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Unknown input fractional-order functional observer design for one-side Lipschitz time-delay fractional-order systems

Mai Viet Thuan[1], Dinh Cong Huong[2,3], Nguyen Huu Sau[4] and Quan Thai Ha[5]

Abstract

This paper addresses the problem of unknown input fractional-order functional state observer design for a class of fractional-order time-delay nonlinear systems. The nonlinearities consist of two parts where one part is assumed to satisfy both the one-sided Lipschitz condition and the quadratically inner-bounded condition and the other is not necessary to be Lipschitz and can be regarded as an unknown input, making the wider class of considered nonlinear systems. By taking the advantages of recent results on Caputo fractional derivative of a quadratic function, we derive new sufficient conditions with the form of linear matrix inequalities (LMIs) to guarantee the asymptotic stability of the systems. Four examples are also provided to show the effectiveness and applicability of the proposed method.

Keywords

Fractional-order systems, fractional-order functional observers, time-delay systems, one-sided Lipschitz condition, linear matrix inequality

Introduction

In recent years, fractional-order systems, which are the generalization of integer-order dynamic systems, provide better mathematical models for some actual physical and engineering systems (Kaczorek, 2011; Kilbas et al., 2006; Petras, 2011; Podlubny, 1999). The fractional-order nonlinear dynamic systems have many dynamic behaviors that are similar to the integer-order systems, such as chaos, bifurcation, and attractor (Deng et al., 2007; Duarte and Macado, 2002; Grigorenko and Grigorenko, 2013; Li and Peng, 2014). Inspired by the advantages of fractional-order systems and their wide applications in interdisciplinary areas, an increasing number of researchers have turned their interests on fractional-order systems and many interesting and important results have been reported on this research issue (Chen et al., 2013; Kaslik and Sivasundaram, 2012; Rakkiyappan et al., 2015; Thuan and Huong, 2018; Wang et al., 2015; Zhang et al., 2015).

In many practical applications, the physical state of a system cannot be determined by direct observation. Therefore, the problem of constructing the state vector and functions of the state vector is of great significance (Teh and Trinh, 2013; Trinh and Fernando, 2012; Wei et al., 2018). It should be noted that, in practice, the information of some individuals of the states may be measured. For these individuals, it is no longer needed to be estimated. It is thus worth studying the problem of design of linear functions of the state vector of a system.

Although the problem of the state observer design for integer-order systems has been intensively investigated in many aspects (Huong, 2018; Huong and Thuan, 2017; Huong and Trinh, 2015, 2016; Huong et al., 2014; Thuan et al., 2012; Trinh et al., 2004, 2006, 2016). The results accounted for fractional-order systems are very few in the literature since this problem is challenging due to the complexity of fractional-order calculus equations and the fact that integer-order algorithms cannot be directly applied to the fractional-order systems. Recently, some interesting results on the observer design for fractional-order systems were reported in Boroujeni and Momeni (2012), Kaczorek (2014, 2017), N'doye et al. (2013, 2017), Huong and Thuan (2018), and Trinh et al. (2019). In particular, a non-fragile fractional-order nonlinear observer design for a class of fractional-order

[1]Department of Mathematics and Informatics, Thainguyen University of Science, Thai Nguyen, Vietnam
[2]Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[3]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[4]Faculty of Fundamental Science, Hanoi University of Industry, Hanoi, Vietnam
[5]Center for High Performance Computing, Hanoi University of Sciences, Hanoi, Vietnam

Corresponding author:
Dinh Cong Huong, Faculty of Mathematics and Statistics, Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
Email: dinhconghuong@tdtu.edu.vn

nonlinear systems was derived in Boroujeni and Momeni (2012); reduced-order fractional descriptor observers for fractional-order descriptor continuous-time linear system were reported in Kaczorek (2014); reduced-order perfect nonlinear observers for fractional descriptor nonlinear discrete-time systems were proposed in Kaczorek (2017); the problem of designing fractional-order observers for continuous-time linear fractional-order systems with unknown inputs was considered in N'doye (2013). In N'doye (2017), the authors proposed an adaptive observer design method for nonlinear fractional-order systems where the nonlinearities of the system satisfying the Lipschitz condition and the unknown parameters are bounded. The problem of designing positive reduced-order distributed functional observers for positive fractional-order interconnected time-delay systems was considered in Trinh et al. (2019) and the problem of designing reduced-order state observers for fractional-order time-delay systems with Lipschitz non-linearities and unknown inputs was investigated in Huong and Thuan (2018). Note that the state observer design methods in Boroujeni and Momeni (2012), Kaczorek (2014), Kaczorek (2017), and N'doye (2013, 2017) only dealt with fractional-order systems without time delays. While the state observer design methods (Huong and Thuan, 2018; Trinh et al., 2019) dealt with a class of linear time-delay fractional-order systems and a class of nonlinear time-delay fractional-order systems where the nonlinearity is assumed to be Lipschitz.

It is worth noticing that the Lipschitz nonlinearities are usually valid only for some classes of nonlinear systems with small Lipschitz constant. Therefore, nonlinear observer designs based on the traditional Lipschitz conditions are unable to deal with large-Lipschitz constant systems. To overcome this drawback, the one-sided Lipschitz condition was introduced to replace the well-known Lipschitz condition (Abbaszadeh and Marquez, 2010). A new concept for nonlinear systems, the quadratic inner-boundedness, was also introduced in the work of Abbaszadeh and Marquez (2010) as a solution to derive tractable conditions for observer designs for one-sided Lipschitz nonlinear systems. The class of nonlinear systems satisfying the one-sided Lipschitz and quadratically inner-bounded conditions is more general than the class of traditional Lipschitz nonlinear systems. It was proved in the work of Abbaszadeh and Marquez (2010) that if a function is Lipschitz, it is both one-sided Lipschitz and quadratically inner-bounded; nevertheless, the converse is not true. The problem of designing observers for nonlinear systems satisfying both the one-sided Lipschitz condition and the quadratically inner-bounded condition has recently attracted much interest (Ahmad et al., 2016; Benallouch et al., 2012; Dong et al., 2017; Nguyen and Trinh, 2016; Nguyen et al., 2018; Wu and Dong, 2017; Zhang et al., 2012, 2014a,b, 2015). However, the results reported in the works by Ahmad et al. (2016), Benallouch et al. (2012), Dong et al. (2017), Nguyen and Trinh (2016), Nguyen et al. (2018), and Zhang et al. (2012, 2014a,b, 2015) only deal with the integer-order nonlinear systems. So far, they have not been extended to the fractional-order ones. Very recently, in Zhan and Ma (2017), a method of designing the unknown input observers for fractional-order one-sided Lipchitz nonlinear systems was considered. By introducing a continuous frequency distributed equivalent model and using the matrix generalized inverse approach, the authors of this work obtained sufficient conditions for asymptotic stability of the observer error dynamic systems, which guarantee the existence of the full-order unknown input observers. Note that the state observer design method in the work of Zhan and Ma (2017) only dealt with fractional-order systems without time delays. While, time delays naturally exist in many dynamical systems and majorly contribute to instability, oscillations, and degradation in system performance. Therefore, it is necessary and important to address the problem of unknown input fractional-order functional observer design for fractional-order time-delay nonlinear systems, where the nonlinearity is assumed to satisfy both the one-sided Lipschitz condition and the quadratically inner-bounded condition.

