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# Input-to-state Stability of Nonlinear Positive Systems

**Yan Zhao and Fanwei Meng***

**Abstract:** In this paper, input-to-state stability (ISS), as a useful tool for robust analysis, is first applied to continuous-time and discrete-time nonlinear positive systems. For continuous-time and discrete-time positive systems, some new definitions of ISS are introduced. Different from the usual ISS definitions for nonlinear systems, our ISS definitions can fully reflect the positiveness requirements of states and inputs of the positive systems. By introducing the max-separable ISS Lyapunov functions, some ISS criterions are given for general nonlinear positive systems. Based on that, the ISS criterions for linear positive systems and affine nonlinear homogeneous systems are given. Through them, the ISS properties can be judged directly from the differential and algebraic characteristics of the systems. Simulation examples verify the validity of our results.

**Keywords:** Input-to-state stability, max-separable Lyapunov functions, nonlinear system, positive system.

## 1. INTRODUCTION

Positive systems are dynamical systems whose, input, output and state variables are constrained to be nonnegative for all time whenever the initial conditions are nonnegative [1]. This kind of systems can be seen in many real-world processes of areas such as economics, biology, ecology and communications. Due to their importance and wide applicability, the analysis and control of positive systems has attracted attention from the control community (see e.g., [2–6] and references therein). In fact, the concept of positive system was first introduced by Luenberger [7] in 1979. In 2000, Farina and Rinaldi systematically studied the theory and application of positive systems in [1], and for the first time introduced quadratic diagonal Lyapunov function to analyze the stability of positive linear systems, revealing the unique and beautiful properties of positive systems. In 2007, Rami et al. [8, 9] used the linear programming algorithm to solve the controller design problem of continuous time and discrete time positive systems, and formally introduced the linear copositive Lyapunov function (LCLF) to discuss the stability of positive systems. This LCLF method can solve the positive system problem more effectively than the quadratic Lyapunov function method. This has promoted the rapid development of positive system theory and application. Until now, for the positive systems, not only the stability [10–12], stabilization [2], but also the problem of observer design [13], fault detection [14], tracking control [15], saturation control [16], model predictive control [17], positive filtering [18, 19] and etc., have been considered. In another, when the positive systems with uncertainty, the robust stability was studied in [20] and [21], etc. However, little research has been done for positive systems, especially for nonlinear positive systems, with disturbance inputs. Only in [5], we consider the boundedness analysis of solutions of nonlinear positive systems with disturbance inputs by using the systems' successive approximation method. The robust analysis of nonlinear positive systems with disturbance inputs, has not been studied.

In fact, since the performance of a real control system is affected more or less by uncertainties such as unmodelled dynamics, parameter perturbations, exogenous disturbances, measurement errors etc., the research on robustness of control systems do always have a vital status in the development of control theory and technology. Aiming at robustness analysis of nonlinear control systems, Sontag's input-to-state stability (ISS) and its various extension such as integral ISS (iISS) and finite-time ISS ( [22–25]) are developed. It has been applied to continuous time dynamical systems [26], discrete-time system [27], switched system [28,29], stochastic systems [29,30], Markovian jump systems [30], time-delay systems [31], sampled-data nonlinear systems [32], networked control systems [33], etc. But, as a useful tool of robust analysis for nonlinear systems, ISS has not been used to positive systems for stability analysis and synthesis.

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Yan Zhao is with the School of Mathematical Sciences, Qufu Normal University, Qufu 273165, P. R. China, and the School of Mathematical Sciences, University of Jinan, Jinan 250022, P. R. China (e-mail: ss_zhaoy@ujn.edu.cn). Fanwei Meng is with the School of Mathematical Sciences, Qufu Normal University, Qufu 273165, P. R. China (e-mail: fwmeng@qfnu.edu.cn).
* Corresponding author.

In this paper, we will introduce some new ISS definitions for positive systems firstly. Different from the usual ISS definitions, our ISS definitions will fully reflect the positiveness requirements of states and inputs of the positive systems. Based on that, some ISS criterions for continuous-time and discrete-time nonlinear positive systems will be provided, respectively. For linear and affine nonlinear positive systems, some ISS criteria will also be given, through which the ISS properties can be judged directly from the differential and algebraic characteristics of the systems. Simulation examples will verify the validity of our results.

The remainder of this paper is organized as follows: Section 2 provides some notations and introduces some preliminary results of positive systems. Section 3 gives the definitions of ISS and max-separable ISS Lyapunov functions for continuous-time positive systems, and provides some ISS criterions by the max-separable ISS Lyapunov function method. Based on the given criterions, the ISS properties for linear positive systems and affine nonlinear homogeneous positive systems are studied. Also, an asymptotical stability criterion for affine nonlinear homogeneous positive system is provided. In Section 4, the ISS properties are studied for discrete-time positive systems. Similar to the study of continuous-time positive systems, a new upper difference operator is defined and some ISS criterions are provided. In Section 5, some simulation examples are provided to illustrate our results. Section 6 includes some concluding remarks.

## 2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all nonnegative real numbers, natural numbers and natural numbers including zero, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, $n$-dimensional real space and $n\times m$ dimensional real matrix space. Especially, for a matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$, it is **Metzler** if and only if its off-diagonal entries $a_{ij}$, $\forall i\neq j$ are nonnegative [1]. The transpose of vectors and matrices are denoted by superscript $T$. The positive orthant $\mathbb{R}^n_+$ in $\mathbb{R}^n$ is the set $\{x=(x_1,\cdots,x_n)^T\in\mathbb{R}^n : x_i\geq 0,\ \forall i\}$. By the boundary of $\mathbb{R}^n_+$, also denoted by $\partial\mathbb{R}^n_+$, we mean the set $\{s\in\mathbb{R}^n_+ : \exists i : s_i=0\}$. The interior of $\mathbb{R}^n_+$ denoted as $\mathcal{V} := int\mathbb{R}^n_+$ which means the set $\{x\in\mathbb{R}^n : x_i>0,\ \forall i\}$. For vectors $x,y\in\mathbb{R}^n$, we write: $x\succeq y$ ($x\preceq y$) if $x_i\geq y_i$ ($x_i\leq y_i$) for $i\in\mathcal{I}_n =: \{1,2,\cdots,n\}$; $x>y$ ($x<y$) if $x\succeq y$ ($x\preceq y$) and $x\neq y$; $x\succ y$ ($x\prec y$) if $x_i>y_i$ ($x_i<y_i$) for $i\in\mathcal{I}_n$. $\mathbf{1}_n\in\mathbb{R}^n$ denotes the vector whose components are all one. The $p$-norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|_p$, where $p$ is usually omitted in the case $p=2$. The max-norm is denoted as $\|\cdot\|_\infty$. For vector $x\in\mathbb{R}^n$, $|x|$ denotes the Euclidean norm $|x|=(\sum_{i=1}^n x_i^2)^{1/2}$. Given a vector $v\in\mathbb{R}^n$, $v\succ 0$, the weighted $l_\infty$ norm is defined by $\|x\|^v_\infty=\max_{1\leq i\leq n}\frac{|x_i|}{v_i}$. All the vectors are column vectors unless otherwise specified. $\mathcal{C}^i$ denotes all the $i$th continuous differential functions; $\mathcal{C}^{i,k}$ denotes all the functions with $i$th continuously differentiable first component and $k$th continuously differentiable second component. Finally, we denote the composition of two functions $\varphi : A\to B$ and $\psi : B\to C$ by $\psi\circ\varphi : A\to C$.

