} &= e^T[\theta_iP_{im} + P_{im}N_iA_i + A_i^T N_i^T P_{im} + \frac{1}{\delta_n}(P_{i1} - P_{i2}) - \eta P_{im} + \epsilon_{imn}F_i^T F_i]e \\
 &\quad + 2e^T P_{im}N_i\Delta f_i - \epsilon_{imn}\Delta f_i^T \Delta f_i \\
 &= [e^T \quad \Delta f_i^T] \tilde{\Phi}_i^{mn} \begin{bmatrix} e \\ \Delta f_i \end{bmatrix},
 \end{aligned}$$

where $\tilde{\Phi}_i^{mn} = \begin{bmatrix} \theta_iP_{im} + P_{im}N_iA_i + A_i^T N_i^T P_{im} + \frac{1}{\delta_n}(P_{i1} - P_{i2}) - \eta P_{im} + \epsilon_{imn}F_i^T F_i & P_{im}N_i \\ * & -\epsilon_{imn}I \end{bmatrix}$.

Substituting $N_i = I - E_iC_i = T_i - G_iH_iC_i$ into $\tilde{\Phi}_i^{mn}$ and in view of Schur Complement Lemma, condition (8) is equivalent to $\tilde{\Phi}_i^{mn} < 0$. It follows from Eq. (13) that

$$\dot{V}_i(t) < \eta V_i(t), \quad t \neq t_k,$$

which implies

$$V_i(t) \leq V_i(t_k) \exp(\eta(t - t_k)), t \in [t_k, t_{k+1}). \tag{17}$$

At the impulsive switching time point $t = t_k$, $V_i(t_k) = \psi(t_k)e^T(t_k)P_i(t_k)e(t_k)$. Denote $\sigma(t) = j, t \in [t_{k-1}, t_k)$, from Eq.(9), it is obtained that

$$\begin{aligned} V_i(t_k) &= \psi(t_k)e^T(t_k)P_i(t_k)e(t_k) \\ &= \rho_i^{-1}e^T(t_k^-)(I - L_iC_i)^T P_{i2}(I - L_iC_i)e(t_k^-) \\ &\leq \mu e^T(t_k^-)P_{j1}e(t_k^-) \\ &= \mu \psi(t_k^-)e^T(t_k^-)P_j(t_k^-)e(t_k^-) \\ &= \mu V_j(t_k^-). \end{aligned} \tag{18}$$

Combining Eq. (17) with above inequality, we have

$$V_i(t_k) \leq \mu V_j(t_k^-) \leq \mu V_j(t_{k-1}) \exp(\eta(t_k - t_{k-1})) \leq V_j(t_{k-1}) \exp(\ln \mu + \eta\delta_2), \tag{19}$$

From Eqs. (10) and (19), it follows that $V_i(t_k) \leq V_j(t_{k-1})$, which implies that $V_{\sigma(t_k)}(t_k), k = 1, 2, \dots$ is strictly decreasing.

Using the same deduction as above leads to

$$V_{\sigma(t_k)}(t_k) \leq \exp(k(\ln \mu + \eta\delta_2))V_{\sigma(t_0)}(t_0) \leq \exp(k(\ln \mu + \eta\delta_2))\max\{\lambda_{\max}(P_{il})\}\|e_0\|^2, l = 1, 2,$$

which implies $\lim_{k \rightarrow +\infty} V_{\sigma(t_k)}(t_k) = 0$ under the condition (10). From Eq. (17), it follows that $V_{\sigma}(t) \leq V_{\sigma}(t_k)\exp(\eta\delta_2)$ for $t \in [t_k, t_{k+1})$, i.e., $\lim_{t \rightarrow +\infty} V_{\sigma}(t) = 0$. From the description of $V_i(t)$, we have $\min\{\lambda(P_{il})\}\|e(t)\|^2 \leq \rho_i V_i(t) \leq \rho_i \max\{\lambda(P_{il})\}\|e(t)\|^2$. Then

$$\lim_{t \rightarrow +\infty} \|e(t)\|^2 \leq \frac{\max\{\rho_i\}}{\min\{\lambda(P_{il})\}} V_i(t) = 0,$$

where $i \in M$ and $l = 1, 2$, $\lambda(P_{il})$ is the eigenvalue of matrix P_{il} . Therefore, system (6) is asymptotically stable. \square

Case b: all error subsystems are stable.

Theorem 3.2. Consider switched system (1) satisfying Assumptions 1 and 2 and the time sequence $t_k \in \Delta(\delta_1, \delta_2)$. For given $\eta > 0, \mu > 0$ and $\rho_i > 1$, if there exist $P_{i1} > 0, P_{i2} > 0, G_i \in \mathbb{R}^{n \times q}, \bar{L}_i \in \mathbb{R}^{n \times q}, \epsilon_{imn} > 0, i, j \in M, i \neq j, m, n = 1, 2$, such that Eq. (9) and the following inequalities hold:

$$\begin{bmatrix} \Omega_{imn} & P_{im}T_i - P_{im}G_iH_iC_i \\ * & -\epsilon_{imn}I \end{bmatrix} < 0, \tag{21}$$

$$\ln \mu - \eta\delta_1 < 0, \tag{22}$$

where $\Omega_{imn} = \theta_i P_{im} + He\{P_{im}(T_i - G_iH_iC_i)A_i\} + \frac{1}{\delta_n}(P_{i1} - P_{i2}) + \eta P_{im} + \epsilon_{imn}F_i^T F_i$ with $\theta_i = \frac{\ln \rho_i}{\delta_1}$,

$$T_i = I - D_i(C_iD_i)^+C_i, H_i = I - C_iD_i(C_iD_i)^+,$$

then the switched observer error dynamic system (6) is asymptotically stable. The observer gains are given by

$$L_i = P_{i2}^{-1}\bar{L}_i, i \in M.$$

Proof. According to the proof in [Theorem 3.1](#), we have

$$\begin{aligned} \dot{V}_i(t) + \eta V_i(t) &\leq \psi(t)\{e^T [\theta_i P_i(t) + 2P_i(t)N_i A_i + \frac{1}{\delta_k} (P_{i1} - P_{i2}) + \eta P_i(t)]e + 2e^T P_i(t)N_i \Delta f_i\} \\ &\leq \psi(t)e^T \{\kappa(t)[v(t)(\theta_i P_{i1} + 2P_{i1}N_i A_i + \frac{1}{\delta_1} (P_{i1} - P_{i2}) + \eta P_{i1}) \\ &\quad + \tilde{v}(t)(\theta_i P_{i1} + 2P_{i1}N_i A_i + \frac{1}{\delta_2} (P_{i1} - P_{i2}) + \eta P_{i1})] \\ &\quad + \tilde{\kappa}(t)[v(t)(\theta_i P_{i2} + 2P_{i2}N_i A_i + \frac{1}{\delta_1} (P_{i1} - P_{i2}) + \eta P_{i2}) \\ &\quad + \tilde{v}(t)(\theta_i P_{i2} + 2P_{i2}N_i A_i + \frac{1}{\delta_2} (P_{i1} - P_{i2}) + \eta P_{i2})]\}e \\ &\quad + 2\psi(t)e^T (\kappa(t)P_{i1} + \tilde{\kappa}(t)P_{i2})N_i \Delta f_i + \epsilon_i(t)(e^T F_i^T F_i e - \Delta f_i^T \Delta f_i) \\ &= \psi(t)\{\kappa(t)v(t)\hat{\Phi}_i^{11} + \kappa(t)\tilde{v}(t)\hat{\Phi}_i^{12} + \tilde{\kappa}(t)v(t)\hat{\Phi}_i^{21} + \tilde{\kappa}(t)\tilde{v}(t)\hat{\Phi}_i^{22}\}, \end{aligned}$$