Motivated by the aforementioned discussion, in this paper, we study the problem of unknown input fractional-order functional observer design for a class of α-fractional-order, $\alpha \in (0, 1)$, nonlinear systems with unknown inputs. The considered systems are more general since the nonlinear term can be divided into two parts where one part is assumed to be under one-sided Lipschitz conditions and the other is not necessary to be one-sided Lipschitz and can be regarded as unknown inputs. Unknown input fractional-order functional observers are proposed and their necessary existence conditions are then derived. Here, to analyze the stability of the system, we first build a simple Lyapunov-like functional and then use recent results on Caputo fractional derivative of a quadratic function found in Duarte-Mermoud et al. (2015) and fractional Razumikhin theorem (Wen et al., 2015) to derive new sufficient conditions guaranteeing the asymptotic stability of the error dynamics. Furthermore, the conditions are with the form of LMIs, which therefore can be effectively solved by using existing convex algorithms. This method is believed to be more desirable when dealing with the systems in the presence of nonlinearities and unknown inputs. Four examples are also provided to show the effectiveness and applicability of the proposed scheme.

The remainder of this paper is organized as follows. The problem statement and preliminaries are given in the next section. Section 3 presents the main results where existence sufficient conditions of the fractional-order functional observer are derived. Four examples with simulation results are given in Section 4 to illustrate the proposed scheme. Finally, Section 5 concludes the paper.

Notation: The following notations will be used in this paper: For a real matrix M, M^T denotes the transpose. $\mathbb{R}^n$ denotes the $n-$ dimensional linear vector space over the reals with the Euclidean norm (two–norm) $\| . \|$ given by $\| x \| = \sqrt{x_1^2 + \ldots + x_n^2}, x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$. $\mathbb{R}^{n \times m}$ denotes the space of $n \times m$ matrices. For a real symmetric matrix $A, \lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. A matrix P is positive definite ($P > 0$) if $x^T P x > 0, \forall x \neq 0$.

Problem statement and preliminaries

We shall begin with recalling the fundamental definition of the fractional calculus that can be found in Kilbas et al.

(2006). The fractional integral with non-integer order $\alpha > 0$ of a function $x(t)$ is defined as follows

$$_{t_0}I_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds,$$

where $x(t)$ is an arbitrary integrable function, $_{t_0}I_t^{\alpha}$ denotes the fractional integral of order α on $[t_0, t]$ and $\Gamma(.)$ represents the gamma function. The Caputo fractional-order derivative of order $\alpha > 0$ for a function $x(t) \in C^{n+1}([t_0, +\infty), \mathbb{R})$ is defined as follows

$$_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, \quad t \geqslant t_0 \geqslant 0,$$

where n is a positive integer such that $n-1 < \alpha < n$. In particular, when $0 < \alpha < 1$, we have

$$_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{\dot{x}(s)}{(t-s)^{\alpha}}ds, \quad t \geqslant t_0 \geqslant 0.$$

Especially, as in Kilbas et al. (2006), we have $_{t_0}^{C}D_t^{0}x(t) = x(t)$ and $_{t_0}^{C}D_t^{1}x(t) = \dot{x}(t)$.

To obtain the main results, we will use the following lemmas:

Lemma 1: Duarte-Mermoud et al. (2015). Let $x(t) \in \mathbb{R}^n$ be a continuous and derivable function. Then, for any time instant $t \geqslant t_0$, the following relationship holds

$$_{t_0}^{C}D_t^{\alpha}(x^T(t)Px(t)) \leqslant 2x^T(t)P\,_{t_0}^{C}D_t^{\alpha}x(t), \quad \forall\alpha \in (0,1), \forall t \geqslant t_0 \geqslant 0,$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 2. Razumikhin-type stability, Wen et al. (2015). Assume that $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s), v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, and $q > 1$. If there exists a continuous function $V(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that:

(i) $u(\parallel x \parallel) \leqslant V(t, x) \leqslant v(\parallel x \parallel), \quad t \in \mathbb{R}^+, x \in \mathbb{R}^n$

(ii) $_{t_0}^{C}D_t^{\alpha}V(t, x(t)) \leqslant -w(\parallel x(t) \parallel)$ if $V(t+s, x(t+s)) < qV(t, x(t)), \quad \forall s \in [-\tau, 0], t \geqslant 0$, then the zero solution of fractional-order system $_{t_0}^{C}D_t^{\alpha}x(t) = f(t, x_t)$ is asymptotically stable.

From now on, for simplicity, we use a notation D^{α} instead of $_{0}^{C}D_t^{\alpha}$.

We consider a Caputo fractional-order nonlinear system with unknown inputs being described as follows

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + A_dx(t-\tau) + Bu(t) + g(x,u,t) + Dd(t), \quad t \geqslant 0, \\ x(\theta) = \phi(\theta), \ \forall\theta \in [-\tau, 0], \\ y(t) = Cx(t), \\ z(t) = Lx(t), \end{cases} \tag{1}$$

where $0 < \alpha < 1$ is the fractional commensurate order of the system, $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $d(t) \in \mathbb{R}^{\ell}$ is the disturbance input vector, $g(x, u, t) \in \mathbb{R}^n$ denotes the nonlinear function, $z(t) \in \mathbb{R}^q$ is the functional state vector which is required to be estimated, $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times \ell}$ and $L \in \mathbb{R}^{q\times n}$ are known real matrices, C and L are assumed to be full-rank matrices, i.e., rank $(C) = p$, rank $(L) = q$ and $p + q \leqslant n$. The time delay τ is assumed to be known positive constant and $\phi(\theta) \in \mathbb{R}^n$ is the initial function. The nonlinear function $g(x, u, t)$ can be decomposed as follows

$$g(x,u,t) = g_1(\varphi,u,t) + Wg_2(x,u,t), \tag{2}$$

where $\varphi(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^{p+q}$, $W \in \mathbb{R}^{n\times d}, 0 \leqslant d \leqslant n$, is a full-column rank matrix, that is, rank $(W) = d$, $g_2(x, u, t) \in \mathbb{R}^d$ is the decomposed nonlinear function that can be regarded as an unknown disturbance of the system (Trinh et al., 2006) and $g_1(\varphi, u, t) \in \mathbb{R}^n$ is a nonlinear function that satisfies the following assumptions

