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Global stabilization of the linearized three-axis axisymmetric spacecraft attitude control system by bounded linear feedback ☆

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ABSTRACT

In this paper, the three-axis attitude stabilization of the axisymmetric spacecraft with bounded inputs is studied. By constructing some novel state transformations, saturated linear state feedback controllers are constructed for the considered attitude control system. By constructing suitable quadratic plus integral Lyapunov functions, globally asymptotic stability of the closed-loop systems is proved if the feedback gain parameters satisfy some explicit conditions. By solving some min–max optimization problems, a global optimal feedback gain for the underactuated attitude stabilization system is proposed such that the convergence rate of the linearized closed-loop system is maximized. Numerical simulations show the effectiveness of the proposed approaches.

1. Introduction

Attitude control systems play fundamental roles in the operation of satellite mission since they constitute a mandatory characteristic for both the survival of satellites and the satisfactory achievement of mission goals. Typical attitude stabilization systems include passive attitude stabilized systems [5,15,35] and active three-axis controlled systems [13,16,20,32,36,39]. It has been realized that the most accurate designs normally include momentum wheels and/or reaction wheels because they allow continuous and smooth control with the lowest possible parasitic disturbing torques [23].

The attitude stabilization problem of axisymmetric spacecraft has been investigated in the literature in the past years (see, [1,27,28,41] and the references therein). The angular velocity equations of an axisymmetric spacecraft were globally asymptotically stabilized in [1] by means of a linear feedback when two control torques act on the body. Optimal control laws for axisymmetric spacecraft with restrictions on initial velocities were presented in [28]. Nonlinear H_∞ control designs with axisymmetric spacecraft control were proposed in [41]. It is noted that the gravity gradient moment and/or actuator saturation were not considered in the aforementioned references.

Saturation nonlinearity exists in every practical control system and makes the overall system inherently nonlinear (see [12,14,17,31,38,40]). Take the spacecraft attitude control system for example. Typical actuators (such as magnetorquers, reaction wheels, or control moment gyros) are subject to saturation due to the physical limitation and energy consumption. Therefore, conventional attitude control schemes may result in control signals beyond the saturation level which can lead to serious differences between the commanded input signal and the actual control effort, and is thus a source of performance degradation or, even worse, instability of the closed-loop system. Therefore, the problem of spacecraft attitude control with actuator saturation has been addressed in the literature [6–8,11,19,21,29,39]. For example, two simple PD controllers were proposed in [24] to address the global asymptotic regulation of rigid spacecraft subject to actuator saturation, the problem of satellite attitude control with actuator saturation was addressed in [12,6,8], a novel observer-controller control scheme was proposed in [12] to solve the output feedback attitude control of a rigid body with bounded input, a simple nonlinear proportional-derivative-type (PD-type) saturated finite-time controller was designed in [6], the attitude tracking of a rigid body by using a quaternion description and global finite time attitude controllers were designed with three types of measurements in [8], discontinuous feedback controllers were designed in [29] to stabilize the attitude of an axisymmetric spacecraft with bounded control, and explicit saturated linear feedback controllers for attitude stabilization were constructed in [39] and global stability for the linearized system was guaranteed.

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Due to the scarce onboard resources and the complex working environment, particularly, for CubeSat-class nanosatellites, the attitude stabilizing controllers must be as simple as possible. Then linear feedback should be the best choice. Noting that in general a linear global stabilizing controller exists if the open-loop system is neutral stable, which is not always satisfied in practice. It is thus challenging to design linear globally stabilizing controllers for the axisymmetric spacecraft since the corresponding open-loop system is not Lyapunov stable [39].

Motivated by the existing results on this topic, in this paper we consider the problem of three-axis attitude control of axisymmetric spacecraft with bounded inputs. By taking the gravity-gradient torques effect into account, a novel state variable transformation on the linearized attitude equation is used to construct saturated linear state feedback controllers for the considered attitude control system. It is shown by constructing explicit Lyapunov function that the proposed controllers guarantee the global stability of the linearized closed-loop systems when the parameters in the feedback gain satisfy some explicit conditions. The optimal linear feedback gain (such that the convergence rate of the linearized closed-loop system is maximized) for the attitude control system is also obtained by studying some min–max optimization problems. We mention that the proposed control approaches are also applicable to nanosatellites since the proposed linear controllers are easy to implement and robust with respect to uncertainties. Numerical simulations show the effectiveness of the proposed approaches.

The remainder of this paper is organized as follows. In Section 2, the model of the attitude control system is introduced. The main results regarding global stabilization of the roll-yaw loop and the pitch loop are then respectively proposed in Sections 3 and 4. A numerical simulation is given to demonstrate the effectiveness of the proposed control law in Section 5. Finally, Section 6 concludes the paper.

2. Model of the attitude control

The attitude motion of a rigid spacecraft can be described in the following reference frames [33,34] (see Fig. 1):

1. Geocentric Equatorial Frame F_i, where the X axis points in the vernal equinox direction, the X–Y plane is the Earth's equatorial plane, and the Z axis coincides with the Earth's axis of rotation and points northward.
2. Orbital Frame F_o, where the origin at the center of mass of the spacecraft, x_o being along the orbit direction, y_o being perpendicular to the orbit plane and z_o being in the nadir direction.
3. Spacecraft-fixed Body Frame F_b, where the origin at the center of mass of the spacecraft.

It follows that if the attitude of the spacecraft is the identity, the body coordinates x_b–y_b–z_b coincide exactly with the orbital coordinates x_o–y_o–z_o [23].

The attitude matrix and attitude kinematics and dynamics can be described respectively as [33]

$$C = \left(q_4^2 - q_v^{\mathrm T} q_v\right) I_3 + 2 q_v q_v^{\mathrm T} - 2 q_4 q_v^{\times} \triangleq \begin{bmatrix} c_x & c_y & c_z \end{bmatrix},$$

$$\dot q = \frac{1}{2}\begin{bmatrix} 0 & \omega_{rz} & -\omega_{ry} & \omega_{rx} \\ -\omega_{rz} & 0 & \omega_{rx} & \omega_{ry} \\ \omega_{ry} & -\omega_{rx} & 0 & \omega_{rz} \\ -\omega_{rx} & -\omega_{ry} & -\omega_{rz} & 0 \end{bmatrix} q, \tag{1}$$

and

$$J\dot\omega + \omega \times J\omega = T_g + T_c, \tag{2}$$

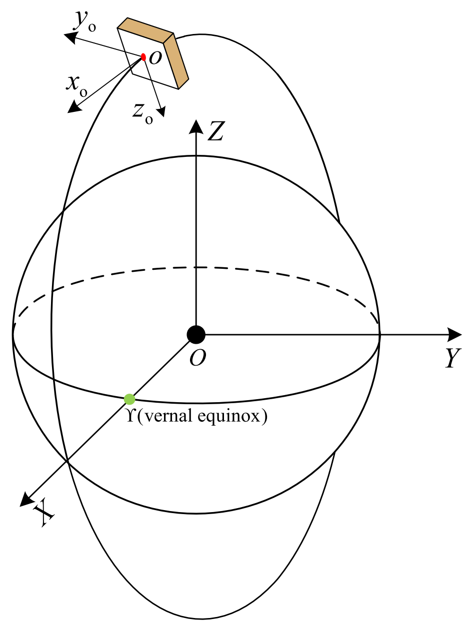

Fig. 1. The geocentric equatorial frame X–Y–Z and the orbital reference frame x_o–y_o–z_o.