where

$$\begin{aligned} \hat{\Phi}_i^{mn} &= e^T [\theta_i P_{im} + P_{im}N_i A_i + A_i^T N_i^T P_{im} + \frac{1}{\delta_n} (P_{i1} - P_{i2}) + \eta P_{im} + \epsilon_{imn} F_i^T F_i]e \\ &\quad + 2e^T P_{im}N_i \Delta f_i - \epsilon_{imn} \Delta f_i^T \Delta f_i \\ &= [e^T \quad \Delta f_i^T] \check{\Phi}_i^{mn} \begin{bmatrix} e \\ \Delta f_i \end{bmatrix}, \end{aligned}$$

where $\check{\Phi}_i^{mn} = \begin{bmatrix} \theta_i P_{im} + P_{im}N_i A_i + A_i^T N_i^T P_{im} + \frac{1}{\delta_n} (P_{i1} - P_{i2}) + \eta P_{im} + \epsilon_{imn} F_i^T F_i & P_{im}N_i \\ * & -\epsilon_{imn} I \end{bmatrix}$.

From condition (21), it is obtained that

$$\dot{V}_i(t) < -\eta V_i(t), \quad t \neq t_k,$$

which implies

$$V_i(t) \leq V_i(t_k) \exp(-\eta(t - t_k)), \quad t \in [t_k, t_{k+1}).$$

At the impulsive switching time point $t = t_k$, based on the condition (22), we have $V_i(t_k) \leq \mu V_j(t_k^-)$. Then it implies that

$$V_i(t_k) \leq \mu V_j(t_{k-1}) \exp(-\eta(t_k - t_{k-1})) \leq V_j(t_{k-1}) \exp(\ln \mu - \eta \delta_1).$$

Combining with Eq. (22), it follows that $V_i(t_k) \leq V_j(t_{k-1})$, and then $V_{\sigma(t_k)}(t_k)$, $k = 1, 2, \dots$ is strictly decreasing. Similar to the proof in [Theorem 3.1](#), system (6) is asymptotically stable.

In order to solve the nonlinear matrix inequalities in [Theorems 3.1](#) and [3.2](#), motivated by the linear programming technique, a new type of algorithm is presented as follows. \square

Algorithm. Step 1) Select the appropriate scalars μ and η satisfying Eq. (10)((22)), by using cvx toolbox in Matlab, equalities (9) can be solved with $P_{i1}, P_{i2} > 0, i \in M$.

Step 2) Substituting the obtained P_{i1}, P_{i2} into Eq. (8)((21)), then one feasible solution can be calculated as G_i . Otherwise, go back to step 1.

Remark 3.3. In [30,31], the linearization method is often used to deal with the term $P_{im}G_i$. If we also take this method in [Theorem 3.1](#), the corresponding result is given as follows:

Consider switched system (1) satisfying Assumptions 1 and 2 and the time sequence $t_k \in \Delta(\delta_1, \delta_2)$. For given $\eta > 0$, $0 < \mu < 1$, $\rho_i > 1$ and constants $k_1, k_2 > 0$, if there exist $P_i > 0$, $\tilde{G}_i \in \mathbb{R}^{n \times q}$, $\tilde{L}_i \in \mathbb{R}^{n \times q}$, $\epsilon_{imn} > 0$, $i, j \in M$, $i \neq j$, $m, n = 1, 2$, such that

$$\begin{bmatrix} \Omega_{imn} & k_m P_i T_i - k_m P_i G_i H_i C_i \\ * & -\epsilon_{imn} I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\mu k_1 P_j & k_2 P_i - C_i^T \tilde{L}_i^T \\ * & -\rho_i k_2 P_i \end{bmatrix} < 0,$$

In $u + \eta \delta_2 < 0$,

where

$\Omega_{imn} = \theta_i k_m P_i + He\{k_m P_i (T_i - G_i H_i C_i) A_i\} + \frac{k_1 - k_2}{\delta_n} P_i - \eta k_m P_i + \epsilon_{imn} F_i^T F_i$ with $\theta_i = \frac{\ln \rho_i}{\delta_1}$, then the switched observer error dynamic system (6) is asymptotically stable. The observer gains are given by $L_i = P_i^{-1} \tilde{L}_i$, $i \in M$ and the matrix $G_i = P_i^{-1} \tilde{G}_i$.

Therefore, above result is a special case of Theorem 3.1, then Theorem 3.1 with the algorithm in Remark 3.3 is less conservatism.

3.2. Reduced-order switched observer

Under Assumption 2, for any $i \in M$, we introduce the time-varying state transformation [16]:

$$K_i = \begin{bmatrix} D_i^\perp \\ (C_i D_i)^+ C_i \end{bmatrix} \triangleq \begin{bmatrix} D_i^\perp \\ \tilde{K}_i \end{bmatrix}, U_i = \begin{bmatrix} (C_i D_i)^\perp \\ (C_i D_i)^+ \end{bmatrix}. \tag{23}$$

Note that K_i is nonsingular and its inverse matrix is

$$K_i^{-1} = [(I - D_i (C_i D_i)^+ C_i) (D_i^\perp)^+ \quad D_i] \triangleq [\hat{K}_i \quad D_i]. \tag{24}$$