The comparison function classes $\mathcal{K}$ and $\mathcal{K}_\infty$ are, respectively, the sets of continuous functions $\{\gamma : \mathbb{R}^n_+, \gamma(0)=0, \gamma \text{ is strictly increasing}\}$ and $\{\gamma\in\mathcal{K} : \gamma \text{ is unbounded}\}$. A function $\beta : \mathbb{R}^2_+\to\mathbb{R}_+$ is of class $\mathcal{KL}$ if for fixed $t\geq 0$ the function $\beta(\cdot,t)$ is of class $\mathcal{K}$ and for fixed $s\geq 0$ the function $\beta(s,\cdot)$ is non-increasing with $\lim_{t\to\infty}\beta(s,t)=0$.

In this paper, we consider the following continuous-time $n$-dimensional nonlinear system

$$\dot{x}=f(x,u),\quad t\geq t_0, \tag{1}$$

where $x\in\mathbb{R}^n_+$, $u\in\mathcal{L}^m_\infty$ are system state and input, respectively; $\mathcal{L}^m_\infty$ denotes the set of all the measurable and locally essentially bounded input $u\in\mathbb{R}^m_+$ on $[t_0,\infty)$ with norm

$$\|u\|=ess\sup_{t\geq t_0}\{\|u(t)\|_\infty\}.$$

$f : \mathbb{R}^n_+\times\mathbb{R}^m_+\to\mathbb{R}^n$ is continuous differentiable in $(x,u)$, satisfies $f(0,0)\equiv 0$; initial data $x_0=(x_{01},\cdots,x_{0n})^T\in\mathbb{R}^n_+$.

For controlled dynamical systems, i.e. systems forced by some exogenous input signal, we assume the given partially ordered input value space $\mathcal{U}\subset\mathbb{R}^m_+$. By an “input" or “control", we will mean a Lebesgue measurable function $u(\cdot) : [t_0,\infty)\to\mathcal{U}$ which is essentially bounded, i.e. there is for each finite interval $[t_0,t_0+T]$ some compact subset $C\subset\mathcal{U}$ such that $u(t)\in C$ for almost all $t\in[t_0,t_0+T]$. We denote $\mathcal{U}_\infty$ as the set of all inputs. Accordingly, for any pair of input values $u_1,u_2\in\mathcal{U}_\infty$, we write $u_1\succeq u_2$ if $u_1(t)\succeq u_2(t)$ for all $t\geq t_0$. We interpret $x(t,t_0,x_0,u)$ as the state at time $t$ with initial data $(t_0,x_0)$ and the external input $u(\cdot)$. Sometimes, when clear from the context, we write $x(t)$ instead of $x(t,t_0,x_0,u)$.

We also consider the discrete-time nonlinear system

$$x(k+1)=f(x(k),u(k)),\ \ k\in\mathbb{N}_0, \tag{2}$$

where $x(k)\in\mathbb{R}^n_+$ is the state vector, $u(k)\in\mathbb{R}^m_+$ is the input. $f : \mathbb{R}^n_+\times\mathbb{R}^m_+\to\mathbb{R}^n_+$, $f(0,0)=0$. Let $x(k,x_0,u)$ denote the solution of system (2) with initial data $(0,x_0)$ and the external input $u(\cdot)$. For brevity, we write $x(k)$ instead of $x(k,x_0,u)$.

For systems (1) and (2), we introduce the following definitions.

**Definition 1** [34]**:** The system (1) ((2)) is called **monotone** if the implication

$$u_1\succeq u_2, x_1\succeq x_2 \Rightarrow \quad x(t,x_1,u_1)\succeq x(t,x_2,u_2) \quad (x(k,x_1,u_1)\succeq x(k,x_2,u_2))$$

holds for all $t\geq t_0$ ($k\in\mathbb{N}_0$).

This definition states that trajectories of monotone systems starting at ordered initial conditions preserve the same ordering for all inputs $u(\cdot)$ and all times $t \geq t_0$. By choosing $x_2 = 0$, $u_2 = 0$, since $x(t,0,0) = 0$ for all $t \geq t_0$ ($x(k,0,0) = 0$ for all $k \in \mathbb{N}_0$), it is easy to see that

$$x_1 \in \mathbb{R}^n_+, u_1 \succeq 0 \Rightarrow \quad x(t,x_1,u_1) \in \mathbb{R}^n_+, \ \ \forall t \geq t_0 \quad (x(k,x_1,u_1) \in \mathbb{R}^n_+, \ \ \forall k \in \mathbb{N}_0).$$

This shows that the positive $\mathbb{R}^n_+$ is an invariant set for the monotone system (1) ((2)). Thus the monotone systems with an equilibrium point at the origin define **positive systems**.

**Proposition 1** [34]**:** System (1) ((2)) is monotone if and only if, for all $x_1, x_2 \in \mathcal{V} := int\mathbb{R}^n_+$,

$$x_1 \succeq x_2, \ \ u_1 \succeq u_2 \Rightarrow f(x_1,u_1) - f(x_2,u_2) \succeq 0. \tag{3}$$

Note that if $f$ is continuously differentiable on $\mathbb{R}^n_+ \times \mathbb{R}^m_+$, then condition (3) is equivalent to the requirement that

$$\frac{\partial f_i}{\partial x_j}(x,u) \geq 0, \qquad \forall x \in \mathcal{V}, \ \forall u \in \mathcal{W}, \ \forall i \neq j, \tag{4}$$

$$\frac{\partial f_i}{\partial u_j}(x,u) \geq 0, \qquad \forall x \in \mathbb{R}^n_+, \ \forall u \in \mathcal{W} \tag{5}$$

for all $i, j \in \mathcal{I}_n$, where $\mathcal{W} = int\mathcal{U}$. If (4) and (5) hold, system (1) ((2)) is called **cooperative**.