where I_3 is the 3×3 identity matrix, $q = [q_v^{\mathrm T}, q_4]^{\mathrm T}$ is the attitude quaternion, $q_v = [q_1, q_2, q_3]^{\mathrm T}$ is its vector part, $\omega_r = \left[\omega_{rx}, \omega_{ry}, \omega_{rz}\right]^{\mathrm T}$ is the (relative) angular velocity of the body frame F_b relative to the orbital frame F_o, [33,34]

$$T_g = \left[T_{gx}, T_{gy}, T_{gz}\right]^{\mathrm T} = 3\omega_0^2 c_z \times J c_z$$

is the gravity-gradient torque vector, in which ω_0 is the orbital rate, the vector $T_c = \left[T_x, T_y, T_z\right]^{\mathrm T}$ denotes the control torque, $J = \operatorname{diag}\left\{J_x, J_y, J_z\right\}$ is the inertia matrix of the spacecraft, and $\omega = \left[\omega_x, \omega_y, \omega_z\right]^{\mathrm T}$ is the angular velocity of the body frame F_b relative to the geocentric inertial frame F_i, and $q_v^{\times}$ is the corresponding cross-product operation defined as [33,34]

$$q_v^{\times} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}.$$

As mentioned in Section 1, this paper focuses on the three-axis attitude control for an axisymmetric spacecraft in a circular orbit. We thus assume that the inertia matrix is symmetric and the axis of symmetric is the minor principal axis, namely,

$$J_x = J_y > J_z. \tag{3}$$

On the other hand, the control input of the attitude control system is subject to saturation, namely,

$$|T_{ci}| \le \varpi_i, \ i = x, y, z,$$

in which $\varpi_i > 0, i \in \{x, y, z\}$, denote the maximal allowable value of controls in the i-axis.

Linearizing the attitude equations (1) and (2) including the gravity-gradient torque at the equilibrium $q^* = [0, 0, 0, 1]^{\mathrm T}$ and $\omega^* = [0, -\omega_0, 0]^{\mathrm T}$ gives [33,34]

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \omega_x + 2\omega_0 q_3 \\ \omega_y + \omega_0 \\ \omega_z - 2\omega_0 q_1 \end{bmatrix},$$

$$\begin{bmatrix} \dot\omega_x \\ \dot\omega_y \\ \dot\omega_z \end{bmatrix} = \begin{bmatrix} -8\omega_0^2 q_1 \sigma_1 - 2\omega_0 \sigma_1 \dot q_3 \\ -6\omega_0^2 \sigma_1 q_2 \\ 0 \end{bmatrix} + J^{-1}\operatorname{sat}_{\varpi}(T_c),$$

namely,

$$\begin{cases} \ddot{q}_1 = -4\sigma_1\omega_0 q_1 + (1-\sigma_1)\,\omega_0\dot{q}_3 + \frac{1}{2J_x}\mathrm{sat}_{\varpi_x}(T_{cx}), \\ \ddot{q}_2 = -3\omega_0^2\sigma_1 q_2 + \frac{1}{2J_y}\mathrm{sat}_{\varpi_y}(T_{cy}), \\ \ddot{q}_3 = -\omega_0\dot{q}_1 + \frac{1}{2J_z}\mathrm{sat}_{\varpi_x}(T_{cz}), \end{cases} \quad (4)$$

where ω_0 is the orbital rate, $\sigma_1 = \frac{J_x - J_z}{J_x}$, and $\mathrm{sat}_{\varpi}(\cdot)$ is the vector-valued saturation function with the saturation level indicated by the vector $\varpi = [\varpi_x, \varpi_y, \varpi_z]^{\mathrm{T}}$, namely,

$$\mathrm{sat}_{\varpi}(T_c) = \left[\ \mathrm{sat}_{\varpi_x}(T_{cx}) \quad \mathrm{sat}_{\varpi_y}(T_{cy}) \quad \mathrm{sat}_{\varpi_z}(T_{cz})\ \right]^{\mathrm{T}},$$

in which $\mathrm{sat}_{\varpi_i}(T_{ci}) = \mathrm{sign}(T_{ci})\min\{|T_{ci}|, \varpi_i\}, i \in \{x, y, z\}$. It follows that $\mathrm{sat}_{\varpi}(a) = \varpi\,\mathrm{sat}\left(\frac{a}{\varpi}\right)$ by defining $\mathrm{sat}(a) = \mathrm{sat}_1(a)$. Here, we should note that $\sigma_1 \in (0, 1)$ from (3). It is clear from the attitude control system (4) that the pitch equation can be decoupled from the roll-yaw equation. Finally, we mention that the roll angle ϕ, the pitch angle θ, and the yaw angle ψ are related with the quaternion q as $q \approx [\frac{1}{2}\phi, \frac{1}{2}\theta, \frac{1}{2}\psi, 1]^{\mathrm{T}}$ (see, for example, [20] and [23]).

The attitude stabilization problem can be described as transforming the state vector $X = [q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3]^{\mathrm{T}}$ from nonzero initial value $X(t_0)$ to the terminal value $X(t_f) = 0$, where t_f is the settling time. The aim of the remaining part of this paper is to designing a globally stabilizing controller for system (4) in the sense that $X(t_0)$ can be arbitrarily chosen. Particularly, we are interested in designing linear feedback controllers since they are relatively easy to be implemented in practice, and, as well known from the literature, they are robust with respect to parameter perturbations and uncertainties [39].

3. Global stabilization of the roll-yaw loop

In this section, we consider the global stabilizing controller design for the roll-yaw loop. Let the state vector and input vector be respectively defined by

$$\chi = \begin{bmatrix} q_1 & q_3 & \dot{q}_1 & \dot{q}_3 \end{bmatrix}^{\mathrm{T}}, u = \begin{bmatrix} \frac{1}{\varpi_x}T_{cx} & \frac{1}{\varpi_z}T_{cz} \end{bmatrix}^{\mathrm{T}}.$$

Then the roll-yaw equation can be written as

$$\dot{\chi} = A\chi + B\mathrm{sat}(u), \quad (5)$$

where A and B are constant matrices given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4\sigma_1\omega_0^2 & 0 & 0 & (1-\sigma_1)\,\omega_0 \\ 0 & 0 & -\omega_0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\varpi_x}{2J_x} & 0 \\ 0 & \frac{\varpi_z}{2J_z} \end{bmatrix} \triangleq \begin{bmatrix} b_1 & b_2 \end{bmatrix}.$$

Since we are seeking global stabilizing controller for system (5), we first need to check whether such a controller exists. It is well known that a linear system can be stabilized globally by bounded controls if and only if the system is asymptotically null controllable with bounded controls (ANCBC), namely, it is stabilizable in the ordinary sense and all the eigenvalues of A are on the closed-left half plane [25]. The matrix pair (A, B) is controllable since

$$\det\begin{bmatrix} b_2 & Ab_2 & A^2b_2 & A^3b_2 \end{bmatrix} = -\frac{4\sigma_1(\sigma_1-1)^2\omega_0^4\varpi_z^4}{J_z^4}, \quad (6)$$

which is nonzero, namely, (A, b_2) is controllable. On the other hand, we can compute

$$\lambda(A) = \left\{0, 0, \pm\mathrm{i}\sqrt{(3\sigma_1+1)}\,\omega_0\right\},$$

which implies that all the eigenvalues of A are on the imaginary axis. Hence (A, B) (actually, (A, b_2)) is ANCBC. Therefore, a globally stabilizing controller for system (5) exists.