By using the state and output transformation

$$\bar{x} = [\bar{x}_1^T \quad \bar{x}_2^T]^T = K_{\sigma(t)} x, \quad \bar{y} = [\bar{y}_1^T \quad \bar{y}_2^T]^T = U_{\sigma(t)} y, \tag{25}$$

then the switched system (1) becomes in the new coordinates

$$\begin{aligned} \dot{\bar{x}}_1 &= A_{\sigma(t),1} \bar{x}_1 + A_{\sigma(t),2} \bar{x}_2 + B_{\sigma(t),1} u + f_{\sigma(t),1}(t, x), \\ \dot{\bar{x}}_2 &= A_{\sigma(t),3} \bar{x}_1 + A_{\sigma(t),4} \bar{x}_2 + B_{\sigma(t),2} u + w + f_{\sigma(t),2}(t, x), \\ \bar{y}_1 &= \tilde{C}_{\sigma(t)} \bar{x}_1, \quad \bar{y}_2 = \bar{x}_2, \end{aligned} \tag{26}$$

where $K_{\sigma(t)} A_{\sigma(t)} K_{\sigma(t)}^{-1} = \begin{bmatrix} A_{\sigma(t),1} & A_{\sigma(t),2} \\ A_{\sigma(t),3} & A_{\sigma(t),4} \end{bmatrix}$, $K_{\sigma(t)} B_{\sigma(t)} = \begin{bmatrix} B_{\sigma(t),1} \\ B_{\sigma(t),2} \end{bmatrix}$, $\bar{x}_1 = D_{\sigma(t)}^\perp x$, $\bar{x}_2 = (C_{\sigma(t)} D_{\sigma(t)})^+ C_{\sigma(t)} x$, $\bar{y}_1 = (C_{\sigma(t)} D_{\sigma(t)})^\perp y$, $\bar{y}_2 = (C_{\sigma(t)} D_{\sigma(t)})^+ y$, $f_{\sigma(t),1}(t, x) = D_{\sigma(t)}^\perp f_{\sigma(t)}(t, x)$, $f_{\sigma(t),2}(t, x) = \tilde{K}_{\sigma(t)} f_{\sigma(t)}(t, x)$, $\tilde{C}_{\sigma(t)} = (C_{\sigma(t)} D_{\sigma(t)})^\perp C_{\sigma(t)} (D_i^\perp)^+$. At switching instants t_k , from [16], the state of system (26) is governed by

$$\bar{x}(t_k) = K_{\sigma(t_k)} x(t_k^-) = K_{\sigma(t_k)} K_{\sigma(t_{k-1})}^{-1} \bar{x}(t_k^-), \quad k = 1, 2, \dots \tag{27}$$

According to Eq. (26), we only need to design observers for \bar{x}_1 , because \bar{x}_2 can be directly obtained by \bar{y}_2 . Furthermore, the dynamics (26) of \bar{x}_1 do not contain the unknown input w .

Consider the following subsystem relevant to \bar{x}_1 in Eq. (26)

$$\begin{aligned} \dot{\bar{x}}_1 &= A_{\sigma(t),1}\bar{x}_1 + A_{\sigma(t),2}\bar{x}_2 + B_{\sigma(t),1}u + f_{\sigma(t),1}(t, x), \\ \dot{\bar{y}}_1 &= \bar{C}_{\sigma(t)}\bar{x}_1, \\ \bar{x}_1(t_k) &= D_{\sigma(t_k)}^\perp x(t_k^-), \quad k = 1, 2, \dots \end{aligned} \tag{28}$$

the proposed observer for system (28) has the form

$$\begin{aligned} \dot{\hat{x}}_1 &= A_{\sigma(t),1}\hat{x}_1 + A_{\sigma(t),2}\bar{y}_2 + B_{\sigma(t),1}u + H_{\sigma(t)}(\bar{y}_1 - \bar{C}_{\sigma(t)}\hat{x}_1) + f_{\sigma(t),1}(t, \hat{x}), \\ \hat{x}_1(t_k) &= D_{\sigma(t_k)}^\perp \hat{x}(t_k^-) = D_{\sigma(t_k)}^\perp \hat{K}_{\sigma(t_{k-1})} \begin{bmatrix} \hat{x}_1(t_{k-1}^-) \\ \hat{y}_2(t_{k-1}^-) \end{bmatrix} \quad k = 1, 2, \dots \end{aligned} \tag{29}$$

where $H_{\sigma(t)}$ are the observer gains to be designed.

Define $e_1 = \bar{x}_1 - \hat{x}_1$, we can obtain the following error system

$$\begin{cases} \dot{e}_1 = (A_{\sigma(t),1} - H_{\sigma(t)}\bar{C}_{\sigma(t)})e_1 + \Delta f_{\sigma(t),1}(t, x, \hat{x}), \\ e_1(t_k) = D_{\sigma(t_k)}^\perp e(t_k^-) = D_{\sigma(t_k)}^\perp \hat{K}_{\sigma(t_{k-1})} e_1(t_{k-1}^-), \quad k = 1, 2, \dots \end{cases} \tag{30}$$

where $\Delta f_{\sigma(t),1}(t, x, \hat{x}) = f_{\sigma(t),1}(t, x) - f_{\sigma(t),1}(t, \hat{x})$. From Assumption 1, one has

$$\|\Delta f_{\sigma(t),1}(t, x, \hat{x})\| \leq \|D_{\sigma(t)}^\perp\| \|F_{\sigma(t)}\hat{K}_{\sigma(t)}e_1\|. \tag{31}$$

In this section, similar to the full-order observer design, two classes of switched systems are considered. One is that all error subsystems are unstable, which implies that there is no restriction of the strong detectability condition for system (1) compared with [26,27]; another case is that all error subsystems are stable.

Case c: all error subsystems are unstable.

Theorem 3.3. Consider switched system (1) satisfying Assumptions 1 and 2 and the time sequence $t_k \in \Delta(\delta_1, \delta_2)$. For given $\eta_1 > 0, 0 < \mu_1 < 1, \alpha_i > 0$ and $\rho_i > 1$, if there exist $P_{i1} > 0, P_{i2} > 0, X_i > 0, Y_i, \epsilon_{imn} > 0, i, j \in M, i \neq j, m, n = 1, 2$, such that

$$\begin{bmatrix} \Xi_{imn} & P_{im} - X_i - \alpha_i \bar{C}_i^T Y_i^T & P_{im} \\ * & -\alpha_i (X_i + X_i^T) & 0 \\ 0 & 0 & -\epsilon_{imn} I \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} -\mu_1 P_{j1} & (D_i^\perp \hat{K}_j)^T P_{i2} \\ * & -\rho_i P_{i2} \end{bmatrix} < 0, \tag{33}$$

$$\ln \mu_1 + \eta_1 \delta_2 < 0, \tag{34}$$

where $\Xi_{imn} = \theta_i P_{im} + P_{im} A_{i,1} + A_{i,1}^T P_{im} - Y_i \bar{C}_i - \bar{C}_i^T Y_i^T + \frac{1}{\delta_n} (P_{i1} - P_{i2}) - \eta_1 P_{im} + \epsilon_{imn} (F_i \hat{K}_i)^T F_i \hat{K}_i$ with $\theta_i = \frac{\ln \rho_i}{\delta_1}$, then the switched observer error dynamic system (30) is asymptotically stable. The observer gains are given by

$$H_i = X_i^{-1} Y_i, \quad i \in M.$$

Proof. Suppose $\sigma(t) = i, t \in [t_k, t_{k+1})$. Let $e_1(t)$ be a solution of system (30) and consider the Lyapunov function $V_{i1}(e_1) = \psi(t) e_1^T P_i(t) e_1, i \in M$, denote $V_{i1}(t) = V_{i1}(e_1(t))$. By pre- and