In this paper, we will consider the ISS properties of nonlinear positive systems. To this end, we introduce the following new definitions on ISS and max-separable ISS Lyapunov function, tailored for positive system.

**Definition 2:** The nonlinear positive system (1) is said to be **input-to-state stable (ISS)** if, there exist scalar functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$x(t) \preceq \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n. \tag{6}$$

**Remark 1:** Under the precondition that positive systems and initial data $x_0 \succ 0$, (6) is equivalent to any of the following three inequalities:

$$\|x(t)\|_\infty \leq \beta(\|x_0\|_\infty, t-t_0) + \gamma(\|u\|),$$
$$\|x(t)\|_1 \leq \frac{1}{n}\beta(\|x_0\|_1, t-t_0) + \frac{1}{n}\gamma(\|u\|),$$
$$\|x(t)\|_2 \leq \sqrt{n}\beta(\sqrt{n}\|x_0\|_2, t-t_0) + \sqrt{n}\gamma(\|u\|).$$

From the properties of $\mathcal{KL}$ functions ( [35]), the nonlinear positive system (1) is also ISS in the sense of $l_\infty$, $l_1$ and $l_2$-norm. So, our new ISS definition is coincident with the usual ISS definitions for general systems.

**Remark 2:** The ISS definition can also be defined in a broader way. If any one of the following conditions (i) and (ii) holds, the nonlinear positive system (1) is said to be **ISS**.

(i) For given positive vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_+$, if there exist scalar functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$x(t) \preceq \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{p} + \gamma(\|u\|)\mathbf{q};$$

(ii) if there exist scalar functions $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}$ ($i = 1, \cdots, n$) such that

$$x(t) \preceq \bar{\beta}(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, t-t_0) + \bar{\gamma}(\|u\|),$$

where $\bar{\beta} = (\beta_1, \cdots, \beta_n)^T$, $\bar{\gamma} = (\gamma_1, \cdots, \gamma_n)^T$.
These definitions cover more situations and are more suitable for practical needs. For example, in the study of multi-agent systems, the position and velocity of agents have different demand. Then, each state component of the state space has different upper bounds. So, we can find that, comparing with the usual ISS definitions, our ISS definition emphasizes the requirement for each component of the system states and is more refined. This idea can also be used to study large-scale systems. Here, for simplicity, we use the special case of $\mathbf{p} = \mathbf{q} = \mathbf{1}_n$ to study the ISS properties of positive systems.

For monotone systems or positive systems, max-separable Lyapunov function is often considered for stability analysis. For the definition and construction of it, we can refer to [36]. Here, for the ISS analysis of positive systems, our new max-separable ISS Lyapunov function is defined as follows.

**Definition 3:** For positive system (1), a function $V(x)$ is called a **max-separable ISS Lyapunov function**

$$V(x) = \max_{i\in\mathcal{I}_n}\{V_i(x_i)\}, \tag{7}$$

with functions $V_i : \mathbb{R}_+ \to \mathbb{R}_+$, if there are functions $\alpha_1$, $\alpha_2$, $\chi$ of class $\mathcal{K}$ and continuous positive definite function $\mu$ such that for all $i \in \mathcal{I}_n$,

$$\alpha_1(x_i) \leq V_i(x_i) \leq \alpha_2(x_i), \tag{8}$$

and

$$V(x) \geq \chi(\|u\|) \Rightarrow D^+V(x) \leq -\mu(x), \tag{9}$$

where the upper-right Dini derivative $D^+V(x)$ along the solutions of (1) is defined as

$$D^+V(x) = \limsup_{h\to 0^+} \frac{V(x+hf(x,u)) - V(x)}{h}. \tag{10}$$

The following result shows that if the functions $V_i$ in (7) are continuous differentiable, then (10) admits an explicit expression.

**Proposition 2** [37]**:** Consider $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ in (7) and let $V_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuously differentiable for all $i \in \mathcal{I}_n$. Then, the upper-right Dini derivative (10) is given by

$$D^+V(x) = \max_{j\in\mathcal{J}(x)} \left\{ \frac{\partial V_j}{\partial x_j}(x_j) f_j(x) \right\},$$

where $\mathcal{J}(x)$ is the set of indices for which the maximum in (7) is attained, i.e.,

$$\mathcal{J}(x) = \{j \in \mathcal{I}_n | V_j(x_j) = V(x)\}.$$

Similar to the continuous-time case, we introduce our new definitions on ISS and max-separable ISS Lyapunov function, tailored for discrete-time nonlinear positive systems.

**Definition 4:** The discrete-time positive system (2) is said to be **input-to-state stable (ISS)** if, there exist scalar functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$x(k) \preceq \beta(\max_{i \in I_n}\{x_{0i}\}, k-k_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n, \ \ k \in \mathbb{N}_0.$$

**Definition 5:** For system (2), a function $V(x)$ is called a **max-separable ISS Lyapunov function**

$$V(x) = \max_{i \in \mathcal{I}_n}\{V_i(x_i)\}, \tag{11}$$

with scalar functions $V_i : \mathbb{R}_+ \to \mathbb{R}_+$, if there are functions $\alpha_1$, $\alpha_2$, $\chi$ of class $\mathcal{K}$ and function $\mu$ of class $\mathcal{K}_\infty$ such that for all $i \in \mathcal{I}_n$,

$$\alpha_1(x_i) \leq V_i(x_i) \leq \alpha_2(x_i), \tag{12}$$

and

$$V(x) \geq \chi(\|u\|) \Rightarrow \Delta^+ V(x) \leq -\mu(x). \tag{13}$$

Similarly to the upper-right Dini derivative for the continuous-time case, for discrete-time systems, the new upper difference operator $\Delta^+$ as in (13) is defined in the form of

$$\Delta^+ V(x) = \max_{j \in \mathcal{J}(x)}\{V_j(f_j(x,u)) - V_j(x_j)\},$$

where $\mathcal{J}(x)$ is the set of indices for which the maximum in (11) is attained, i.e.,

$$\mathcal{J}(x) = \{j \in \mathcal{I}_n | V_j(x_j) = V(x)\}.$$

## 3. INPUT-TO-STATE STABILITY OF CONTINUOUS-TIME NONLINEAR POSITIVE SYSTEMS

In this section, based on our new definitions on ISS and max-separable ISS Lyapunov function, we will state and prove the following ISS criterion for continuous-time nonlinear positive systems.