Even a globally stabilizing controller for (5) exits, it may possess nonlinear from [25]. Existing results show that a linear global stabilizing controller exists if A is neutral stable, namely, all the eigenvalues of A are simple. However, this is not satisfied for system (5) since A possesses a zero eigenvalue with multiplicity two. In fact, the Jordan form J_A of A is given by

$$J_A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \mathrm{i}\sqrt{(3\sigma_1+1)}\,\omega_0 & \\ & & & -\mathrm{i}\sqrt{(3\sigma_1+1)}\,\omega_0 \end{bmatrix}.$$

As a result, global stabilizing linear controller can not be obtained by using positive definite solutions to Lyapunov equations associated with the open-loop system [19,39]. Generally, for a single-input planar linear system having only zero eigenvalues with multiplicity two, a linear global stabilizing controller exists [25]. However, for a general multiple-inputs linear system having both zero eigenvalues with multiplicity two and nonzero imaginary eigenvalues, it is not clear whether a linear global stabilizing controller exists [22]. One of the main contribution of this paper is that we can give a positive answer to this problem associated with system (5) and a constructive solution is provided, as shown below.

Theorem 1. *The closed-loop system consisting of the linear system (5) and the linear controller*

$$u = F\chi,$$

$$F = 2J_z\begin{bmatrix} -\frac{4\sigma_1\omega_0^2}{(1-\sigma_1)^2\varpi_x}k_2 & 0 & -\frac{\omega_0}{\varpi_x(1-\sigma_1)}k_1 & \frac{\omega_0}{\varpi_x(1-\sigma_1)}k_2 \\ -\frac{\omega_0^2}{\varpi_z}k_4 & -\frac{\omega_0^2(3\sigma_1+1)}{\varpi_z}k_3 & \frac{\omega_0}{\varpi_z}(k_3-k_5) & -\frac{\omega_0}{\varpi_z}k_4 \end{bmatrix}, \quad (7)$$

is globally asymptotically stable and locally exponentially stable, where $k_1 > 0, k_2 \geq 0, k_3 > 0$ and $k_5 > 0$ are any scalars and k_4 satisfies

$$k_4 > \frac{\left((3\sigma_1+1)k_5k_2^2 + \left((\sigma_1-1)^2k_1^2 + (3\sigma_1+1)k_2^2\right)(k_3-k_5)\right)^2}{4\left((\sigma_1-1)^2k_1^2 + (3\sigma_1+1)k_2^2\right)(1-\sigma_1+k_2)(3\sigma_1+1)k_1k_5}. \quad (8)$$

Proof. To perform the controller design and stability analysis of the closed-loop system, a state transformation is useful. It can be verified by direct multiplication that the matrix

$$T = 2J_z\begin{bmatrix} 0 & -\frac{\omega_0^2(3\sigma_1+1)}{\varpi_z} & \frac{\omega_0}{\varpi_z} & 0 \\ \frac{\omega_0^2}{\varpi_z} & 0 & 0 & \frac{\omega_0}{\varpi_z} \\ 0 & 0 & \frac{\omega_0}{\varpi_z} & 0 \\ \frac{4\sigma_1\omega_0^2}{(\sigma_1-1)\varpi_z} & 0 & 0 & \frac{\omega_0}{\varpi_z} \end{bmatrix}, \quad (9)$$

is nonsingular since

$$\det(T) = -\frac{16(3\sigma_1+1)^2J_z^4\omega_0^6}{(\sigma_1-1)\varpi_z^4} \neq 0.$$

We then make the transformation of state variables $x = T\chi$ such that the roll-yaw equation (5) is transformed into

$$\dot{x} = \omega_0 A_0 x + \omega_0 B_0 \text{sat}(u), \tag{10}$$

where $(A_0, B_0) = \frac{1}{\omega_0}\left(TAT^{-1}, TB\right)$ is independent of ω_0 and are given by

$$A_0 = \begin{bmatrix} 0 & -4\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sigma_1 \\ 0 & 0 & \frac{3\sigma_1+1}{\sigma_1-1} & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} \mu & 0 \\ 0 & 1 \\ \mu & 0 \\ 0 & 1 \end{bmatrix}, \tag{11}$$

in which

$$\mu = \frac{\varpi_x}{\varpi_z}(1-\sigma_1) \geq 0.$$

In the following we want to find linear state feedback controllers of the form

$$u = F_0 x, \; F_0 = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix} \triangleq \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{12}$$

in which the parameters $f_{ij}, i \in \{1,2\}, j \in \{1,2,3,4\}$ are to be specified, such that system (10) is globally asymptotically stabilized. To this end, we consider the following Lyapunov function candidate [30,40]:

$$V(x) = x^{\mathrm{T}} P_0 x + 2\rho_1 \int_0^{f_1 x} \text{sat}(s)\,\mathrm{d}s + 2\rho_2 \int_0^{f_2 x} \text{sat}(s)\,\mathrm{d}s, \tag{13}$$

where P_0 is a positive semi-definite matrix satisfying the Lyapunov matrix equation

$$A_0^{\mathrm{T}} P_0 + P_0 A_0 = 0,$$

and the parameters $\rho_1 \geq 0$ and $\rho_2 > 0$ are to be specified. It can be shown that P_0 takes the form