post multiplying both sides of (32) with $\begin{bmatrix} I & -(H_i\bar{C}_i)^T & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & -(H_i\bar{C}_i)^T & 0 \\ 0 & 0 & I \end{bmatrix}^T$, one has

$$\Psi_{imn} = \begin{bmatrix} \Lambda_{imn} - \eta_1 P_{im} & P_{im} \\ * & -\epsilon_{imn} I \end{bmatrix} < 0, \tag{35}$$

where $\Lambda_{imn} = \theta_i P_{im} + He\{P_{im}(A_{i,1} - H_i\bar{C}_i) + \frac{1}{\delta_n}(P_{i1} - P_{i2}) + \epsilon_{imn} F_i^T \hat{K}_i^T F_i \hat{K}_i\}$. Similar to the proof of Theorem 3.1, we obtain that

$$\dot{V}_{1i}(t) - \eta_1 V_{1i}(t) \leq \psi(t)\{\kappa(t)v(t)\Pi_i^{11} + \kappa(t)\tilde{v}(t)\Pi_i^{12} + \tilde{\kappa}(t)v(t)\Pi_i^{21} + \tilde{\kappa}(t)\tilde{v}(t)\Pi_i^{22}\}, \tag{36}$$

where $\Pi_i^{mn} = [e_1^T \quad \Delta f_{i,1}^T] \Psi_{imn} \begin{bmatrix} e_1 \\ \Delta f_{i,1} \end{bmatrix}$. Combining Eq. (35) with Eq. (36), it follows that

$$V_{1i}(t) \leq V_{1i}(t_k) \exp(\eta_1(t - t_k)), t \in [t_k, t_{k+1}). \tag{37}$$

From Eq. (33), then

$$\begin{aligned} V_{1i}(t_k) &= \psi(t_k) e_1^T(t_k) P_i(t_k) e_1(t_k) \\ &= \rho_i^{-1} e_1^T(t_k^-) (D_i^\perp \hat{K}_j)^T P_{i2} (D_i^\perp \hat{K}_j) \\ &\leq \mu_1 e^T(t_k^-) P_{j1} e(t_k^-) \\ &= \mu_1 V_{1j}(t_k^-). \end{aligned}$$

According to the condition (34), using the similar method in Theorem 3.1, we have $e_1(t) \rightarrow 0$ when $t \rightarrow +\infty$. The proof is completed. \square

Case d: all error subsystems are stable.

Theorem 3.4. Consider the switched system (1) satisfying Assumptions 1 and 2 and the time sequence $t_k \in \Delta(\delta_1, \delta_2)$. For given $\eta_1 > 0, \mu_1 > 0, \alpha_i > 0$ and $\rho_i > 1$, if there exist $P_{i1} > 0, P_{i2} > 0, X_i > 0, Y_i, \epsilon_{imn} > 0, i, j, l \in M, i \neq j, m, n = 1, 2$, such that Eq. (33) and the following inequalities hold:

$$\begin{bmatrix} \Xi_{imn} & P_{im} - X_i - \alpha_i \bar{C}_i^T Y_i^T & P_{im} \\ * & -\alpha_i (X_i + X_i^T) & 0 \\ * & 0 & -\epsilon_{imn} I \end{bmatrix} < 0, \tag{38}$$

$$\ln u_1 - \eta_1 \delta_1 < 0, \tag{39}$$

where $\Xi_{imn} = \theta_i P_{im} + P_{im} A_{i,1} + A_{i,1}^T P_{im} - Y_i \bar{C}_i - \bar{C}_i^T Y_i^T + \frac{1}{\delta_n}(P_{i1} - P_{i2}) + \eta_1 P_{im} + \epsilon_{imn} F_i^T \hat{K}_i^T F_i \hat{K}_i$ with $\theta_i = \frac{\ln \rho_i}{\delta_1}$, then the switched observer error dynamic system (30) is asymptotically stable. The observer gains are given by

$$H_i = X_i^{-1} Y_i, i \in M.$$

Proof. The proof can be easily derived by the methodology given earlier. So we omit it here. \square

Remark 3.4. From Eq. (25), we have $\hat{x}(t) = K_{\sigma(t)}^{-1} \hat{\hat{x}} = K_{\sigma(t)}^{-1} \begin{bmatrix} \hat{x}_1 \\ \hat{y}_2 \end{bmatrix}$. Define $e = x - \hat{x}$, then

$e(t) = K_{\sigma(t)}^{-1} \begin{bmatrix} e_1 \\ 0 \end{bmatrix}$. According to Theorems 3.3 and 3.4, $e_1(t)$ asymptotically converges to zero,

it follows that $e(t)$ asymptotically converges to zero, which means the system states in Eq. (1) can be estimated effectively by the proposed reduced-order observers.

Remark 3.5. In [26–28], the observer design is based on the assumption that the matrix triplets (A_i, C_i, D_i) have negative real part. However, in the reduced order design of our paper, this assumption is removed, which means that the observer can also be designed when none of the matrix pairs $(A_{i,1}, \bar{C}_i)$ is required to be detectable, i.e., all error subsystems are unstable. This can be verified in following example.

Remark 3.6. When not all error subsystems are stable, the key point for designing full-order and reduced-order observers is how to ensure the stability of the whole error dynamic systems. In this case, according to the above four theorems, for $\sigma(t) = i, t \in [t_k, t_{k+1})$, $\sigma(t) = j, t \in [t_{k-1}, t_k)$, we have $\dot{V}_i(t) < \eta_i V_i(t)$ and $V_i(t_k) \leq \mu V_j(t_k^-)$, where η_i can be positive or negative. Choose $\eta_{\sigma(t_k)} \leq \eta, k = 0, 1, 2, \dots$, for $t \in [t_k, t_{k+1})$, then

$$\begin{aligned} V_{\sigma(t)}(t) &\leq \mu^{N_\delta(t, t_0)} V_{\sigma(t_0)}(t_0) \exp[\eta_{\sigma(t_1)}(t_1 - t_0) + \eta_{\sigma(t_2)}(t - t_1) + \dots + \eta_{\sigma(t_{k+1})}(t - t_k)] \\ &\leq V_{\sigma(t_0)}(t_0) \exp[N_\delta(t, t_0) \ln \mu + \eta(t - t_0)], \end{aligned}$$

where $N_\delta(T, t)$ denotes the number of impulsive times of the impulsive sequence δ on the interval (t, T) . Moreover, the average impulsive interval of the impulsive sequence $\delta = \{t_1, t_2, \dots\}$ is equal to T_a if there exist positive integer N_0 and positive number T_a , such that

$$\frac{T - t}{T_a} - N_0 \leq N_\delta(T, t) \leq \frac{T - t}{T_a} + N_0, \quad \forall T \geq t \geq 0.$$