**Theorem 1:** If there exists a max-separable ISS-Lyapunov function for nonlinear positive system (1), system (1) is ISS.

**Proof:** Let $\tau \in [t_0, \infty)$ denotes a time at which the trajectory enters the set $\mathcal{B} = \{x \in \mathbb{R}^n_+ | V(x) < \chi(\|u\|)\}$ for the first time. Let us complete the proof by considering the following two cases: $x_0 \in \mathcal{B}^c$ and $x_0 \in \mathcal{B}\backslash\{0\}$, respectively.

**Case 1:** $x_0 \in \mathcal{B}^c$. In this case, for any $t \in [t_0, \tau]$, $V(x) \geq \chi(\|u\|)$. From (9), we have

$$D^+ V(x(t)) \leq -\mu(x(t)), \ \ t \in [t_0, \tau].$$

By Theorem 4.19 in [35], there exists a $\mathcal{KL}$ function $\hat{\beta}$ such that

$$V(x(t)) \leq \hat{\beta}(V(x_0), t-t_0), \quad t \in [t_0, \tau].$$

From the definition of max-separable ISS Lyapunov function and the property of $\mathcal{K}$ functions,

$$\begin{aligned}\alpha_1(\max_{i \in \mathcal{I}_n}\{x_i\}) = &\max_{i \in \mathcal{I}_n}\{\alpha_1(x_i)\} \leq \max_{i \in \mathcal{I}_n}\{V_i(x_i)\} = V(x) \\ &\leq \hat{\beta}(V(x_0), t-t_0) \\ &\leq \hat{\beta}(\alpha_2(\max_{i \in \mathcal{I}_n}\{x_{0i}\}), t-t_0), \ \ t \in [t_0, \tau].\end{aligned}$$

So,

$$\begin{aligned}x(t) \preceq &\alpha_1^{-1} \circ \hat{\beta}(\alpha_2(\max_{i \in \mathcal{I}_n}\{x_{0i}\}), t-t_0)\mathbf{1}_n \\ := &\beta(\max_{i \in \mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{1}_n, \ \ t \in [t_0, \tau],\end{aligned}$$

where $\beta(r,s) \in \mathcal{KL}$ can be known from Lemma 4.2 in [35].

When $t \in (\tau, \infty)$, since $D^+ V(x)$ is negative for $x(t)$ outside the set $\mathcal{B}$, we have $V(x) \leq \chi(\|u\|)$. Thus, we get

$$x(t) \preceq \beta(\max_{i \in \mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n, \ \ t \in [t_0, \infty),$$

where $\gamma = \alpha_1^{-1} \circ \chi \in \mathcal{K}$ can be known from Lemma 4.2 in [35].

**Case 2:** $x_0 \in \mathcal{B}\backslash\{0\}$. In this case, $\tau = t_0$. When $t > t_0$, following the proof of *Case 1.*, we obtain that

$$x(t) \preceq \beta(\max_{i \in \mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n, \ \ t \in (t_0, \infty).$$

When $t = t_0$, by the definition of the set $\mathcal{B}$ and the definition of $\gamma$, we obtain

$$\begin{aligned}x(t_0) \preceq &\gamma(\|u\|)\mathbf{1}_n \\ \preceq &\beta(\max_{i \in \mathcal{I}_n}\{x_{0i}\}, t-t_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n, \ \ t \in [t_0, \infty).\end{aligned}$$

In all, system (1) is ISS. □

Consider the nonlinear positive system system (1) with $f(x,u) = Ax + u$, where $A \in \mathbb{R}^{n \times n}$. Then, it reduces to the linear positive system

$$\dot{x} = Ax + u, \tag{14}$$

where the function $u : [t_0, \infty) \to \mathbb{R}^n_+$ is the disturbance. Using the max-separable ISS-Lyapunov function method shown in the proof of Theorem 1, we can get the following proposition.

**Proposition 3:** For the linear positive system (14), suppose that $A$ is asymptotically stable Metzler matrix, and further, there exist number $\varepsilon_0 > 0$ and vector $v \succ 0$ such that for any $i \in \mathcal{I}_n$,

$$-a_{ii} \geq \sum_{j=1, j\neq i}^{n} a_{ij}\frac{v_j}{v_i} + \varepsilon_0,$$

then system (14) is ISS.

**Proof:** See Appendix A. □

Consider the affine nonlinear positive system

$$\dot{x} = f(x) + g(x)u, \quad (15)$$

where the function $u : [0, +\infty) \to \mathbb{R}_+$ is the disturbance. For the convenience of ISS study, we introduce the following definitions and assumptions on system (15).

**Definition 6** [38]**:** A continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, which is $\mathcal{C}^1$ on $\mathbb{R}^n \backslash \{0\}$, is said to be **cooperative** if the Jacobian matrix $(\partial f/\partial x)(a)$ is Metzler for all $a \in \mathbb{R}^n_+ \backslash \{0\}$.

Cooperative vector fields satisfy the following property.

**Proposition 4** [39]**:** Let vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be cooperative. For any two vectors $x, y \in \mathbb{R}^n_+$, with $x_i = y_i$, and $x \succeq y$, we have $f_i(x) \succeq f_i(y)$.

**Definition 7** [10]**:** $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **homogeneous** of degree $\alpha$ if for all $x \in \mathbb{R}^n$ and all real $\lambda > 0$, $f(\lambda x) = \lambda^{\alpha} f(x)$.

When $\alpha = 1$, $f$ is called homogeneous of degree one.

**Definition 8** [10]**:** $g : \mathbb{R}^n \to \mathbb{R}^n$ is **order-preserving** on $\mathbb{R}^n_+$ if $g(x) \succeq g(y)$ for any $x, y \in \mathbb{R}^n_+$ such that $x \succeq y$.

**Assumption 1:** i) $f$ is cooperative, continuous on $\mathbb{R}_n$, continuously differentiable on $\mathbb{R}_n \backslash \{0\}$, and homogeneous of degree one; ii) $g$ is continuous and bounded by $M$ on $\mathbb{R}^n_+$; iii) There exists a vector $v \succ 0$ such that $f(v) \prec 0$.