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \frac{3\sigma_1+1}{(\sigma_1-1)^2}\alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_1 \end{bmatrix},$$

where the parameters $\alpha_i, i = 1, 2$, are any positive constants.
The time-derivative of $V(x)$ along the trajectories of the closed-loop system consisting of (10) and (12) can be computed as

$$\begin{aligned}
\dot{V}(x) &= 2\dot{x}^{\mathrm{T}} P_0 x + 2\rho_1 \text{sat}(f_1 x) f_1 \dot{x} + 2\rho_2 \text{sat}(f_2 x) f_2 \dot{x} \\
&= 2\dot{x}^{\mathrm{T}} P_0 x + 2\dot{x}^{\mathrm{T}} F_0^{\mathrm{T}} D_0 \text{sat}(u) \\
&= \omega_0 x^{\mathrm{T}} \left(A_0^{\mathrm{T}} P_0 + P_0 A_0\right) x + 2\omega_0 \text{sat}^{\mathrm{T}}(u) B_0^{\mathrm{T}} P_0 x \\
&\quad + 2\omega_0 x^{\mathrm{T}} A_0^{\mathrm{T}} F_0^{\mathrm{T}} D_0 \text{sat}(u) \\
&\quad + \omega_0 \text{sat}^{\mathrm{T}}(u) \left(D_0 F_0 B_0 + B_0^{\mathrm{T}} F_0^{\mathrm{T}} D_0^{\mathrm{T}}\right) \text{sat}(u) \\
&\leq 2\omega_0 \text{sat}^{\mathrm{T}}(u) \left(B_0^{\mathrm{T}} P_0 + D_0 F_0 A_0\right) x \\
&\quad + \omega_0 \text{sat}^{\mathrm{T}}(u) \left(D_0 F_0 B_0 + B_0^{\mathrm{T}} F_0^{\mathrm{T}} D_0^{\mathrm{T}}\right) \text{sat}(u) \\
&\quad + 2\omega_0 \text{sat}^{\mathrm{T}}(u) T_0 (u - \text{sat}(u)) \\
&= 2\omega_0 \text{sat}^{\mathrm{T}}(u) \left(B_0^{\mathrm{T}} P_0 + D_0 F_0 A_0 + T_0 F_0\right) x \\
&\quad + \omega_0 \text{sat}^{\mathrm{T}}(u) \left(D_0 F_0 B_0 + B_0^{\mathrm{T}} F_0^{\mathrm{T}} D_0^{\mathrm{T}} - 2T_0\right) \text{sat}(u) \\
&\triangleq 2\omega_0 \text{sat}^{\mathrm{T}}(u) R_0 x - \omega_0 \text{sat}^{\mathrm{T}}(u) S_0 \text{sat}(u),
\end{aligned} \tag{14}$$

where

$$D_0 = \text{diag}\{\rho_1, \rho_2\},$$
$$R_0 = B_0^{\mathrm{T}} P_0 + D_0 F_0 A_0 + T_0 F_0,$$
$$S_0 = 2T_0 - \left(D_0 F_0 B_0 + B_0^{\mathrm{T}} F_0^{\mathrm{T}} D_0^{\mathrm{T}}\right),$$

and the inequality [26]

$$2\text{sat}^{\mathrm{T}}(u) T_0 (u - \text{sat}(u)) \geq 0,$$

has been used. Here T_0 is any positive semi-definite diagonal matrix. Suppose that there exists a P_0 satisfying $R_0 = 0$ and $S_0 > 0$. Then inequality (14) can be continued as

$$\dot{V}(x) \leq -\omega_0 \text{sat}^{\mathrm{T}}(F_0 x) S_0 \text{sat}(F_0 x). \tag{15}$$

Let $T_0 = \text{diag}\{1, 0\}$. Then

$$R_0 = \begin{bmatrix} f_{11} & -4\sigma_1\rho_1 f_{11} + f_{12} & \gamma_{13} & (1-\sigma_1)\rho_1 f_{13} + f_{14} \\ 0 & \alpha_2 - 4\sigma_1\rho_2 f_{21} & \frac{3\sigma_1+1}{\sigma_1-1}\rho_2 f_{24} & \alpha_1 + (1-\sigma_1)\rho_2 f_{23} \end{bmatrix},$$

where

$$\gamma_{13} = \frac{3\sigma_1+1}{(\sigma_1-1)^2}\mu\alpha_1 + \frac{3\sigma_1+1}{\sigma_1-1}\rho_1 f_{14} + f_{13}.$$

Consequently, $R_0 = 0$ if and only if the following five equations are satisfied

$$\begin{cases} 0 = f_{11} = f_{12} = f_{24}, \\ 0 = \gamma_{13}, \\ 0 = (1-\sigma_1)\rho_1 f_{13} + f_{14}, \\ 0 = \alpha_2 - 4\sigma_1\rho_2 f_{21}, \\ 0 = \alpha_1 + (1-\sigma_1)\rho_2 f_{23}. \end{cases} \tag{16}$$

Denote

$$\rho_1 = \frac{k_2}{(1-\sigma_1)k_1},$$
$$\rho_2 = \frac{(\sigma_1-1)^2 k_1^2 + (3\sigma_1+1)k_2^2}{\mu^2(1-\sigma_1)(3\sigma_1+1)k_1 k_5},$$
$$\alpha_1 = (1-\sigma_1)k_5\rho_2,$$
$$\alpha_2 = 4\sigma_1 k_3 \rho_2,$$

in which $k_1 > 0$, $k_2 \geq 0$, $k_3 > 0$, and $k_5 > 0$ are any scalars. Then we solve (16) to get $f_{13} = -\frac{k_1}{\mu}$, $f_{14} = \frac{k_2}{\mu}$, $f_{21} = k_3$, and $f_{23} = -k_5$. Let $f_{22} = -k_4$. Then we have

$$S_0 = \begin{bmatrix} 2 + 2\rho_1 k_1 & -\left(\frac{\rho_1 k_2}{\mu} + \mu\rho_2(k_3 - k_5)\right) \\ -\left(\frac{\rho_1 k_2}{\mu} + \mu\rho_2(k_3 - k_5)\right) & 2\rho_2 k_4 \end{bmatrix}.$$

Therefore, S_0 is positive definite if and only if (8) is satisfied. Hence, it can be verified that P_0 satisfies $R_0 = 0$ and $S_0 > 0$, if $k_1 > 0, k_2 \geq 0, k_3 > 0$ and $k_5 > 0$ are any scalars, and k_4 satisfies (8). So the gain matrix F_0 in the controller (12) can be expressed as

$$F_0 = \begin{bmatrix} 0 & 0 & -\frac{k_1}{\mu} & \frac{k_2}{\mu} \\ k_3 & -k_4 & -k_5 & 0 \end{bmatrix}. \tag{17}$$

Now we show that the Lyapunov function candidate (13) is positive definite. Clearly, $V(x)$ is positive semi-definite and $V(x) = 0$ if and only if

$$x^{\mathrm{T}} P_0 x = 0, \; \rho_1 x^{\mathrm{T}} f_1^{\mathrm{T}} f_1 x = 0, \; \rho_2 x^{\mathrm{T}} f_2^{\mathrm{T}} f_2 x = 0. \tag{18}$$