Based on this definition, if $\mu \geq 1$, then

$$\left(\frac{t - t_0}{T_a} - N_0\right) \ln \mu \leq N_\delta(t, t_0) \ln \mu \leq \left(\frac{t - t_0}{T_a} + N_0\right) \ln \mu, \quad \forall t \geq t_0,$$

which implies

$$\begin{aligned} V_{\sigma(t)}(t) &\leq V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{t - t_0}{T_a} + N_0\right) \ln \mu + \eta(t - t_0)\right] \\ &= \mu^{N_0} V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{\ln \mu}{T_a} + \eta\right)(t - t_0)\right]. \end{aligned}$$

If $0 < \mu < 1$, then we have

$$\begin{aligned} V_{\sigma(t)}(t) &\leq V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{t - t_0}{T_a} - N_0\right) \ln \mu + \eta(t - t_0)\right] \\ &= \mu^{-N_0} V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{\ln \mu}{T_a} + \eta\right)(t - t_0)\right]. \end{aligned}$$

Let $\bar{\mu} = \max\{\mu^{-N_0}, \mu^{N_0}\}$, it is obtained that

$$V_{\sigma(t)}(t) \leq \bar{\mu} V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{\ln \mu}{T_a} + \eta\right)(t - t_0)\right].$$

Thus, from $\min\{\lambda(P_{ii})\} \|e(t)\|^2 \leq \rho_i V_i(t) \leq \rho_i \max\{\lambda(P_{ii})\} \|e(t)\|^2$, it follows that

$$\|e(t)\|^2 \leq \frac{\max\{\rho_i\}\bar{\mu}}{\min\{\lambda(P_{il})\}} V_{\sigma(t_0)}(t_0) \exp\left[\left(\frac{\ln \mu}{T_a} + \eta\right)(t - t_0)\right],$$

where $i \in M$ and $l = 1, 2$, $\lambda(P_{il})$ is the eigenvalue of matrix P_{il} . If $\frac{\ln \mu}{T_a} + \eta < 0$, the error dynamic system is asymptotically stable. Therefore, when some error subsystems are stable and some are unstable, the full-order and reduced-order observer can also be constructed by designing the suitable impulsive switching rule that satisfies $\frac{\ln \mu}{T_a} + \eta < 0$.

4. Examples

In this section, two examples are studied to show the effective of the proposed hybrid observers for switched nonlinear systems with unknown inputs.

Example 1. Consider the following two subsystems:

$$A_1 = \begin{bmatrix} -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & -2 & -1 \\ 1 & 1 & -1 & -1 & -0.1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 3 & 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & -1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{bmatrix}, B_1 = B_2 = 0,$$

$f_1(t, x) = (\tanh(x_1), \tanh(x_2), \tanh(x_3), \tanh(x_4), \tanh(x_5))^T$, $f_2(t, x) = (\sin(x_1), \sin(x_2), \sin(x_3), \sin(x_4), \sin(x_5))^T$. It can be verified that $rank(C_i D_i) = rank(D_i) = 2$, $i = 1, 2$, which satisfies Assumption 2. Let $\delta_1 = 0.02$, $\delta_2 = 0.04$. The switching and impulsive signal are shown in Fig. 1.

Case a: When all error subsystems are not stable, an impulsive full-order observer is designed.

According to algorithm for Theorem 3.1, choose $\rho_1 = \rho_2 = 1.01$, $\eta = 1$, $\mu = 0.39$, we have

$$L_1 = \begin{bmatrix} -0.6662 & 0.3329 & -0.0005 \\ 0.2955 & -0.2955 & 0.2045 \\ -0.3329 & 0.6662 & -0.0005 \\ 0.3333 & 0.3333 & -0.0000 \\ 0.2946 & -0.2946 & -0.7946 \end{bmatrix}, L_2 = \begin{bmatrix} 0.5680 & 0.1361 & -0.5680 \\ 0.4311 & 0.8621 & -0.4311 \\ -0.4320 & 0.1361 & 0.4320 \\ 1.0000 & -0.0000 & -0.0000 \\ 0.2950 & 0.59000 & -0.2950 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.5000 & -0.5000 & 0.1265 & 0.6265 & 0.5000 \\ -0.5000 & 0.5000 & -1.3332 & -0.8332 & 0.5000 \\ -0.5000 & 0.5000 & -0.2159 & -0.7159 & -0.5000 \\ 0.5000 & -0.5000 & -0.6898 & -0.1898 & 0.5000 \\ -0.5000 & 0.5000 & -1.1064 & -0.6064 & 0.5000 \end{bmatrix},$$

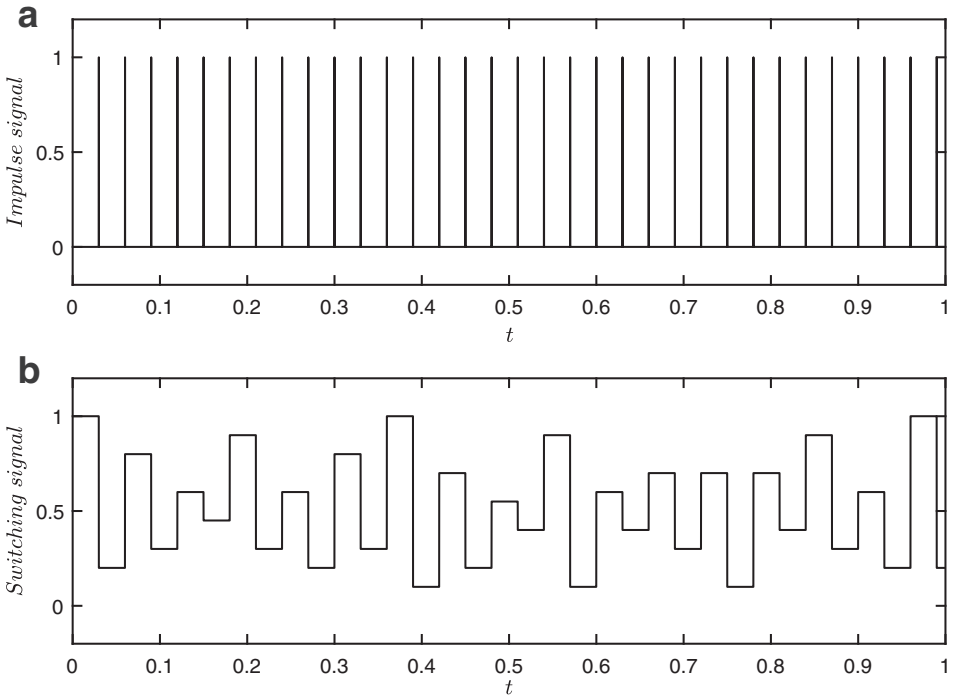


Fig. 1. (a) Impulse signal. (b) Switching signal.

$$N_2 = \begin{bmatrix} 1 & 0 & 0 & 0.9114 & 0 \\ -1 & 0 & 0 & -0.4091 & 0 \\ 1 & 0 & 0 & -0.0867 & 0 \\ 0 & 0 & 0 & -0.7861 & 0 \\ -1 & -1 & 0 & -1.2242 & 1 \end{bmatrix}.$$

It should be noted that each subsystem of switched error system is unstable, since $eig(N_1A_1) = \{1.2702 + 0.0000i, -0.4588 \pm 1.6358i, -0.0000 \pm 0.0000i\}$, $eig(N_2A_2) = \{0, -3.0000, 1.7842, -1.0000, -0.0000\}$, which means that the methods in [21,24,25,27] cannot be used to design the observer for the studied switched system.