**Assumption 2:** i) $f$ and $g$ are continuous on $\mathbb{R}_n$, continuously differentiable on $\mathbb{R}_n \backslash \{0\}$, and homogeneous of degree one; ii) $f$ is cooperative and $g$ is order-preserving on $\mathbb{R}^n_+$; iii) There exists a vector $v \succ 0$ such that $f(v) \prec 0$.

Under Assumption 1 and Assumption 2, we have the following two propositions, respectively.

**Proposition 5:** If Assumption 1 holds, the system (15) is ISS.

**Proof:** See Appendix B. □

**Proposition 6:** Suppose that Assumption 2 holds and there exists $\eta \succ 0$ such that $\frac{f_i(v)+g_i(v)\|u\|}{v_i} + \eta_i = 0$ for any $i \in \mathcal{I}_n$, then the system (15) is asymptotically stable.

**Proof:** See Appendix C. □

**Remark 3:** Using our ISS criterion, Theorem 1, or max-separable ISS-Lyapunov function method for continuous positive systems, Propositions 3, 5, and 6 can be got. From them, the ISS or asymptotical stability properties can be got, just from the differential and algebraic characteristics of the systems.

## 4. INPUT-TO-STATE STABILITY OF DISCRETE-TIME NONLINEAR POSITIVE SYSTEMS

In this section, we will study the ISS of discrete-time positive systems. Based on the preliminary results of Section 2, we can give our main result as follows for the ISS analysis of discrete-time nonlinear positive systems.

**Theorem 2:** If there exists a max-separable ISS-Lyapunov function for the discrete-time nonlinear positive system (2), the system (2) is ISS.

**Proof:** Let $j_0$ denotes a time at which the trajectory enters the set $\mathcal{B} = \{x \in \mathbb{R}^n_+ | V(x) < \chi(\|u\|)\}$ for the first time. Let us complete the proof by considering the following two cases: $x_0 \in \mathcal{B}^c$ and $x_0 \in \mathcal{B}\backslash\{0\}$, respectively.

**Case 1:** $x_0 \in \mathcal{B}^c$. In this case, for any $k < j_0$, $V(x) \geq \chi(\|u\|)$. From (13), we have

$$\Delta^+ V(x(k)) \leq -\mu(x(k)), \quad k < j_0.$$

By the proof of Lemma 3.5 in [27], there exists a $\mathcal{KL}$ function $\hat{\beta}$ such that

$$V(x(k)) \leq \hat{\beta}(V(x_0), k - k_0), \quad k < j_0.$$

From the definition of max-separable ISS Lyapunov function and the property of $\mathcal{K}$ functions,

$$\begin{aligned}
&\alpha_1(\max_{i\in\mathcal{I}_n}\{x_i\}) = \max_{i\in\mathcal{I}_n}\{\alpha_1(x_i)\} \leq \max_{i\in\mathcal{I}_n}\{V_i(x_i)\} = V(x) \\
&\leq \hat{\beta}(V(x_0), k - k_0) = \hat{\beta}(\alpha_2(\max_{i\in\mathcal{I}_n}\{x_{0i}\}), k - k_0), \\
&\quad k < j_0.
\end{aligned}$$

So,

$$\begin{aligned}
x(k) &\preceq \alpha_1^{-1} \circ \hat{\beta}(\alpha_2(\max_{i\in\mathcal{I}_n}\{x_{0i}\}), k - k_0)\mathbf{1}_n \\
&:= \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, k - k_0)\mathbf{1}_n, \quad k < j_0,
\end{aligned}$$

where $\beta(r,s) \in \mathcal{KL}$ can be known from Lemma 4.2 in [35].

When $k \geq j_0$, since $\Delta^+ V(x)$ is negative for $x(k)$ outside the set $\mathcal{B}$, we have $V(x(k)) \leq \chi(\|u\|)$. Thus, we get

$$x(k) \preceq \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, k - k_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n, \quad k \in \mathbb{N}_0.$$

**Case 2:** $x_0 \in \mathcal{B}\backslash\{0\}$. In this case, $k_0 = 0$. When $k > k_0$, following the proof of *Case 1.*, we obtain that

$$x(k) \preceq \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, k - k_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n.$$

When $k=k_0$, by the definition of the set $\mathcal{B}$ and the definition of $\gamma$, we obtain

$$\begin{aligned} x(k_0) \preceq & \gamma(\|u\|)\mathbf{1}_n \\ \preceq & \beta(\max_{i\in\mathcal{I}_n}\{x_{0i}\}, k-k_0)\mathbf{1}_n + \gamma(\|u\|)\mathbf{1}_n. \end{aligned}$$

In all, the system (2) is ISS. □

Consider the ISS of discrete-time nonlinear positive system (2) with $f(x,u)=Ax+u$, where $A\in\mathbb{R}^{n\times n}$. The system (2) reduces to the discrete-time linear positive system

$$x(k+1) = Ax(k) + u(k), \tag{16}$$

where the function $u:\mathbb{N}_0\to\mathbb{R}^n_+$ is the disturbance.

Using the max-separable ISS-Lyapunov function method shown in the proof of Theorem 2, we can get the following proposition, from which the ISS property of discrete-time linear positive system can be got just from the characteristics of the system matrix.

**Proposition 7:** For the discrete-time linear positive system (16), suppose that $A$ is asymptotically stable Metzler matrix, and further, there exists $\varepsilon_0>0$ such that $-(a_{ii}-1)\geq\sum_{j=1, j\neq i}^{n} a_{ij}\frac{v_j}{v_i}+\varepsilon_0$ for $i=1,\cdots,n$, then system (16) is ISS.

**Proof:** See Appendix D. □

Consider the discrete-time affine nonlinear positive system

$$x(k+1) = f(x(k)) + g(x(k))u(k), \tag{17}$$

where the function $u:\mathbb{N}_0\to\mathbb{R}_+$ is the disturbance.

Using the max-separable ISS-Lyapunov function method shown in the proof of Theorem 2, we can also get the following Proposition 8, which can be proved by combining the proof of Proposition 5 and 7. Here, the detailed proof of it is omitted.

**Proposition 8:** For the discrete-time nonlinear positive system (17), suppose that i) $f$ is cooperative, continuous on $\mathbb{R}_n$, continuously differentiable on $\mathbb{R}_n\backslash\{0\}$, and homogeneous of degree one; ii) $g$ is continuous and bounded by $M$ on $\mathbb{R}^n_+$; iii) there exists a vector $v\succ 0$ such that $f(v)-v\prec 0$, then the system (17) is ISS.