The leading principal minors of the matrix

$$\begin{aligned} P_0 + F_0^{\mathrm{T}} F_0 &= P_0 + f_1^{\mathrm{T}} f_1 + f_2^{\mathrm{T}} f_2 \\ &= \begin{bmatrix} k_3^2 & -k_3 k_4 & -k_3 k_5 & 0 \\ -k_3 k_4 & 4\sigma_1 k_3 \rho_2 + k_4^2 & k_4 k_5 & 0 \\ -k_3 k_5 & k_4 k_5 & \frac{(3\sigma_1+1)k_5}{(1-\sigma_1)}\rho_2 + \frac{k_1^2}{\mu^2} + k_5^2 & -\frac{k_1 k_2}{\mu^2} \\ 0 & 0 & -\frac{k_1 k_2}{\mu^2} & (1-\sigma_1)k_5\rho_2 + \frac{k_2^2}{\mu^2} \end{bmatrix}, \end{aligned}$$

are given by

$$\begin{cases} \Delta_1 = k_3^2 > 0, \\ \Delta_2 = 4\sigma_1 \rho_2 k_3^3 > 0, \\ \Delta_3 = 4\sigma_1 \rho_2 k_3^3 \left(\frac{3\sigma_1+1}{1-\sigma_1} \rho_2 k_5 + \frac{1}{\mu^2} k_1^2 \right) > 0, \\ \Delta_4 = 4\sigma_1 \rho_2^2 k_3^3 k_5 \left((3\sigma_1 + 1) \rho_2 k_5 + \frac{3\sigma_1+1}{\mu^2 (1-\sigma_1)} k_2^2 + \frac{1-\sigma_1}{\mu^2} k_1^2 \right) > 0. \end{cases}$$

It can be concluded that $P_0 + F_0^{\mathrm{T}} F_0 > 0$. Therefore, as $\rho_1 > 0$ and $\rho_2 > 0$, the unique vector x satisfying (18) is zero, namely, $V(x)$ is positive definite. Similarly, if $k_2 = 0$, it can also be shown that $V(x)$ is positive definite since $P_0 + f_2^{\mathrm{T}} f_2$ is positive definite, as the leading principal minors of $P_0 + f_2^{\mathrm{T}} f_2$ are given by

$$\begin{cases} \Delta_1 = k_3^2 > 0, \\ \Delta_2 = \frac{4\sigma_1 (1-\sigma_1)}{\mu^2 (3\sigma_1+1) k_5} k_1 k_3^3 > 0, \\ \Delta_3 = \frac{4\sigma_1 (1-\sigma_1)}{\mu^4 (3\sigma_1+1) k_5} k_1^2 k_3^3 > 0, \\ \Delta_4 = \frac{4\sigma_1 (1-\sigma_1)^3}{\mu^6 (3\sigma_1+1)^2 k_5} k_1^3 k_3^3 > 0. \end{cases}$$

By the LaSalle invariant principle [10], it follows from (15) that the state converges to the set $\Sigma = \{x | F_0 x = 0\}$ eventually. Noting that in the set Σ, the closed-loop system becomes $\dot{x} = \omega_0 A_0 x$. As

$$\operatorname{rank} \begin{bmatrix} F_0 \\ F_0 A_0 \\ F_0 A_0^2 \\ F_0 A_0^3 \end{bmatrix} = 4,$$

namely, the matrix pair (A_0, F_0) is observable for any $\mu \geq 0$, the only element in the set Σ is 0 and the stability of system (10) is guaranteed. This proves globally asymptotic stability of the closed-loop system. The locally exponential stability follows from the fact that the linearized closed-loop system

$$\dot{x} = \omega_0 (A_0 + B_0 F_0) x = \omega_0 A_{0c} x,$$

is asymptotically stable, namely, A_{0c} is Hurwitz. Finally, the proof is finished by noting that $F = F_0 T$. □

From Theorem 1 and (6), we immediately obtain the following corollary.

Corollary 1. *The closed-loop system consisting of the following linear system with a single input*

$$\dot{\chi} = A\chi + b_2 \mathrm{sat}(u_2), \tag{19}$$

and the following linear state feedback

$$u_2 = f\chi, \quad f = 2J_z \left[-\frac{\omega_0^2}{\varpi_z} k_4 \quad -\frac{\omega_0^2 (3\sigma_1+1)}{\varpi_z} k_3 \quad \frac{\omega_0}{\varpi_z} (k_3 - k_5) \quad -\frac{\omega_0}{\varpi_z} k_4 \right], \tag{20}$$

is globally asymptotically stable and locally exponentially stable, where $k_3 > 0$, $k_4 > 0$ and $k_5 > 0$ are any positive scalars.

Proof. Consider the state of transformation $x = T\chi$, where T is given by (9). Then system (19) can be written as

$$\dot{x} = \omega_0 A_0 x + \omega_0 b_0 \mathrm{sat}(u_2), \tag{21}$$

where $b_0 = [0, 1, 0, 1]^{\mathrm{T}}$ is the second column of B_0 defined in (11). Let $k_1 = \mu^2 l_1$, $k_2 = \mu^2 l_2$, in which $l_1 > 0$ and $l_2 \geq 0$. Then the gain matrix F_0 in the controller (12) can be also expressed as

$$F_0 = \begin{bmatrix} 0 & 0 & -\mu l_1 & \mu l_2 \\ k_3 & -k_4 & -k_5 & 0 \end{bmatrix}, \tag{22}$$

and the leading principal minors of $P_0 + f_2^{\mathrm{T}} f_2$ are given by

$$\begin{cases} \Delta_1 = k_3^2 > 0, \\ \Delta_2 = \frac{4\sigma_1 (1-\sigma_1)}{(3\sigma_1+1) k_5} l_1 k_3^3 > 0, \\ \Delta_3 = \frac{4\sigma_1 (1-\sigma_1)}{(3\sigma_1+1) k_5} l_1^2 k_3^3 > 0, \\ \Delta_4 = \frac{4\sigma_1 (1-\sigma_1)^3}{(3\sigma_1+1)^2 k_5} l_1^3 k_3^3 > 0. \end{cases}$$

Then, by simply setting $\mu = 0$ (namely, $\varpi_x = 0$) in (22) and noting that

$$\det \begin{bmatrix} f_2 \\ f_2 A_0 \\ f_2 A_0^2 \\ f_2 A_0^3 \end{bmatrix} = \frac{4\sigma_1 (3\sigma_1 + 1)}{\sigma_1 - 1} k_3^2 k_5^2 < 0,$$

it follows from the proof of Theorem 1 that the linear system (21) is globally stabilized by the linear state feedback

$$u_2 = f_2 x, \quad f_2 = \begin{bmatrix} k_3 & -k_4 & -k_5 & 0 \end{bmatrix}. \tag{23}$$

Hence, the global stability of the closed-loop system consisting of (19) and (20) is guaranteed by noting that $f = f_2 T$, where T is given by (9). □

We mention that in Corollary 1 the roll-yaw equation (5) is underactuated and the roll-axis is the underactuated axis (i.e., $T_{cx} = 0$). Even for this difficult case, the global stability of the closed-loop system controlled by the bounded linear feedback (20) can be guaranteed.

For the closed-loop system consisting of (21) and (23), its convergence rate relies heavily on the eigenvalues of the linearized closed-loop system $\dot{\chi} = (A + b_2 f)\chi$ (or, equivalently (19) and (20)) since it will eventually works in linear region. Since $\lambda(A + b_2 f) = \omega_0 \lambda(A_0 + b_0 f_2)$, it is expected to choose properly $k_i, i = 3, 4, 5$ such that the following min–max optimization problem is solved

$$\min_{k_j, j \in \{3,4,5\}} \max_{i \in \{1,2,3,4\}} \mathrm{Re}\{\lambda_i (A_0 + b_0 f_2)\}. \tag{24}$$

Since the above min–max optimization problem is independent of ω_0 and μ, a global optimal solution may be found. This is indeed the case as stated by the following proposition.