Case b: When all error subsystems are stable. Choose $\eta = 4$, $\mu = 1.01$, solving inequalities in Theorem 3.2, the impulsive gain matrices are given by

$$L_1 = \begin{bmatrix} -0.3906 & 0.1321 & 0.0379 \\ 0.2255 & -0.2386 & 0.1261 \\ -0.2107 & 0.4688 & -0.0367 \\ 0.2101 & 0.1313 & 0.03755 \\ 0.2252 & -0.2382 & -0.4735 \end{bmatrix}, L_2 = \begin{bmatrix} 0.3140 & 0.0291 & -0.4127 \\ 0.2877 & 0.5721 & -0.1878 \\ -0.2869 & 0.0282 & 0.1876 \\ 0.6012 & 0.0009 & -0.0002 \\ 0.0600 & 0.3876 & -0.0946 \end{bmatrix}.$$

Next, the reduced-order switched observer is designed for above presented switched system. Following the time varying state transformation [16], we have

$$A_{1,1} = \begin{bmatrix} -2.3199 & -0.6217 & 2.1880 \\ 1.3607 & 0.5309 & -1.1972 \\ 1.2311 & 2.3774 & 0.2890 \end{bmatrix}, A_{2,1} = \begin{bmatrix} -2 & -2 & -1.4142 \\ -0.0000 & -0.0000 & -0.0000 \\ -0.7071 & -2.8284 & -2 \end{bmatrix},$$

$$\bar{C}_1 = [1 \quad 1 \quad 0], \bar{C}_2 = [0 \quad -1 \quad 0].$$

Case c: When all the error systems are unstable. Let $\alpha_1 = 5.15$, $\alpha_2 = 5.05$, $\eta_1 = 5$, $\mu_1 = 0.2$, $\rho_1 = \rho_2 = 1.39$. Solving LMIs in Theorem 3.3, we obtain observer gains for each subsystem:

$$H_1 = [0.3730 \quad 0.0921 \quad 0.4289]^T, H_2 = [-0.0003 \quad 0.0000 \quad 0.0008]^T.$$

It is verified that each subsystem of switched error system is unstable, since

$$\text{eig}(A_{11} - H_1 \bar{C}_1) = \{-2.1112 + 0.0000i, 0.0731 \pm 0.5565i\}$$

and

$$\text{eig}(A_{21} - H_2 \bar{C}_2) = \{-3.0000, -1.0000, 0.0001\}.$$

From [32], we can find the strong detectability of above two subsystems is not satisfied, which means the methods in [21,25–29] cannot be used to design observers for the studied switched system. Because in our paper, each error subsystem does not need to satisfy the strong detectability condition in the case of observer design.

Case d: When all the error systems are stable. Set $\alpha_1 = \alpha_2 = 0.05$, $\eta_1 = 10.5$, $\mu_1 = 1.001$, solving LMIs in Theorem 3.4, the switched observer gains are given by

$$H_1 = [14.9009 \quad 5.0992 \quad 15.1352]^T, H_2 = [20.0638 \quad -20 \quad 0.5292]^T.$$

Let $e(t_0) = (0.3, 0.2, -0.5, 0.2, 0.1)^T$. From Fig. 2, both subsystems of the switched error system are unstable in the case of full-order impulsive observer design. For the reduced-order switched observer design, two subsystem state responses perform similarly as it is shown in Fig. 2 with state divergence. Under the signal rule in Fig. 2, the state trajectories of the whole dynamic of the observation error are presented in Figs. 3 and 4. It can be clearly observed that the asymptotical stability of the switched error systems (6) and (30) have been achieved. Thus, the simulation results well show the effectiveness of the proposed full-order and reduced-order observers.

Example 2. Consider a practical electronic circuit in Fig. 5, where C_i ($i = 1, 2$) are capacitors, V_i are the corresponding voltages that can be regarded as the measurement outputs. R_i ($i = 1, 2, 3$) are resistors, L is an inductor, $w(t)$ is unknown value of the voltage resource. The Chua's diode is a nonlinear resistor [29] that satisfies the following condition

$$I = f(u) = \begin{cases} -1.143U - 0.429, & U \leq 1 \\ -0.714U, & -1 < U \leq 1 \\ -1.143U + 0.429, & U > 1, \end{cases}$$

where U , I stand for the voltage and the current of the Chua's diode, respectively. Then the circuit system is written by

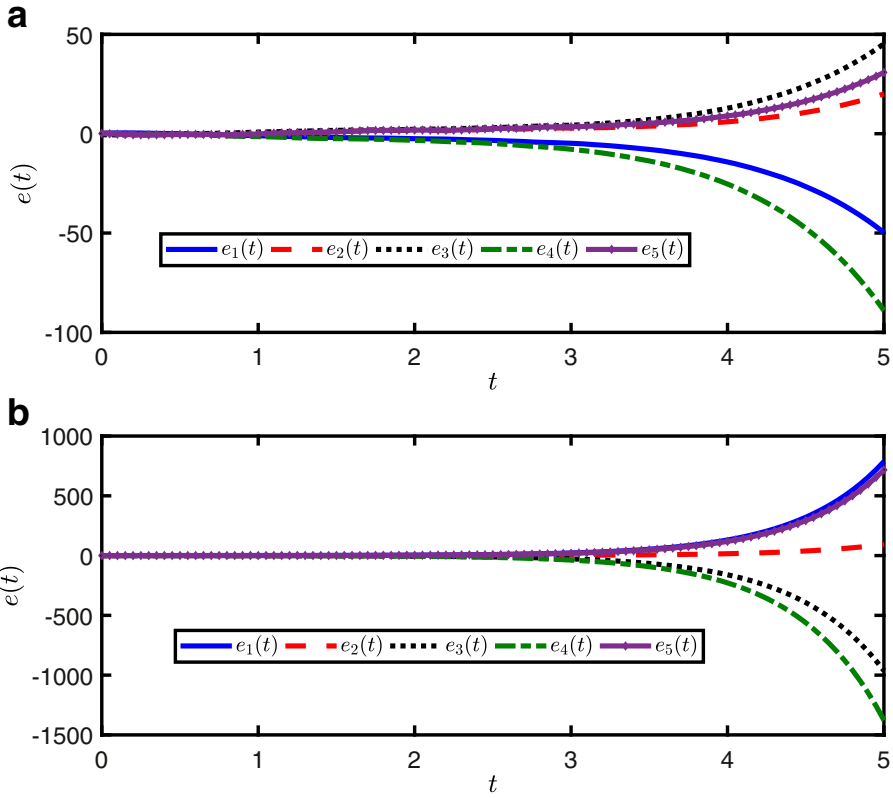


Fig. 2. Trajectories of the error state e of subsystems. (a) Error subsystem 1. (b) Error subsystem 2.