## 5. SIMULATION EXAMPLES

In this section, three examples are presented to demonstrate the effectiveness and usefulness of our main results.

**Example 1** (Linear transportation network [40])**:** Consider a dynamical system model which describe a transportation network connecting four buffers with disturbances,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = A\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}, \tag{18}$$

where

$$A = \begin{bmatrix} -1-l_{31} & l_{12} & 0 & 0\\ 0 & -l_{12}-l_{32} & l_{23} & 0\\ l_{31} & l_{32} & -l_{23}-l_{43} & l_{34}\\ 0 & 0 & l_{43} & -4-l_{34}\end{bmatrix}.$$

The states $x_1$, $x_2$, $x_3$, $x_4$ represent the contents of the buffers, $L=(l_{ij})\in\mathbb{R}^{4\times 4}_+$ is the transfer rate matrix, with the gain $l_{ij}$ determines the rate of transfer from buffer $j$ to buffer $i$. Such transfer between buffers is necessary to stabilize the system. Notice that the dynamics has the form $\dot{x}=Ax+u$ where $A$ is a Metzler matrix provided that every $l_{ij}$ is nonnegative, and $u$ is the disturbance. Hence, by Proposition 3, ISS is equivalent to there exists $\varepsilon_0>0$ and vector $\mathbf{v}_4\in\mathbb{R}^4_+$ such that $A\mathbf{v}_4+\varepsilon_0\mathbf{v}_4\preceq 0$. There exist many solutions for this inequality. Here, as an example, we take $\varepsilon_0=1$ and $\mathbf{v}_4=\mathbf{1}_4$, $L=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&1&0&1\\0&0&3&0\end{bmatrix}$, system (18) is input-to-state stable. Moreover, if we take $u=\mathbf{1}_4$, the simulation curves of $(x_1,x_2,x_3,x_4)^T$ with initial value $(1,1,1,1)^T$ are shown in Fig. 1. From Fig. 1, it can be seen that, the states are ultimately bounded but don't converge to zero. When $u\equiv 0$, the simulation curves of $(x_1,x_2,x_3,x_4)^T$ with initial value $(1,1,1,1)^T$ are shown

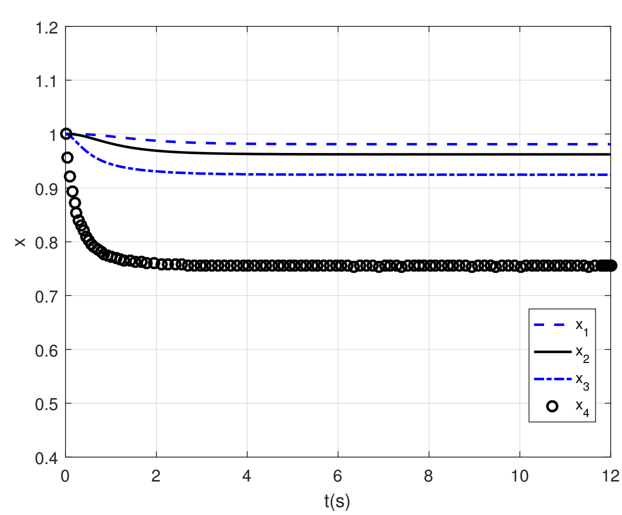


Fig. 1. Response of linear transportation network with disturbance.

in Fig. 2. From Fig. 2, it can be seen that, the states will converge to zero.

**Example 2:** Consider the nonlinear dynamical system given by (15) with

$$f(x)=\begin{bmatrix}-3x_1+6x_2-3\sqrt{x_1^2+x_2^2}\\ 2x_1-2x_2-\sqrt{x_1^2+x_2^2}\end{bmatrix},\quad g(x)=\begin{bmatrix}\sin(x_1)\\ \cos(x_2)\end{bmatrix},$$

$u$ is the disturbance input bounded by 1. From Definition 6 and Definition 7, $f$ is cooperative and homogeneous of degree one. $g$ is continuous and bounded by 1 on $\mathbb{R}^n_+$. There exists a vector $v=(1,1)^T$ such that $f(v)\prec 0$. From Proposition 5, the above system is ISS. When $u\equiv 1$, the simulation curves of $(x_1,x_2)^T$ with initial value $(1,2)^T$ are shown in Fig. 3. From Fig. 3, it can be seen that, the states are ultimately bounded but don't converge to zero. When $u\equiv 0$, the simulation curves of $(x_1,x_2)^T$ with initial value $(1,2)^T$ are shown in Fig. 4. From Fig. 4, it can be seen that, the states will converge to zero. In the above system, if

$$g(x)=\begin{bmatrix}\sqrt{x_1^2+x_2^2}\\ \sqrt{x_1^2+x_2^2}\end{bmatrix},$$

$g$ is homogeneous and order-preserving, Assumption 2 holds. From Proposition 6, for any given $u$ bounded by $\frac{\sqrt{2}}{2}$, there exists $\eta=(0.2426,0.4142)^T\succ 0$ such that $\frac{f_i(v)+g_i(v)\|u\|}{v_i}+\eta_i=0$ holds for any $i\in\{1,2\}$, where $v=(1,1)^T$, the system is asymptotically stable. When $u\equiv\frac{\sqrt{2}}{2}$, the simulation curves of $(x_1,x_2)^T$ with initial value $(1,2)^T$ are shown in Fig. 5. From Fig. 5, it can be seen that, the states will converge to zero.

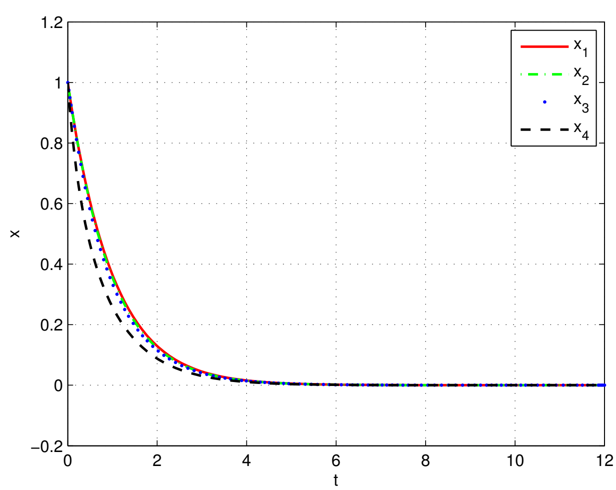


Fig. 2. Response of linear transportation network without disturbance.