Proposition 1. *Let A_0, b_0, and f_2 be stated in (21) and (23). Then*

$$\min_{k_j, j \in \{3,4,5\}} \max_{i \in \{1,2,3,4\}} \mathrm{Re}\{\lambda_i (A_0 + b_0 f_2)\} = -\sqrt{3\sigma_1 + 1},$$

and the optimal solution is given by

$$k_3 = \frac{3\sigma_1 + 1}{4\sigma_1}, \quad k_4 = 4\sqrt{3\sigma_1 + 1}, \quad k_5 = \frac{4(3\sigma_1 + 1)}{1 - \sigma_1}. \tag{25}$$

Proof. By noting that

$$A_0 + b_0 f_2 = \begin{bmatrix} 0 & -4\sigma_1 & 0 & 0 \\ k_3 & -k_4 & -k_5 & 0 \\ 0 & 0 & 0 & 1 - \sigma_1 \\ k_3 & -k_4 & -k_5 + \frac{3\sigma_1+1}{\sigma_1-1} & 0 \end{bmatrix},$$

the characteristic equation of $A_0 + b_0 f_2$ is given by

$$\begin{aligned}0 &= \det(sI_4 - (A_0 + b_0 f_2)) \\ &= s^4 + k_4 s^3 + (4\sigma_1 k_3 + (1-\sigma_1) k_5 + 3\sigma_1 + 1) s^2 \\ &\quad + (3\sigma_1 + 1) k_4 s + 12\sigma_1^2 k_3 + 4\sigma_1 k_3. \end{aligned} \tag{26}$$

By the change of variable $s = t - \sqrt{3\sigma_1 + 1}$, (26) can be expressed as

$$\begin{aligned}0 &= t^4 + \left(k_4 - 4\sqrt{3\sigma_1+1}\right) t^3 \\ &\quad + \left(-3\sqrt{3\sigma_1+1}k_4 + 4\sigma_1 k_3 + (1-\sigma_1) k_5 + 7(3\sigma_1+1)\right) t^2 \\ &\quad + \sqrt{3\sigma_1+1}\left(4\sqrt{3\sigma_1+1}k_4 - 2(4\sigma_1 k_3 + (1-\sigma_1)k_5) \right. \\ &\quad \left. -6(3\sigma_1+1)\right) t \\ &\quad + (3\sigma_1+1)\left(-2\sqrt{3\sigma_1+1}k_4 + 8\sigma_1 k_3 + (1-\sigma_1) k_5 \right. \\ &\quad \left. + 2(3\sigma_1+1)\right). \end{aligned} \tag{27}$$

Hence, all the roots s_i for (26) satisfy $\mathrm{Re}\{s_i\} \le -\sqrt{3\sigma_1+1}$ if and only if all the roots t_i for (27) satisfy $\mathrm{Re}\{t_i\} \le 0$. It is well known that all the zeros of the polynomial equation (27) has non-positive real parts only if

$$\begin{cases} 0 \le k_4 - 4\sqrt{3\sigma_1+1}, \\ 0 \le -3\sqrt{3\sigma_1+1}k_4 + 4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1), \\ 0 \le 2\sqrt{3\sigma_1+1}k_4 - 4\sigma_1 k_3 - (1-\sigma_1)k_5 - 3(3\sigma_1+1), \\ 0 \le -2\sqrt{3\sigma_1+1}k_4 + 8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1). \end{cases}$$

Solve the above inequalities to give

$$\begin{cases} k_4 \ge 4\sqrt{3\sigma_1+1}, \\ k_4 \le \frac{1}{3\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1)), \\ k_4 \ge \frac{1}{2\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1)), \\ k_4 \le \frac{1}{2\sqrt{3\sigma_1+1}}(8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1)), \end{cases}$$

from which the following four relations can be derived:

$$\begin{cases} 4\sqrt{3\sigma_1+1} \\ \quad \le k_4 \le \frac{1}{3\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1)), \\ \frac{1}{2\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1)) \\ \quad \le k_4 \le \frac{1}{3\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1)), \\ 4\sqrt{3\sigma_1+1} \\ \quad \le k_4 \le \frac{1}{2\sqrt{3\sigma_1+1}}(8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1)), \\ \frac{1}{2\sqrt{3\sigma_1+1}}(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1)) \\ \quad \le k_4 \le \frac{1}{2\sqrt{3\sigma_1+1}}(8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1)). \end{cases}$$

Since $\sigma_1 \in (0,1)$, the above inequalities can be rearranged as

$$\begin{cases} 12(3\sigma_1+1) \le k_4 \le 4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1), \\ 3(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1)) \\ \quad \le k_4 \le 2(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1)), \\ 8(3\sigma_1+1) \le k_4 \le 8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1), \\ 4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1) \\ \quad \le k_4 \le 8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1). \end{cases}$$

Then we have

$$\begin{cases} 12(3\sigma_1+1) \le 4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1), \\ 3(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1)) \\ \quad \le 2(4\sigma_1 k_3 + (1-\sigma_1)k_5 + 7(3\sigma_1+1)), \\ 8(3\sigma_1+1) \le 8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1), \\ 4\sigma_1 k_3 + (1-\sigma_1)k_5 + 3(3\sigma_1+1) \\ \quad \le 8\sigma_1 k_3 + (1-\sigma_1)k_5 + 2(3\sigma_1+1). \end{cases}$$

Simplifying the above four relations yields

$$\begin{cases} 5(3\sigma_1+1) \le 4\sigma_1 k_3 + (1-\sigma_1)k_5, \\ 5(3\sigma_1+1) \ge 4\sigma_1 k_3 + (1-\sigma_1)k_5, \\ 6(3\sigma_1+1) \le 8\sigma_1 k_3 + (1-\sigma_1)k_5, \\ \quad 3\sigma_1 + 1 \le 4\sigma_1 k_3. \end{cases}$$

Hence the following conditions can be obtained:

$$\begin{cases} 5(3\sigma_1+1) = 4\sigma_1 k_3 + (1-\sigma_1)k_5, \\ 4\sqrt{3\sigma_1+1} = k_4, \\ \quad 3\sigma_1 + 1 \le 4\sigma_1 k_3. \end{cases}$$