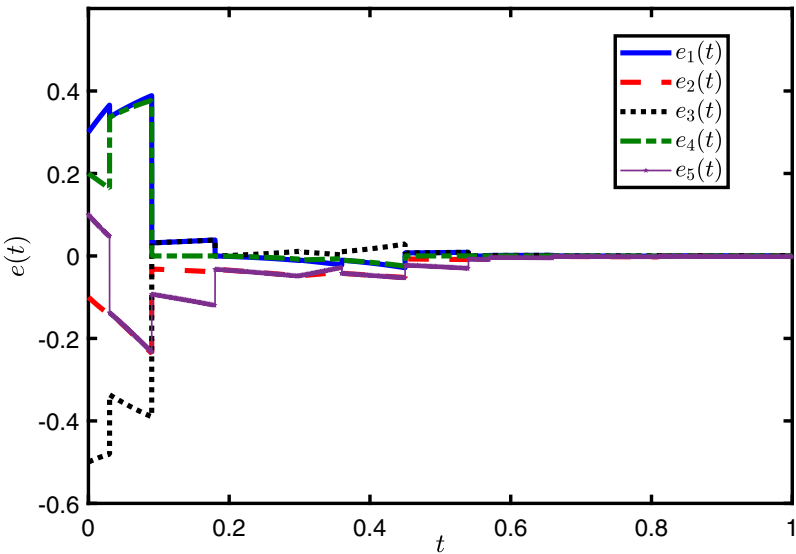


Fig. 3. Trajectories of the switched error system with full-order impulsive observer.

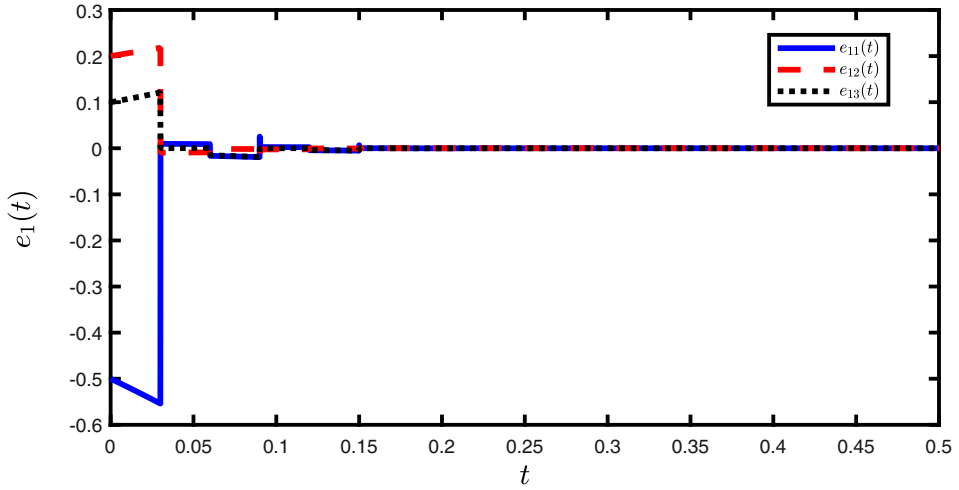


Fig. 4. Trajectories of the switched error system with reduced-order switched observer.

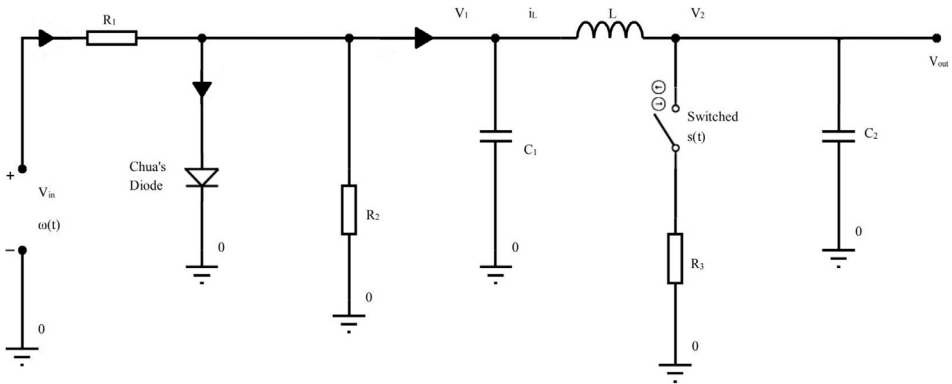


Fig. 5. The electronic circuit.

$$\begin{cases} \dot{i}_L(t) = \frac{1}{L}V_1(t) - \frac{1}{L}V_2(t), \\ \dot{V}_1(t) = -\frac{1}{C_1}i_L(t) - \frac{1}{C_1}f(V_1) - \frac{1}{C_1R_2}V_1(t) + \frac{1}{C_1R_1}w(t), \\ \dot{V}_2(t) = \frac{1}{C_2}i_L(t) - \frac{s(t)}{C_2R_3}V_2(t), \end{cases}$$

where $s(t) = \begin{cases} 1, & \text{switch on} \\ 0, & \text{switch off} \end{cases}$ is used to describe the switching between the two modes of the electronic circuit. Let $x = [x_1 \ x_2 \ x_3]^T = [i_L(t) \ V_1(t) \ V_2(t)]^T$, then we have the system

$$\begin{cases} \dot{x}(t) = A_i x(t) + D_i w(t) + f_i(t, x(t)), \\ y(t) = C_i x(t), \quad i = 1, 2, \end{cases}$$

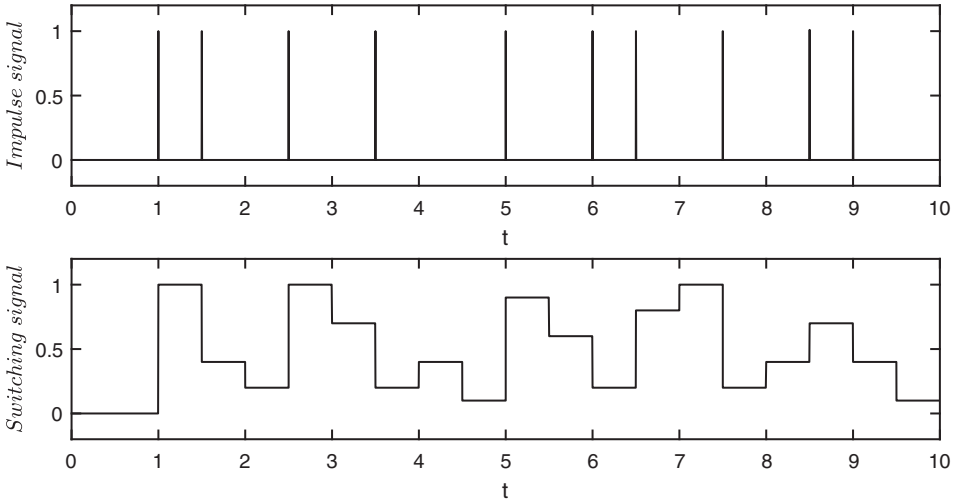


Fig. 6. (a) Impulse signal. (b) Switching signal.