Assumption 1: $g_1(\varphi, u, t)$ is one-sided Lipschitz with respect to x in region $\mathcal{D}$ (Abbaszadeh and Marquez, 2010), that is

$$\langle g_1(\varphi,u,t) - g_1(\hat{\varphi},u,t), (\varphi - \hat{\varphi})\rangle \leqslant \rho||\varphi - \hat{\varphi}||^2,$$
$$\forall\varphi, \hat{\varphi} \in \mathcal{D}, \ u \in \mathbb{R}^m,$$

where ρ is one-sided Lipschitz constant.

Assumption 2: $g_1(\varphi, u, t)$ is quadratically inner-bounded in region $\mathcal{D}$ (Abbaszadeh and Marquez, 2010), that is

$$||g_1(\varphi,u,t) - g_1(\hat{\varphi},u,t)||^2 \leqslant \beta||\varphi - \hat{\varphi}||^2$$
$$+ \gamma\langle g_1(\varphi,u,t) - g_1(\hat{\varphi},u,t), (\varphi - \hat{\varphi})\rangle, \ \forall\varphi, \hat{\varphi} \in \mathcal{D}, \ u \in \mathbb{R}^m,$$

where β and γ are known constants.

Remark 1: It was reported in Abbaszadeh and Marquez (2010) that the constants ρ, β, γ in Assumption 1 and Assumption 2 can be positive, negative or zero, while the well-known Lipschitz constant must be positive. The one-sided Lipschitz constants are usually much smaller than the classical Lipschitz constants. Moreover, if the vector function $g_1(\varphi, u, t)$ satisfies the Lipschitz condition, then it is also both the one-sided Lipschitz and quadratically inner-bounded. However, the converse is not true.

The following assumption will be used in this paper:

Assumption 3: $x(t) \in \mathcal{D}, \ \forall t \geqslant 0$ for $\phi(\theta) \in \mathbb{R}^n, \ \forall\theta \in [-\tau, 0]$, where $\mathcal{D}$ is the region in Assumption 1 and Assumption 2.

The aim of this work is to design an asymptotic estimator of the required state function $z(t)$ of (1). For this, we propose the following fractional-order functional observer

$$\begin{cases} D^{\alpha}\omega(t) = N\omega(t) + N_d\omega(t-\tau) + Hu(t) + Gy(t) + G_dy(t-\tau) + Mg_1(\hat{\varphi},u,t), \\ \hat{z}(t) = \omega(t) + Ky(t), \end{cases} \tag{3}$$

where $\omega(t) \in \mathbb{R}^q$ is the observer state vector, $g_1(\hat{\varphi}, u, t) = g_1(\begin{bmatrix} y \\ \hat{z} \end{bmatrix}, u, t) \in \mathbb{R}^n$, matrices N, N_d, H, G, G_d, M and K are the observer parameters with appropriate dimensions.

Let us denote the following error vectors $e(t) \in \mathbb{R}^q$ and $\epsilon(t) \in \mathbb{R}^q$ as

$$\epsilon(t) = \omega(t) - Mx(t),\ M \in \mathbb{R}^{q \times n}, \tag{4}$$

$$e(t) = \hat{z}(t) - z(t). \tag{5}$$

Definition 1: *The fractional-order functional observer described in the form of (3) is said to be an asymptotic estimator if $\hat{z}(t)$ is an asymptotic estimate of the functional state vector $z(t)$, that is, $e(t) = \hat{z}(t) - z(t) \to 0$ as $t \to \infty$.*

Main results

From the fractional-order functional observer (3) and Definition 1, the design of the observer is reduced to find matrices N, N_d, H, G, G_d, M and K such that $e(t) = \hat{z}(t) - z(t) \to 0$ as $t \to \infty$. The following theorem provides a sufficient condition guaranteeing the existence of the observer design.

Theorem 1: An asymptotic estimate $\hat{z}(t)$ of the state function $z(t)$ exists with any given initial conditions of $x(\theta)$, $\omega(\theta)$ and any $u(t)$ if the following conditions hold

Condition 1: The error dynamic

$$D^\alpha \epsilon(t) = N\epsilon(t) + N_d\epsilon(t-\tau) + M(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)) \\ \textit{is asymptotically stable}. \tag{6}$$

Condition 2: The following matrix equations hold

$$NM + GC - MA = 0, \tag{7a}$$

$$N_dM + G_dC - MA_d = 0, \tag{7b}$$

$$MW = 0, \tag{7c}$$

$$MD = 0, \tag{7d}$$

$$H = MB, \tag{7e}$$

$$M = L - KC. \tag{7f}$$

Proof: First, by taking the fractional-order derivative of (4) and using the information of (1) and (3), we obtain the following fractional-order error dynamic system

$$\begin{aligned} D^\alpha \epsilon(t) = &\ D^\alpha \omega(t) - MD^\alpha x(t) \\ = &\ N\epsilon(t) + N_d\epsilon t - \tau) + (NM + GC - MA)x(t) \\ &+ (N_dM + G_dC - MA_d)x(t-\tau) \\ &+ (H - MB)u(t) + M(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)) \\ &- MWg_2(x, u, t) - MDd(t). \end{aligned} \tag{8}$$

By substituting $z(t)$ and $\hat{z}(t)$ from (1) and (3) into (5), we have the estimate error vector $e(t)$ as follows

$$e(t) = \epsilon(t) + (M + KC - L)x(t). \tag{9}$$

According to (9), if matrix equation (7f) of Condition 2 holds, then estimate error $e(t) \to 0$ as $t \to \infty$. Thus, based on Definition 1, $\hat{z}(t)$ is an asymptotic estimator of the state function $z(t)$. This completes the proof of Theorem 1.