By this, the polynomial equation (27) can be simplified as

$$0 = t^4 + (3\sigma_1+1)(4\sigma_1 k_3 - (3\sigma_1+1)).$$

If $k_3 > \frac{3\sigma_1+1}{4\sigma_1}$, then all roots of the above polynomial equation are given by

$$\left\{\frac{1\pm\mathrm{i}}{2}\sqrt{2}\sqrt[4]{(3\sigma_1+1)(4\sigma_1 k_3 - 3\sigma_1 - 1)},\right. \\ \left. -\frac{1\pm\mathrm{i}}{2}\sqrt{2}\sqrt[4]{(3\sigma_1+1)(4\sigma_1 k_3 - 3\sigma_1 - 1)}\right\},$$

which contains roots having positive real parts, namely, the polynomial equation (26) have roots with real parts being larger than $-\sqrt{3\sigma_1+1}$. Hence, there must hold $k_3 = \frac{3\sigma_1+1}{4\sigma_1}$. Consequently, $k_5 = \frac{4(3\sigma_1+1)}{1-\sigma_1}$, and all the roots the polynomial equation (26) are $-\sqrt{3\sigma_1+1}$. The proof is finished. □

For the closed-loop system consisting of (5) and (7), it is noticed that

$$\lambda(A+BF) = \lambda(\omega_0 A_0 + \omega_0 B_0 F_0) = \omega_0 \lambda(A_0 + B_0 F_0),$$
$$A_0 + B_0 F_0 = \begin{bmatrix} 0 & -4\sigma_1 & -k_1 & k_2 \\ k_3 & -k_4 & -k_5 & 0 \\ 0 & 0 & -k_1 & k_2 + 1 - \sigma_1 \\ k_3 & -k_4 & -k_5 + \frac{3\sigma_1+1}{\sigma_1 - 1} & 0 \end{bmatrix},$$

in which $A_0 + B_0 F_0$ is independent of ω_0 and μ. Similarly to (24), the following min–max problem can be proposed

$$\min_{k_j, j\in\{1,2,3,4,5\} \text{ satisfying } (8)} \max_{i\in\{1,2,3,4\}} \mathrm{Re}\{\lambda_i(A_0 + B_0 F_0)\}. \tag{28}$$

However, similar to problem (29) in [40], the above min–max optimization problem (28) is a nonlinear and non-convex optimization problem. Hence, in general only a local optimal solution can be computed by a linear search technique (see examples in Section 5).

4. Global stabilization of the pitch loop

We write the pitch equation in (4) as

$$\dot{Z} = \Phi Z + \Psi \mathrm{sat}(v), \tag{29}$$

where $Z = [q_2, \dot{q}_2]^{\mathrm{T}}$, $v = \frac{1}{\varpi_y} T_{cy}$ and Φ and Ψ are given by

$$\Phi=\begin{bmatrix}0 & 1\\ -3\omega_0^2\sigma_1 & 0\end{bmatrix},\ \Psi=\begin{bmatrix}0\\ \frac{\varpi_y}{2J_y}\end{bmatrix}.$$

Similarly to the roll-yaw loop, we have the following proposition.

Proposition 2. *The closed-loop system consisting of the linear system (29) and the linear state feedback*

$$v=HZ,\ H=\begin{bmatrix}-\frac{6\sigma_1\omega_0^2J_y}{\varpi_y}h_1 & -\frac{2\omega_0J_y}{\varpi_y}h_2\end{bmatrix}, \tag{30}$$

is globally asymptotically stable and locally exponentially stable, where $h_1\geq 0$ and $h_2>0$ are any scalars.

Proof. By the following state of transformation

$$z=\Pi Z=2J_y\begin{bmatrix}\frac{\omega_0^2}{\varpi_y} & 0\\ 0 & \frac{\omega_0}{\varpi_y}\end{bmatrix}Z,$$

system (29) can be expressed as

$$\dot z=\omega_0\Phi_0z+\omega_0\Psi_0\text{sat}(v), \tag{31}$$

where $(\Phi_0,\Psi_0)=\frac{1}{\omega_0}\left(\Pi\Phi\Pi^{-1},\Pi\Psi\right)$ is independent of ω_0 and are given by

$$\Phi_0=\begin{bmatrix}0 & 1\\ -3\sigma_1 & 0\end{bmatrix},\ \Psi_0=\begin{bmatrix}0\\ 1\end{bmatrix}.$$

We now look for linear control law $v=H_0z$ such that system (31) is globally stabilized, where $H_0=[-3\sigma_1h_1,-h_2]$, in which $h_1\geq 0$ and $h_2>0$ are any scalars. Similarly to the proof of Theorem 1, consider the following Lyapunov function candidate:

$$W(z)=z^{\mathrm T}Q_0z+2E_0\int_0^{H_0z}\text{sat}(s)\,\mathrm ds,$$

where $E_0=\frac{h_1}{h_2}\geq 0$ and the positive definite matrix Q_0 takes the form

$$Q_0=\frac{3\sigma_1h_1^2+h_2^2}{h_2}\begin{bmatrix}3\sigma_1 & 0\\ 0 & 1\end{bmatrix}.$$

Then it can be verified that

$$\Phi_0^{\mathrm T}Q_0+Q_0\Phi_0=0,$$
$$\Psi_0^{\mathrm T}Q_0+E_0H_0\Phi_0+\Pi_0H_0=0,$$
$$\Theta=2+2h_1>0,$$

where $\Pi_0=1$ and $\Theta=2\Pi_0-\left(E_0H_0\Psi_0+\Psi_0^{\mathrm T}H_0^{\mathrm T}E_0^{\mathrm T}\right)$. The time-derivative of $W(z)$ along the trajectories of the closed-loop system consisting of (31) and $v=H_0z$ can be evaluated as

$$\dot W(z)\leq -\omega_0\text{sat}^{\mathrm T}(H_0z)\,\Theta\text{sat}(H_0z). \tag{32}$$

Notice that the matrix

$$Q_0+H_0^{\mathrm T}H_0=\begin{bmatrix}3\sigma_1\gamma_1+9\sigma_1^2h_1^2 & 3\sigma_1h_1h_2\\ 3\sigma_1h_1h_2 & \gamma_1+h_2^2\end{bmatrix},$$

in which $\gamma_1=\frac{3\sigma_1h_1^2+h_2^2}{h_2}$, is positive definite since $\det\left(Q_0+H_0^{\mathrm T}H_0\right)>0$. It follows that $W(z)$ is positive definite. Then the stability of the closed-loop system follows from (32) and the LaSalle invariant principle, as the matrix pair (Φ_0,H_0) is observable for any $h_1\geq 0$ and $h_2>0$. The proof is finished. □

The eigenvalue set of the linearized closed-loop system consisting of (29) and (30) is given by

$$\omega_0\lambda(\Phi_0+\Psi_0H_0)=\omega_0\lambda\left(\begin{bmatrix}0 & 1\\ -3\sigma_1(1+h_1) & -h_2\end{bmatrix}\right).$$

Different from the roll-yaw loop discussed in the previous section, the real parts of the elements in $\lambda(\Phi_0+\Psi_0H_0)$ can be made arbitrarily negative by choosing $h_i, i=1,2$ properly (sufficiently large). However, it should be noted that the actuator will be saturated and the transient performance can be very poor if $h_i, i=1,2$ are too large. Therefore, there is a trade-off in choosing $h_i, i=1,2$.