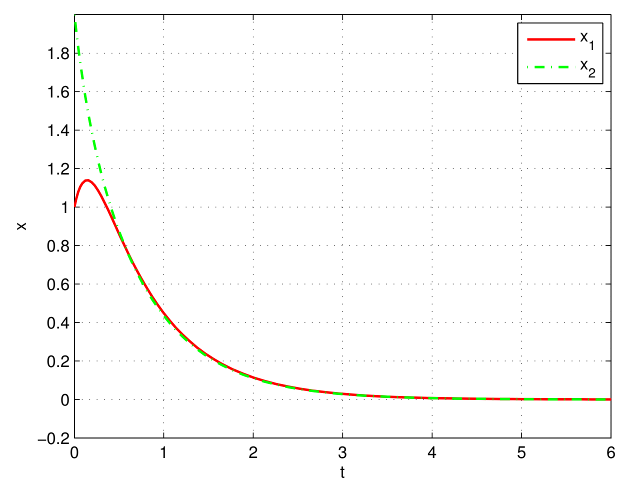


Fig. 4. Response of nonlinear affine system without disturbance input.

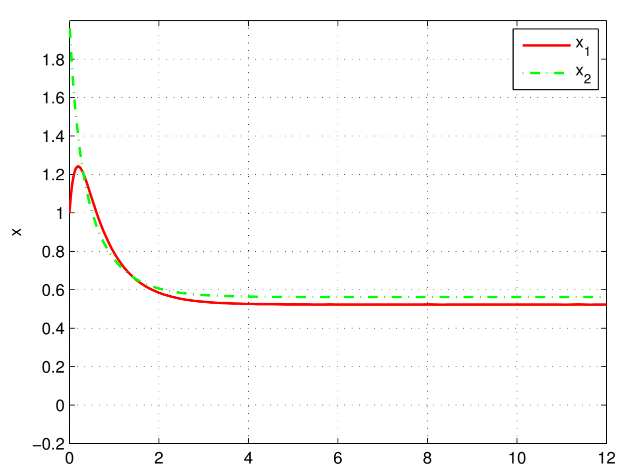


Fig. 3. Response of nonlinear affine system with disturbance input.

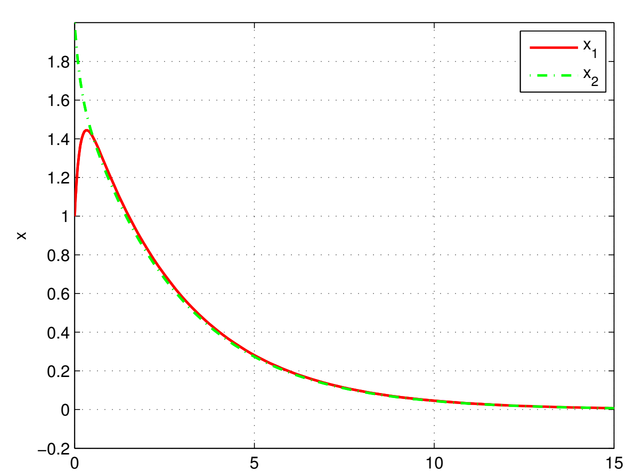


Fig. 5. Response of nonlinear affine system with disturbance input.

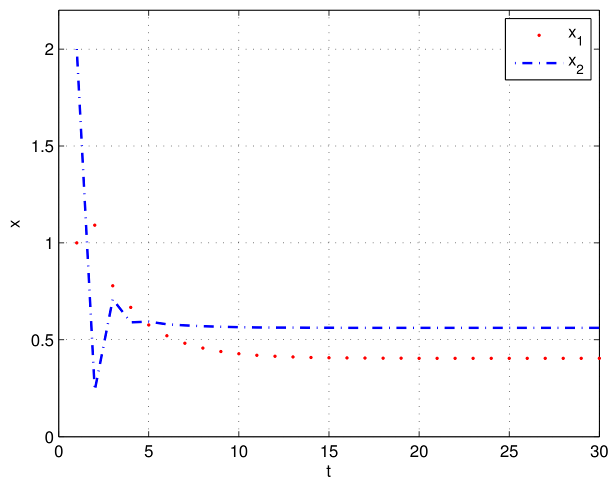


Fig. 6. Response of discrete-time nonlinear positive system with disturbance input.

**Example 3:** Consider the discrete-time nonlinear dynamical systems given by (17) with

$$f(x_1,x_2)=\begin{pmatrix} 0.3\sqrt{x_1^2+x_2^2} \\ 0.2\sqrt{x_1^2+x_2^2} \end{pmatrix},$$
$$g(x_1,x_2)=0.5\begin{pmatrix} \sin(x_1) \\ \cos(x_2) \end{pmatrix},$$

input disturbance $u(k)\equiv 1$. It is easy to verify that $f$ and $g$ satisfy conditions of Example 1. Especially, there exists a vector $v=(1,1)^T$ such that $f(v)-v\prec 0$. When $u\equiv 1$, the simulation curves of $(x_1,x_2)^T$ with initial value $(1,2)^T$ are shown in Fig. 6. From Fig. 6, it can be seen that, the system states are ultimately bounded but don't converge to zero.

## 6. CONCLUSION

In this paper, ISS analysis has been first done for continuous-time and discrete-time nonlinear positive systems. By introducing the max-separable ISS Lyapunov functions, the ISS criterions of continuous-time and discrete-time nonlinear positive systems are provided. Based on those, for linear positive systems and affine nonlinear homogeneous positive systems, some useful criterions are given. From them, the systems' ISS properties can be got only from the differential and algebraic characteristics of the systems. Simulation examples including an application example verify the validity of our results.