Now, the design of fractional-order functional observer $\hat{z}(t)$ reduces to determining the observer parameters such that all the conditions in Theorem 1 are satisfied. Observing that matrix H can be directly calculated from (7e) if M is found. The remaining task is to solve unknown observer matrices to satisfy conditions (7a)–(7d) and (7f).

Let us denote $T = G - NK$, $T_d = G_d - N_dK$ and L^+ is the generalized inverse of L. Since L is assumed to be a full row-rank matrix, that is, rank $(L) = q$, resulting in $LL^+ = I$ and $[L^+ \quad (I - L^+L)]$ is a full row-rank matrix.

By substituting M from (7f) into (7d), (7c), (7b) and (7a), we obtain

$$LW = KCW, \tag{10}$$

$$LD = KCD, \tag{11}$$

$$NL = LA - KCA - TC, \tag{12}$$

$$N_dL = LA_d - KCA_d - T_dC. \tag{13}$$

By post-multiplying both sides of (12) and (13) with $[L^+ \quad (I - L^+L)]$ and after some rearrangement, the following equations are obtained

$$N = (LA - KCA - TC)L^+, \tag{14}$$

$$N_d = (LA_d - KCA_d - T_dC)L^+, \tag{15}$$

$$LA(I - L^+L) = KCA(I - L^+L) + TC(I - L^+L), \tag{16}$$

$$LA_d(I - L^+L) = KCA_d(I - L^+L) + T_dC(I - L^+L). \tag{17}$$

Equations (10), (11) and (16)–(17) can be expressed into the following form

$$\chi\Phi = \Psi, \tag{18}$$

where

$$\chi = [K \quad T \quad T_d], \tag{19}$$

$$\Phi = \begin{bmatrix} CW & CD & CA(I - L^+L) & CA_d(I - L^+L) \\ 0 & 0 & C(I - L^+L) & C(I - L^+L) \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \in \mathbb{R}^{3p \times (d + \ell + 2n)}, \tag{20}$$

$$\Psi = [LW \quad LD \quad LA(I - L^+L) \quad LA_d(I - L^+L)] \in \mathbb{R}^{q \times (d + \ell + 2n)}. \tag{21}$$

Since Φ and Ψ are two known constant matrices, a solution for χ always exists if and only if

$$\operatorname{rank}\begin{bmatrix}\Phi\\ \Psi\end{bmatrix} = \operatorname{rank}(\Phi). \tag{22}$$

Under condition (22), a general solution for χ is given by

$$\chi = \Psi\Phi^{+} + Z\left(I_{3p} - \Phi\Phi^{+}\right), \tag{23}$$

where $\Phi^{+} \in \mathbb{R}^{(d+\ell+2n)\times 3p}$ is the Moor-Penrose-inverse of Φ and $Z \in \mathbb{R}^{q\times 3p}$ is an arbitrary matrix to be determined. Moreover, matrices K, T, T_d can now be extracted from (23) and are expressed as

$$K = K_1 + ZK_2,\ T = T_1 + ZT_2,\ T_d = T_{d1} + ZT_{d2}, \tag{24}$$

where

$$K_1 = \Psi\Phi^{+}e_K,\ K_2 = (I_{3p} - \Phi\Phi^{+})e_K, \tag{25}$$

$$T_1 = \Psi\Phi^{+}e_T,\ T_2 = (I_{3p} - \Phi\Phi^{+})e_T, \tag{26}$$

$$T_{d1} = \Psi\Phi^{+}e_{T_d},\ T_{d2} = (I_{3p} - \Phi\Phi^{+})e_{T_d} \tag{27}$$

and $e_K, e_T, e_{T_d} \in \mathbb{R}^{3p\times p}$ are the following

$$e_K = \begin{bmatrix} I_p\\ 0\\ 0\end{bmatrix}, e_T = \begin{bmatrix} 0\\ I_p\\ 0\end{bmatrix}, e_{T_d} = \begin{bmatrix} 0\\ 0\\ I_p\end{bmatrix}. \tag{28}$$

By substituting (24) into (14), (15) and (7f), the following equations are obtained

$$N = N_1 + ZN_2,\ N_d = N_{d1} + ZN_{d2},\ M = M_1 + ZM_2, \tag{29}$$

where

$$N_1 = (LA - K_1CA - T_1C)L^{+},\ N_2 = -(K_2CA + T_2C)L^{+},$$

$$N_{d1} = (LA_d - K_1CA_d - T_{d1}C)L^{+},\ N_{d2} = -(K_2CA_d + T_{d2}C)L^{+},$$

$$M_1 = L - K_1C,\ M_2 = -K_2C.$$

By substituting N, N_d and M from (29) into (6), we obtain

$$\begin{aligned} D^{\alpha}\epsilon(t) &= (N_1 + ZN_2)\epsilon(t) + (N_{d1} + ZN_{d2})\epsilon(t-\tau)\\ &\quad + (M_1 + ZM_2)(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)). \end{aligned} \tag{30}$$

Therefore, the remaining task is to determine of a matrix Z such that the time-delay system (30) is asymptotically stable. The following theorem provides a sufficient condition guaranting that the system (30) is asymptotically stable.

Theorem 2: The zero solution of system (30) is asymptotically stable if there exist three positive scalars ν, λ, μ, a symmetric positive definite matrix P, a matrix Y with appropriate dimensions such that the following LMI holds

$$\Pi = \begin{bmatrix} \mathcal{M}_{11} & PN_{d1} & YN_{d2} & PM_1 & YM_2\\ * & -P & 0 & 0 & 0\\ * & * & -P & 0 & 0\\ * & * & * & -\lambda I_q & 0\\ * & * & * & * & -\mu I_q \end{bmatrix} < 0, \tag{31}$$

where

$\mathcal{M}_{11} = PN_1 + N_1^TP + YN_2 + N_2^TY^T + 2(1+\nu)P + (\lambda+\mu)(\beta+\gamma\rho)I_q$.
Moreover, the matrix Z is obtained as follows
$Z = P^{-1}Y$.