where

$$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{C_1} & -\frac{1}{C_1 R_2} & 0 \\ \frac{1}{C_2} & 0 & -\frac{1}{C_2 R_3} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{C_1} & -\frac{1}{C_1 R_2} & 0 \\ \frac{1}{C_2} & 0 & 0 \end{bmatrix}, B_1 = B_2 = 0,$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 0 \\ \frac{1}{C_1 R_1} \\ 0 \end{bmatrix}, f_1(t, x) = f_2(t, x) = \begin{bmatrix} 0 \\ -\frac{1}{C_1} f(x_2) \\ 0 \end{bmatrix}.$$

Choose $L = 1, C_1 = 2, C_2 = 1, R_1 = 1, R_2 = 0.5, R_3 = 1,$ and $w(t) = 2 \sin(5t).$ Assumption 2 is satisfied, and then denote $\delta_1 = 0.5, \delta_2 = 1.5.$ Based on Algorithm for the full-order observer design, $\eta = 1.3, \mu = 0.001, \rho_1 = \rho_2 = 0.01,$ then the condition $\ln \mu + \eta \delta_2 = -4.9578 < 0,$ we have

$$L_1 = \begin{bmatrix} -0.0184 & 0.0184 \\ 3.3010 & 0.0000 \\ -0.000 & 3.3010 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0007 & 35.1172 \\ 3.3010 & -0.0000 \\ 0.0000 & 3.3010 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 & 0.8454 \\ 0 & -0.8887 \\ 0 & 1.1980 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & -1.7976 \\ 0 & 1.2928e - 04 \\ 0 & 0.9999 \end{bmatrix}.$$

For the reduced order observer design, according to the time varying state transformation,

$$A_{1,1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, A_{2,1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \bar{C}_1 = \bar{C}_2 = [0 \quad 1].$$

Choose $\alpha_1 = 5, \alpha_2 = 16, \eta_1 = 4, \mu_1 = 0.002, \rho_1 = \rho_2 = 1,$ the equality $\ln \mu_1 + \eta_1 \delta_2 = -0.2146 < 0$ is satisfied, and then solving LMIs in Theorem 3.3, the observer gain matrices are given by

$$H_1 = \begin{bmatrix} -0.0163 \\ 0.2000 \end{bmatrix}, H_2 = \begin{bmatrix} -0.0000 \\ -0.0019 \end{bmatrix}.$$

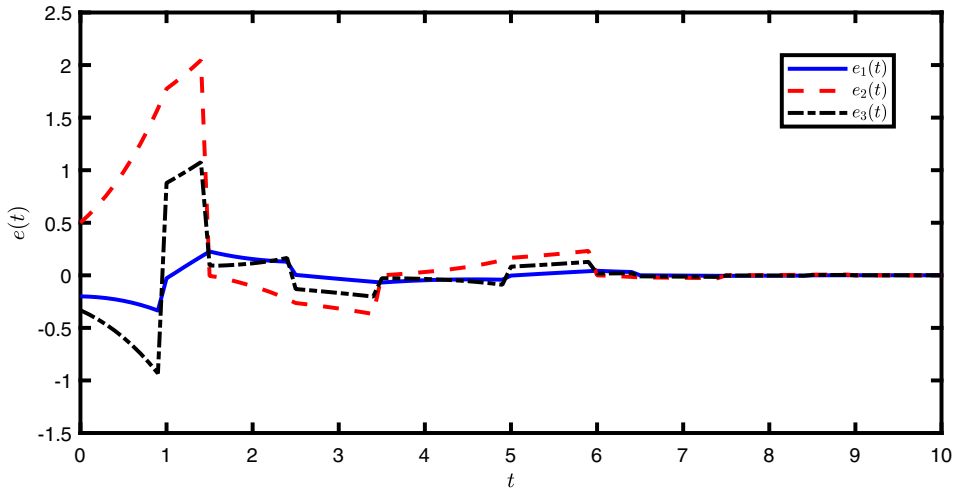


Fig. 7. Trajectories of the switched error system with full-order impulsive observer.

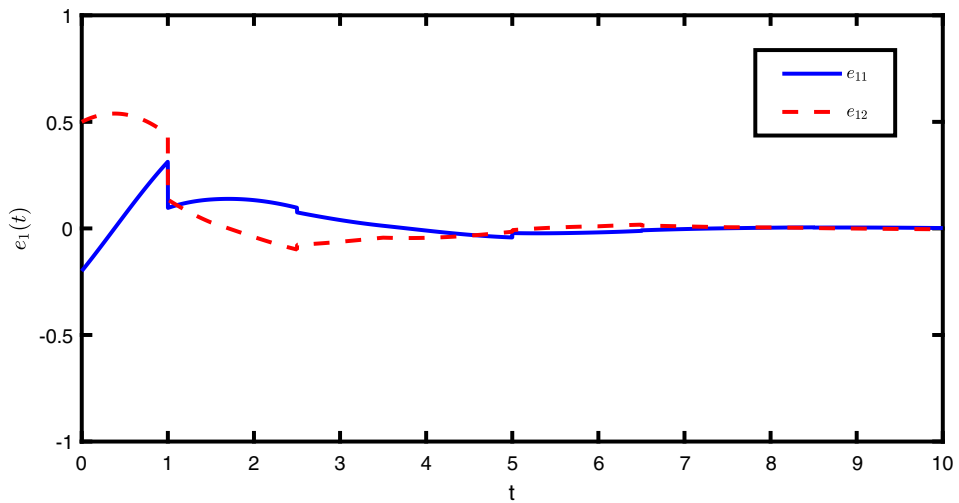


Fig. 8. Trajectories of the switched error system with reduced-order switched observer.

In this example, the corresponding switched impulsive signal is presented in Fig. 6. Under the initial condition $e(t_0) = (-0.2, 0.5, -0.4)^T$, Figs. 7 and 8 show that the switched error dynamic systems (6) and (30) are asymptotically stable under the designed full-order and reduced-order observer respectively, which implies that both the full-order and reduced-order observers have good estimation performances.

5. Conclusion

This paper has investigated a full-order impulsive and reduced-order switched observer via dwell-time switchings without requiring the stability of each error subsystem of the switched

systems. In spite of the instability of individual subsystems, the stability of the error switched system is achieved by confining the dwell time within an upper bounds. In addition, the multiple time-varying Lyapunov functions method is also used to analyze the stability analysis of a class of switched error systems with stable subsystems. Based on the estimated state information produced by the designed observer in this paper, a future research will focus on some practical control problems [33,34]. Finally, the simulation results have shown the effectiveness of the proposed criteria.

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