## APPENDIX A

### A.1. Proof of Proposition 3

**Proof:** Because $A$ is asymptotically stable Metzler matrix, there exists $v\succ 0$ such that $Av\prec 0$. Define function $V(x)$ in the form of

$$V(x)=\max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}.$$

From Proposition 2,

$$D^+V(x)=\max_{j\in\mathcal{J}(x)}\frac{1}{v_j}(A_jx+u_j).$$

Let the subscript $m$ denotes the index which make the upper-right Dini derivative of $V$ maximal.

$$\begin{aligned} D^+V(x)=&\max_{j\in\mathcal{J}(x)}\{\frac{1}{v_j}(A_jx+u_j)\}\\ =&\frac{1}{v_m}(A_mx+u_m)\\ =&a_{mm}\frac{x_m}{v_m}+\sum_{j=1,j\neq m}^{n}a_{mj}\frac{x_j}{v_m}+\frac{u_m}{v_m}\\ =&a_{mm}\frac{x_m}{v_m}+\sum_{j=1,j\neq m}^{n}a_{mj}\frac{v_j}{v_m}\frac{x_j}{v_j}+\frac{u_m}{v_m}\\ \leq&a_{mm}\frac{x_m}{v_m}+\sum_{j=1,j\neq m}^{n}a_{mj}\frac{v_j}{v_m}\frac{x_m}{v_m}+\frac{u_m}{v_m}\\ =&(a_{mm}+\sum_{j=1,j\neq m}^{n}a_{mj}\frac{v_j}{v_m})\frac{x_m}{v_m}+\frac{u_m}{v_m}\\ \leq&-\varepsilon_0\frac{x_m}{v_m}+\frac{u_m}{v_m}. \end{aligned}$$

If we let $\chi(\cdot)=\frac{1}{\varepsilon}\times\cdot$,

$$\|x\|_\infty^v\geq\|u\|_\infty^v\Rightarrow D^+V(x)\leq-(\varepsilon_0-\varepsilon)\frac{x_m}{v_m}=-(\varepsilon_0-\varepsilon)V.$$

It's easy to see that (8) holds. So, $V(x)=\max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}$ is a max-separable ISS Lyapunov function for positive system (14). Therefore, system (14) is ISS. □

## APPENDIX B

### B.1. Proof of Proposition 5

**Proof:** By Assumption 1, there exists a vector $v\succ 0$ such that $f(v)\prec 0$. Define function $V(x)$ in the form of

$$V(x)=\max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}.$$

From Proposition 2,

$$D^+V(x)=\max_{j\in\mathcal{J}(x)}\frac{1}{v_j}(f_j(x(t))+g_j(x(t))u).$$

Let the subscript $m$ denotes the index which make the upper-right Dini derivative of $V$ maximal. Then, by $x\preceq\|x\|_\infty^v v$,

$$D^+V(x)\leq\frac{1}{v_m}(f_m(x(t))+g_m(x(t))u)$$

$$\leq \frac{1}{v_m}(\|x\|_\infty^v f_m(v) + M\|u\|).$$

If we take $\chi(\cdot) = \frac{2M}{\max\limits_{i=1,\cdots,n}\{|f_i(v)|\}} \times \cdot$,

$$\|x\|_\infty^v \geq \frac{2M}{\max\limits_{i=1,\cdots,n}\{|f_i(v)|\}}\|u\|$$
$$\Rightarrow D^+V \leq \frac{f_m(v)}{2v_m}\|x\|_\infty^v = \frac{f_m(v)}{2v_m}V < 0.$$

Also, (8) holds. So, $V(x) = \max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}$ is a max-separable ISS-Lyapunov function for positive system (15). Therefore, the system (15) is ISS. □

## APPENDIX C

### C.1. Proof of Proposition 6

**Proof:** Define function $V(x)$ in the form of

$$V(x) = \max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}.$$

From Proposition 2,

$$D^+V(x) = \max_{j\in\mathcal{J}(x)}\frac{1}{v_j}(f_j(x(t)) + g_j(x(t))u).$$

Let the subscript $m$ denotes the index which make the upper-right Dini derivative of $V$ maximal. Then, by $x \preceq \|x\|_\infty^v v$,

$$\begin{aligned} D^+V(x) \leq& \frac{f_m(x(t)) + g_m(v)u}{v_m} \\ \leq& \frac{\|x\|_\infty^v(f_m(v) + g_m(v)u)}{v_m} \\ \leq& -\eta_m\|x\|_\infty^v \\ =& -\eta_m\frac{x_m}{v_m}. \end{aligned}$$

So,

$$\frac{x_m(t)}{v_m} \leq \frac{x_m(t_0)}{v_m}e^{-\eta_m(t-t_0)} \leq \frac{x_m(t_0)}{v_m}e^{-\eta_0(t-t_0)},$$
$$x(t) \preceq V(t_0)v_{\max}e^{-\eta_0(t-t_0)}\mathbf{1}_n, \quad \forall t \in [t_0,\infty),$$

where $v_{\max} = \max\{v_1,\cdots,v_n\}$. Therefore, the system (15) is asymptotically stable. □

## APPENDIX D

### D.1. Proof of Proposition 7

**Proof:** Because $A$ is asymptotically stable Metzler matrix, there exists $v \succ 0$ such that $Av \prec 0$. Define function $V(x)$ in the form of

$$V(x) = \max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}.$$

Let the subscript $m$ denotes the index which make the difference of $V$ maximal. Similar to the proof of Proposition 3,

$$\begin{aligned} V(Ax+u) - V(x) =& \frac{(Ax+u)_m}{v_m} - \frac{x_m}{v_m} \\ =& \frac{A_m x + u_m}{v_m} - \frac{x_m}{v_m} \\ \leq& (a_{mm} - 1 + \sum_{j=1,j\neq m}^{n} a_{mj}\frac{v_j}{v_m})\frac{x_m}{v_m} + \frac{u_m}{v_m} \\ \leq& -\varepsilon_0\frac{x_m}{v_m} + \frac{u_m}{v_m}. \end{aligned}$$

If we let $\chi(\cdot) = \frac{1}{\varepsilon} \times \cdot$,

$$\|x\|_\infty^v \geq \|u\|_\infty^v \Rightarrow \Delta^+V(x) \leq -(\varepsilon_0 - \varepsilon)\frac{x_m}{v_m} = -(\varepsilon_0 - \varepsilon)V.$$

Also, (12) holds. So, $V(x) = \max_{i\in\mathcal{I}_n}\{\frac{x_i}{v_i}\}$ is a max-separable ISS Lyapunov function for positive system (14). Therefore, the system (14) is ISS. □

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**Yan Zhao** received her B.S. degree from the University of Jinan, China, in 2002 and her M.S. degree from the Qufu Normal University, China, in 2005. She is currently a doctoral student of the Qufu Normal University and a teacher of the University of Jinan. Her research interests are in the stability theory and nonlinear systems.

**Fanwei Meng** received his B.S. and M.S. degrees from the School of Mathematical Sciences, Qufu Normal University, China, in 1983 and 1988, respectively. He received his Ph.D. degree from China Academy of Engineering Physics in 2003. He is a professor with the School of Mathematical Science, Qufu Normal University since 1995, and the director of School of Mathematical Science, Qufu Normal University since 2009. His research interests include stability theory and qualitative theory of order differential equation, Hamiltonian systems and Fractional order differential equation.