Proof: We choose a Lyapunov function candidate as follows

$$V(t, \epsilon(t)) = \epsilon^T(t)P\epsilon(t).$$

It is easy to verify that

$$\lambda_{\min}(P)\ \|\ \epsilon(t)\|^2 \leq V(t, \epsilon(t)) \leq \lambda_{\max}(P)\ \|\ \epsilon(t)\|^2. \tag{32}$$

Therefore, condition (i) in Lemma 2 is satisfied. It follows from Lemma 1 that we obtain the α-order Caputo derivative of $V(t, \epsilon(t))$ along the trajectories of the system (30) as follows

$$\begin{aligned} D^{\alpha}V(t, \epsilon(t)) &\leq 2\epsilon^T(t)PD^{\alpha}\epsilon(t)\\ &= \epsilon^T(t)\left[PN_1 + N_1^TP + PZN_2 + N_2^TZ^TP\right]\epsilon(t)\\ &\quad + 2\epsilon^T(t)PN_{d1}\epsilon(t-\tau) + 2\epsilon^T(t)PZN_{d2}\epsilon(t-\tau)\\ &\quad + 2\epsilon^T(t)PM_1(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t))\\ &\quad + 2\epsilon^T(t)PZM_2(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)). \end{aligned} \tag{33}$$

By using the Cauchy matrix inequality, we have

$$2\epsilon^T(t)PN_{d1}\epsilon(t-\tau) \leq \epsilon^T(t)PN_{d1}P^{-1}N_{d1}^TP\epsilon(t) + \epsilon^T(t-\tau)P\epsilon(t-\tau), \tag{34}$$

$$\begin{aligned} 2\epsilon^T(t)PZN_{d2}\epsilon(t-\tau) &\leq \epsilon^T(t)PZN_{d2}P^{-1}N_{d2}^TZ^TP\epsilon(t)\\ &\quad + \epsilon^T(t-\tau)P\epsilon(t-\tau), \end{aligned} \tag{35}$$

$$\begin{aligned} &2\epsilon^T(t)PM_1(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)) \leq \lambda^{-1}\epsilon^T(t)PM_1M_1^TP\epsilon(t)\\ &+ \lambda||g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t)||^2\\ &\leq \lambda^{-1}\epsilon^T(t)PM_1M_1^TP\epsilon(t) + \lambda\beta||\hat{\varphi} - \varphi||^2 + \lambda\gamma < g_1(\hat{\varphi}, u, t)\\ &- g_1(\varphi, u, t), \hat{\varphi} - \varphi >\\ &\leq \lambda^{-1}\epsilon^T(t)PM_1M_1^TP\epsilon(t) + \lambda(\beta+\gamma\rho)||\hat{\varphi} - \varphi||^2\\ &\leq \lambda^{-1}\epsilon^T(t)PM_1M_1^TP\epsilon(t) + \lambda(\beta+\gamma\rho)||\epsilon(t)||^2 \end{aligned} \tag{36}$$

$$\begin{aligned} &2\epsilon^T(t)PZM_2(g_1(\hat{\varphi}, u, t) - g_1(\varphi, u, t))\\ &\leq \mu^{-1}\epsilon^T(t)PZM_2M_2^TZ^TP\epsilon(t) + \mu(\beta+\gamma\rho)||\epsilon(t)||^2. \end{aligned} \tag{37}$$

Since $V(t, \epsilon(t)) = \epsilon^T(t)P\epsilon(t)$, in the light of the Razumikhin theorem (Lemma 2), we assume that for some real number $\nu > 0$ such that

$$\epsilon^T(t+s)P\epsilon(t+s) < (1+\nu)\epsilon^T(t)P\epsilon(t), \quad \forall s \in [-\tau, 0].$$

Therefore

$$\epsilon^T(t-\tau)P\epsilon(t-\tau) < (1+\nu)\epsilon^T(t)P\epsilon(t). \tag{38}$$

Submitting inequalities (34)–(38) into (33), we obtain

$$D^{\alpha}V(t, \epsilon(t)) \leq \epsilon^T(t)\mathcal{M}\epsilon(t), \tag{39}$$

where

$$\mathcal{M} = PN_1 + N_1^T P + PZN_2 + N_2^T Z^T P + PN_{d1}P^{-1}N_{d1}^T P + PZN_{d2}P^{-1}N_{d2}^T Z^T P + 2(1+\nu)P + \lambda^{-1}PM_1M_1^T P + \mu^{-1}PZM_2M_2^T Z^T P.$$

Setting $Y = PZ$, and using the Schur Complement Lemma, we see that the condition $\mathcal{M} < 0$ is equivalent to

$$\Pi < 0. \tag{40}$$

Therefore, we have

$$D^\alpha V(t, \epsilon(t)) \leq \lambda_{\max}(\mathcal{M}) \parallel \epsilon(t) \parallel^2. \tag{41}$$

Since $\mathcal{M} < 0$, $\lambda_{\max}(\mathcal{M}) < 0$. Hence, the condition (ii) in Lemma 2 is also satisfied. Therefore, the system (30) is asymptotically stable. This completes the proof of the theorem.

Remark 2: Note that (31) in Theorem 2 contains variable νP, which leads (31) to be a bilinear matrix inequality (BMI) with respect to ν and P. Condition (31) in Theorem 2 reduced to a linear matrix inequality (LMI) conditon when we fixed $\nu > 0$. Therefore, we can combine a one-dimensional search method with a convex optimisation algorithm such as MATLAB LMI Control Toolbox (Gahinet et al., 1995) to solve (31).

Accordingly, we now propose an effective algorithm to obtain the unknown observer parameters.

Algorithm 1

Step 1: Obtain matrices Φ and Ψ from (20)–(21). Check the existence condition (22).
Step 2: For some constant $\nu > 0$, BMI (31) becomes LMI.
Step 3: Solve the LMI (31). Obtain matrix $Z = P^{-1}Y$.
Step 4: Obtain matrices K, T, T_d from (24) and matrices N, N_d, M from (29). And then obtain matrices G, G_d from equations $T = G - NK$, $T_d = G_d - N_dK$, respectively.
Step 5: From (7e), compute matrix H. The observer design is complete.

Numerical examples

In this section, four examples with simulations are presented to illustrate the effectiveness of the proposed method.

Example 1: Consider the nonlinear system described in (1), where $\alpha = 0.85$, $x(t) = x_1(t), x_2(t), x_3(t), x_4(t)^T, \mathbb{R}^4$

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & -9 \end{bmatrix}, A_d = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, D = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, g(x,u,t) = \begin{bmatrix} x_1x_2 \\ -2x_1x_2 \\ 0 \\ -1.33\sin(x_3) \end{bmatrix}, \tag{42}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Note that for this example, the methods in Abbaszadeh and Marquez (2010), Ahmad et al. (2016), Benallouch et al. (2012), Dong et al. (2017), Nguyen and Trinh (2016), Nguyen et al. (2018), Zhang et al. (2012, 2014a,b, 2015) cannot be applied because they only deal with the integer-order systems.

For illustrative purpose, let us design a nonlinear unknown input functional observer to estimate $x_3(t) + 2x_4(t)$ and $x_4(t)$. Hence, matrix L for the functional state equation in (1) is defined as $L = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

From decomposition (2), the nonlinear function $g(x,u,t)$ can be decomposed into two portions as

$$g_1(\varphi,u,t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1.33\sin(x_3(t)) \end{bmatrix}, W = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad g_2(x,u,t) = x_1(t)x_2(t), \tag{43}$$

where the nonlinear function $g_1(\varphi,u,t)$ is under one-sided Lipschitz condition in Assumption 1 with the constant $\rho = 1.33$, by using the well-known mean value theorem, $g_1(\varphi,u,t)$ also satisfies Assumption 2 with the constants $\beta = 1.7689$ and $\gamma = 0$.

According to Step 1 of Algorithm 1, we obtain matrices Φ and Ψ from equations (20)–(21). Since $\text{rank}\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = 4 = \text{rank}[\Phi]$, condition (22) is satisfied. By solving the LMI in (31), matrix Z can be obtained as $Z = \begin{bmatrix} 0 & 0 & 23.1746 & 0 & 0 & 127.2654 & 0 & 0 & -69.5238 \\ 0 & 0 & 10.9428 & 0 & 0 & 58.7967 & 0 & 0 & -32.8285 \end{bmatrix}$. According to Step 3 and Step 4 of Algorithm 1, we obtain matrices K, T, T_d, N, N_d, M, G, G_d and H as below

$$K = \begin{bmatrix} 0 & 0 & 23.1746 \\ 0 & 0 & 10.9428 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 127.2654 \\ 0 & 0 & 58.7967 \end{bmatrix}, T_d = \begin{bmatrix} 0 & 0 & -69.5238 \\ 0 & 0 & -32.8285 \end{bmatrix}, \tag{44}$$

$$N = \begin{bmatrix} -16.3924 & -7.3897 \\ -4.0826 & -11.7777 \end{bmatrix}, N_d = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}, M = \begin{bmatrix} 0 & 0 & -22.1746 & 2 \\ 0 & 0 & -10.9428 & 1 \end{bmatrix}, \tag{45}$$

$$G = \begin{bmatrix} 0 & 0 & -333.487 \\ 0 & 0 & -164.6964 \end{bmatrix}, G_d = \begin{bmatrix} 0 & 0 & 21.8857 \\ 0 & 0 & 10.9428 \end{bmatrix}, H = \begin{bmatrix} -58.5238 \\ -28.8285 \end{bmatrix}. \tag{46}$$

Thus, from Steps 2–4 above, we have completed the design of a second-order nonlinear unknown input functional observer to estimate two linear functions $z_1(t) = x_3 + 2x_4(t)$ and $z_2(t) = x_4(t)$. To show the performance of the designed observer, let us consider the input $u(t) = 0.1$, $0 \leq t \leq 10$, $d(t) = 5\sin t$, $0 \leq t \leq 10$, $\tau = 1s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $x_3(\theta) = 3$, $x_4(\theta) = 4$, $\omega_1(\theta) = 0.1$, $\omega_2(\theta) = 0.2$ for $\theta \in [-1, 0]$. Figures 1–2 show the simulated responses of the two functional states, $z_1(t)$ and $z_2(t)$, and

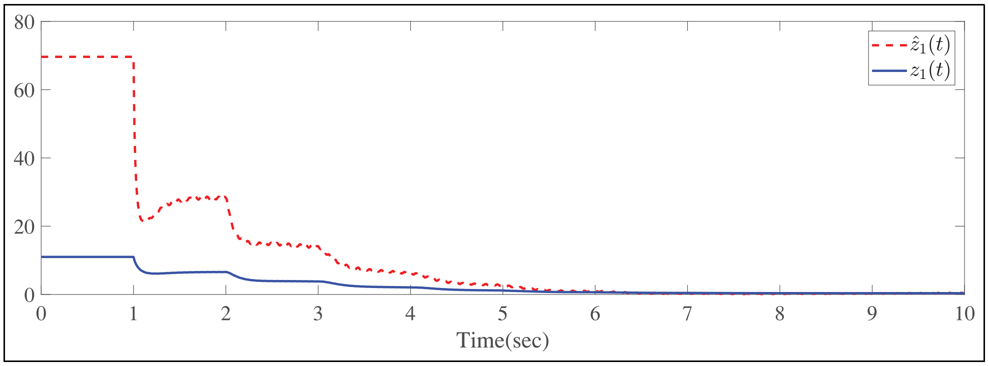

Figure 1. Response of $z_1(t)$ and its estimation with order $\alpha = 0.85$.

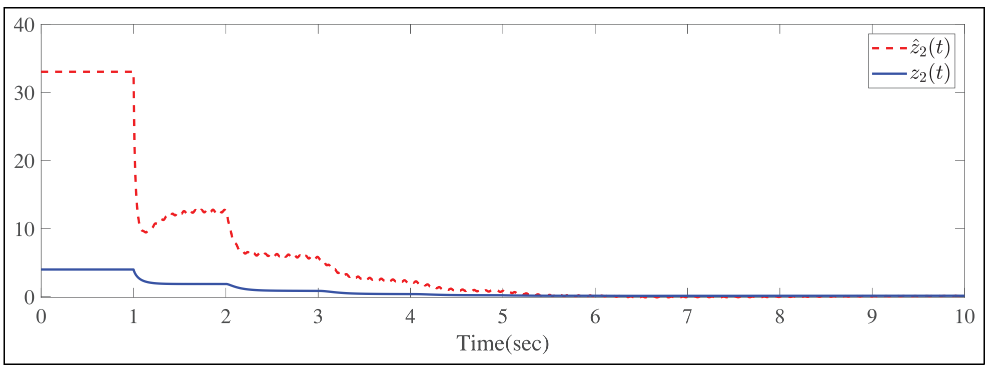

Figure 2. Response of $z_2(t)$ and its estimation with order $\alpha = 0.85$.

their estimations, which show that $\hat{z}_1(t) \to z_1(t)$ and $\hat{z}_2(t) \to z_2(t)$. It is clear from Figures 1–2 that the states of the functional observer converge to the true states of the system, and thus the design of nonlinear unknown input functional observer is now completed.

Example 2: Consider the nonlinear time-delay fractional-order systems described in (1), where $\alpha = 0.91$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2,\ A = \begin{bmatrix} 1 & 0.5 \\ 0 & -5 \end{bmatrix},\ A_d = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{47}$$

$$C = [1 \quad 2],\ L = [0 \quad 1],\ g(x,u,t) = \begin{bmatrix} -x_1(t)(x_1^2(t) + x_2^2(t)) \\ -x_2(t)(x_1^2(t) + x_2^2(t)) \end{bmatrix}. \tag{48}$$

It should be emphasized that the methods in Abbaszadeh and Marquez (2010), Ahmad et al. (2016), Benallouch et al. (2012), Dong et al. (2017), Nguyen and Trinh (2016), Nguyen et al. (2018), Zhang et al. (2012, 2014a, b, 2015) only deal with the integer-order systems and hence they cannot be applied to this example.

As reported in Abbaszadeh and Marquez (2010), the nonlinear function $g_1(\varphi, u, t) = \begin{bmatrix} -x_1(t)(x_1^2(t) + x_2^2(t)) \\ -x_2(t)(x_1^2(t) + x_2^2(t)) \end{bmatrix}$ is globally one-sided Lipschitz with constant $\rho = 0$. Consider region

$$S = \{x \in \mathbb{R}^2 : ||x|| \leq 2\},$$

$g_1(\varphi, u, t)$ is quadratically inner-bounded in S with constants $\beta = -48$ and $\gamma = -16$. In region S, $g_1(\varphi, u, t)$ is locally Lipschitz with the Lipschitz constant 12.

According to Step 1 of Algorithm 1, we obtain matrices Φ and Ψ from equations (20)–(21). Since $\text{rank}\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = 2 = \text{rank}[\Phi]$, condition (22) is satisfied. By solving the LMI in (31), matrix Z can be obtained as $Z = 10^3[1.7691 \quad 0.7740 \quad 0]$. According to Step 3 and Step 4 of Algorithm 1, we obtain matrices K, T, T_d, N, N_d, M, G, G_d and H as below

$$\begin{aligned} &K = T = T_d = 0,\ N = -5,\ N_d = -2, \\ &M = [0 \quad 1],\ G = G_d = 0,\ H = 2. \end{aligned} \tag{49}$$

Simulation results

We now let the input $u(t) = 0.01\sin(0.01t)$, $0 \leq t \leq 10$, $d(t) = 6\sin 2t$, $0 \leq t \leq 10$, $\tau = 1s$ and the initial conditions are $x_1(\theta) = 0.1$, $x_2(\theta) = 0.2$, $\omega(\theta) = -0.2$ for $\theta \in [-1, 0]$. Figure 3 shows the response of $z(t) = x_2(t)$ and its estimation with order $\alpha = 0.91$.

Example 3: Consider the nonlinear time-delay fractional-order systems described in (1), where $\alpha = 0.7$

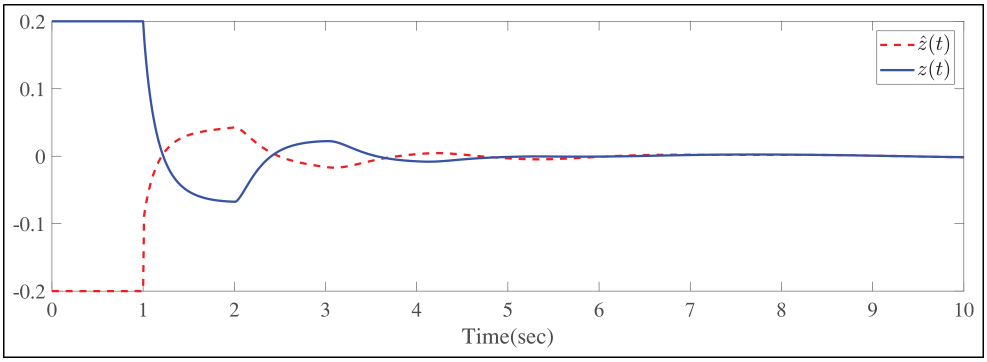

Figure 3. Response of $z(t)$ and its estimation with order $\alpha = 0.91$.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2,\ A = \begin{bmatrix} -4 & 1 \\ 0 & -6 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 2 & -5 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{50}$$

$$C = [1 \quad 1],\ L = [0 \quad 1],\ g(x,u,t) = \begin{bmatrix} x_1(t)x_2(t) \\ -\operatorname{sgn}(x_2)\sqrt{|x_2|} \end{bmatrix}, \tag{51}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign (signum) function.

Note that for this example, the nonlinear observer designs reported in Abbaszadeh and Marquez (2010), Ahmad et al. (2016), Benallouch et al. (2012), Dong et al. (2017), Nguyen and Trinh (2016), Nguyen et al. (2018), and Zhang et al. (2012, 2014a,b, 2015) do not exist. Let us now consider the method in this paper.

From decomposition (2), the nonlinear function $g(x,u,t)$ can be decomposed into two portions as

$$g_1(\varphi,u,t) = \begin{bmatrix} 0 \\ -\operatorname{sgn}(x_2)\sqrt{|x_2|} \end{bmatrix},\ W = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$g_2(x,u,t) = x_1(t)x_2(t). \tag{52}$$

Consider region

$$S = \{x \in \mathbb{R}^2 : ||x|| \leqslant 4\}.$$

Note that in region $S_1 = \{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S : x_1 = 0\} \subset S$, $g_1(x,u,t)$ is not Lipschitz. Indeed, we have

$$\frac{||g_1(x,u,t) - g_1(0,u,t)||}{||x - 0||} = \sqrt{\frac{|x_2|}{x_2^2}} = \sqrt{\frac{1}{|x_2|}}, \tag{53}$$

which is unbounded as $x_2 \to 0$.

It is not hard to check that, the nonlinear function $g_1(\varphi,u,t) = \begin{bmatrix} 0 \\ -\operatorname{sgn}(x_2)\sqrt{|x_2|} \end{bmatrix}$ is one-sided Lipschitz in S with constant $\rho = -\frac{1}{4}$ and is quadratically inner-bounded in S with constants $\beta = 0$ and $\gamma = \rho = -\frac{1}{4}$.

According to Step 1 of Algorithm 1, we obtain matrices Φ and Ψ from equations (20)–(21). Since $\operatorname{rank}\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = 2 = \operatorname{rank}[\Phi]$, condition (22) is satisfied. By solving the LMI in (31), matrix Z can be obtained as $Z = 10^3[-1.1014 \quad 0 \quad 0]$. According to Step 3 and Step 4 of Algorithm 1, we obtain matrices K, T, T_d, N, N_d, M, G, G_d and H as below

$$K = T = T_d = 0,\ N = -6,\ N_d = -1,$$
$$M = [0 \quad 1],\ G = G_d = 0,\ H = 2. \tag{54}$$

Simulation results

We now let the input $u(t) = 0.01\sin(0.01t)$, $0 \leqslant t \leqslant 10$, $d(t) = 10\sin t$, $0 \leqslant t \leqslant 10$, $\tau = 1s$ and the initial conditions are $x_1(\theta) = 1$, $x_2(\theta) = 2$, $\omega(\theta) = 1$ for $\theta \in [-1,0]$. Figure 4 shows the response of $z(t) = x_2(t)$ and its estimation with order $\alpha = 0.7$.

Example 4: (A practical example). Let us now consider the following fractional-order model of a moving object in the cartesian coordinates (Lan and Zhou, 2013; Lan et al., 2016), which is of the form (1), where $\alpha = 0.85$ (Lan et al., 2016) and

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2,\ A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{55}$$

$$C = [1 \quad 0],\ g(x,u,t) = \begin{bmatrix} -x_1(t)(x_1^2(t) + x_2^2(t)) \\ -x_2(t)(x_1^2(t) + x_2^2(t)) \end{bmatrix}. \tag{56}$$

We now design a nonlinear unknown input functional observer to estimate $x_1(t) + 2x_2(t)$. Hence, matrix L for the functional state equation in (1) is defined as $L = [1 \quad 2]$. Note that different from the observers reported in Lan and Zhou (2013) and Lan et al. (2016), where a full-order Luenberger-type fractional-order non-fragile observer was designed (see, Lan and Zhou (2013)) and full-order and reduced-order observers for fractional-order one-sided Lipschitz nonlinear systems were proposed (see, Lan et al. (2016)), the design of nonlinear unknown input functional observer is proposed in this example.

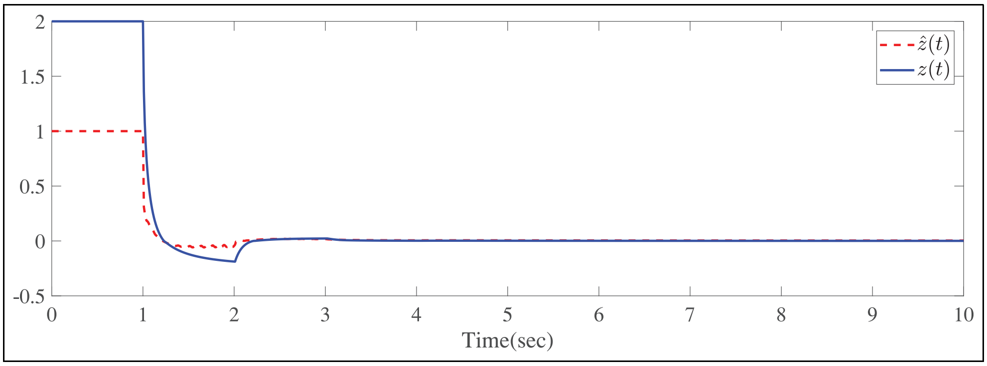

Figure 4. Response of z(t) and its estimation with order $\alpha = 0.7$.

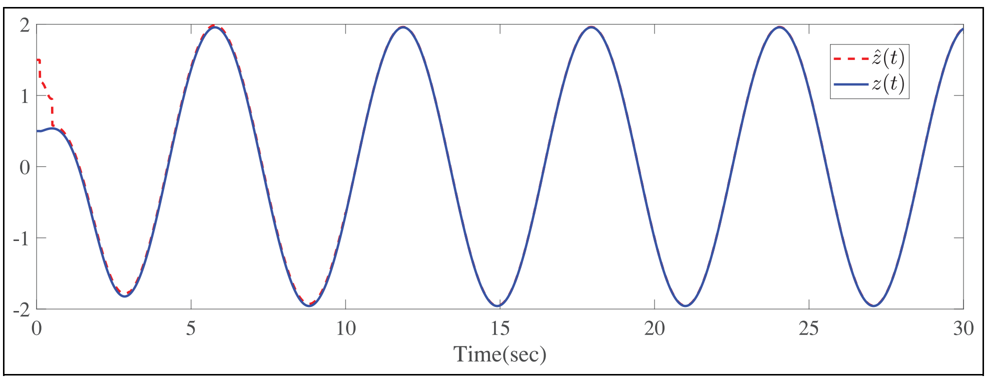

Figure 5. Response of z(t) and its estimation with order $\alpha = 0.85$.

It follows from Example 2 that the function $g_1(\varphi, u, t) = \begin{bmatrix} -x_1(t)(x_1^2(t) + x_2^2(t)) \\ -x_2(t)(x_1^2(t) + x_2^2(t)) \end{bmatrix}$ is globally one-sided Lipschitz with constant $\rho = 0$ and is quadratically inner-bounded in S with constants $\beta = -48$ and $\gamma = -16$ in region $S = \{x \in \mathbb{R}^2 : ||x|| \leqslant 2\}$.

According to Step 1 of Algorithm 1, we obtain matrices Φ and Ψ from equations (20)–(21). Since $\text{rank}\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = 2 = \text{rank}[\Phi]$, condition (22) is satisfied. By solving the LMI in (31), matrix Z can be obtained as $Z = 10^3[2.3307 \quad 1.5809 \quad 0]$. According to Step 3 and Step 4 of Algorithm 1, we obtain matrices K, T, T_d, N, N_d, M, G, G_d and H as below

$$\begin{aligned} &K = -5,\ T = T_d = 0,\ N = 4,\ N_d = 0,\ M = [6 \quad 2], \\ &G = -20,\ G_d = 0,\ H = 0. \end{aligned} \tag{57}$$

Simulation results

We now let the initial conditions are $x_1(0) = 0.1$, $x_2(0) = 0.2$, $\omega(0) = 2$. Figure 5 shows the response of $z(t) = x_1(t) + 2x_2(t)$ and its estimation with order $\alpha = 0.85$.

Conclusion

The problem of unknown input fractional-order functional state observer design for a class of one-side Lipschitz time-delay fractional-order systems has been investigated in this paper. By taking the advantages of recent results on Caputo fractional derivative of a quadratic function, we obtain new sufficient conditions with the form of a linear matrix inequality to guarantee the asymptotic stability of the fractional-order error dynamic system. Four examples have been provided to show the effectiveness and applicability of the proposed method.

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ORCID iD

Dinh Cong Huong https://orcid.org/0000-0003-2028-3784

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