We mention that the proposed bounded linear feedback controllers (7), (20) and (30) can stabilize the linear system (5) and (19) globally while it is not clear whether it can also stabilize globally the original nonlinear models (1) and (2), which is a hard problem and deserves a further study.

5. Simulations

In this section, some simulations will be carried out by applying the saturated linear controllers on the *original* nonlinear plant (1) and (2).

The development of nanosatellites is currently a significant trend in the area of space science and engineering [2,18,37]. One unit (U) of a CubeSat is defined by $10\times 10\times 10$ cm^3 with mass of less than 1.33 kg per 1U [9]. In general, the inertia of CubeSat-class nanosatellite is required as axisymmetric and the axis of symmetric is the minor principal axis. Since full-scale attitude control systems of large satellites are generally too complicated or too expensive to be installed in CubeSat-class nanosatellites, the proposed bounded linear controllers may be helpful for CubeSat-class nanosatellites as they are very easy to be implemented.

Let the nominal inertia matrix of the UYS-1 nanosatellite be given by [3,4]

$$J=\text{diag}\{0.1521,0.1521,0.0375\}\ \text{kg}\cdot\text{m}^2,$$

which is subject to the following parameter uncertainties

$$\Delta J=\begin{bmatrix}\Delta J_{xx} & \Delta J_{xy} & \Delta J_{xz}\\ \Delta J_{xy} & \Delta J_{yy} & \Delta J_{yz}\\ \Delta J_{xz} & \Delta J_{yz} & \Delta J_{zz}\end{bmatrix}\ \text{kg}\cdot\text{m}^2,$$

where $|\Delta J_{ii}|\leq 0.1J_i$ and $|\Delta J_{ij}|\leq 0.05\max\{J_i,J_j\},\ i\in\{x,y,z\}$. For simulation, we let $\Delta J_{xx}=0.1J_x$, $\Delta J_{yy}=0.1J_y$, $\Delta J_{zz}=0.1J_z$, $\Delta J_{xy}=0.02J_x$, $\Delta J_{yz}=-0.02J_x$ and $\Delta J_{xz}=-0.05J_x$. Its orbit is circular with an altitude of 700 km and an inclination of 98°. The maximal magnitude of the control signal is assumed to be $T_{ci\max}=2$ mN·m, $i\in\{x,y,z\}$. Then $\mu=0.7535$. For simulation purpose, the initial condition are chosen as $\phi(t_0)=\theta(t_0)=\psi(t_0)=10°$ and $\dot\phi(t_0)=\dot\theta(t_0)=\dot\psi(t_0)=0.01°/\text{s}$.

For the pitch loop, we use the controller (30) where $h_1=70$, $h_2=25$, which is such that $\lambda(\Phi_0+\Psi_0H_0)=\{-12.5000\pm 2.0580\text{i}\}$. For the roll-yaw loop, both the global stabilizing controllers (7) and (20) are considered.

1. For the controller (7), to get a satisfactory control performance, we first solve the optimization problem (28) in the interval

$$(k_1,k_2,k_3,k_4,k_5)\in((0,100]\times(0,100]\times(0,100]$$
$$\times(p(k),p(k)+100]\times(0,100]),$$

where

$$p(k)=\frac{\left((3\sigma_1+1)k_5k_2^2+\left((\sigma_1-1)^2k_1^2+(3\sigma_1+1)k_2^2\right)(k_3-k_5)\right)^2}{4\left((\sigma_1-1)^2k_1^2+(3\sigma_1+1)k_2^2\right)(1-\sigma_1+k_2)(3\sigma_1+1)k_1k_5}.$$

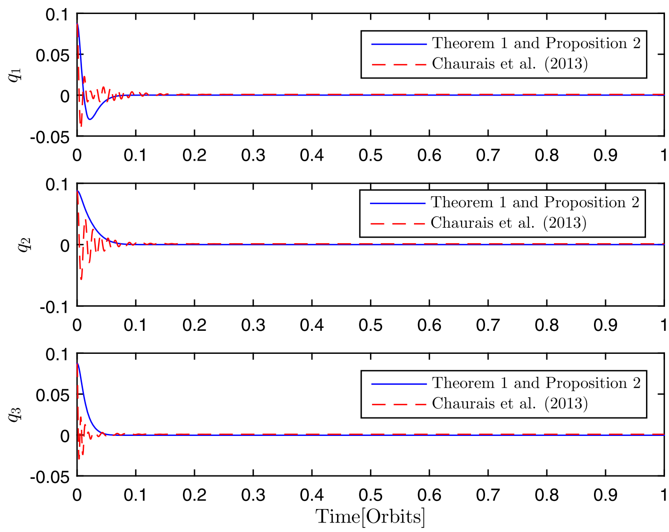

Fig. 2. The attitude quaternion time histories with 10° and 0.01°/s initial errors on all three axes with controllers (7) and (30).

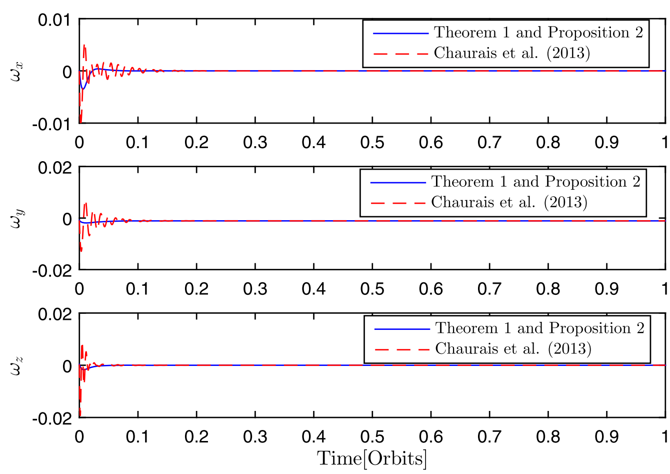

Fig. 3. The attitude angular rate time histories with 10° and 0.01°/s initial errors on all three axes with controllers (7) and (30).

By a linear search technique, a local optimal solution to the min–max optimization problem (28) can be found as

$$(k_1, k_2, k_3, k_4, k_5) = (60, 75, 95, 29.2, 95)\,, \tag{33}$$

which is such that

$$\lambda\left(A_0 + B_0 F_0\right) = \{-22.1797 \pm 7.9490\mathrm{i}, -22.4409 \pm 3.1010\mathrm{i}\}\,.$$

We set the parameters $k_i, i \in \{1, 2, 3, 4, 5\}$, as (33) in controller (7). The attitude quaternions, the attitude angular velocities and control signals are respectively recorded in Figs. 2–4. From these figures we can see that the system converges to its equilibrium in less than 0.4 orbit period, which is quite satisfactory.

2. For the controller (20), we take the parameters k_3, k_4 and k_5 as in (25). In this case, the attitude quaternions, the attitude angular velocities and control signals are respectively shown in Figs. 5–7. From these figures we can observe that it takes more than 1 orbit period to transfer the initial condition to the desired equilibrium. This regulation time is about twice of the regulation time needed by the attitude stabilizing controller (7), which is reasonable since we use only one actuator in this case.

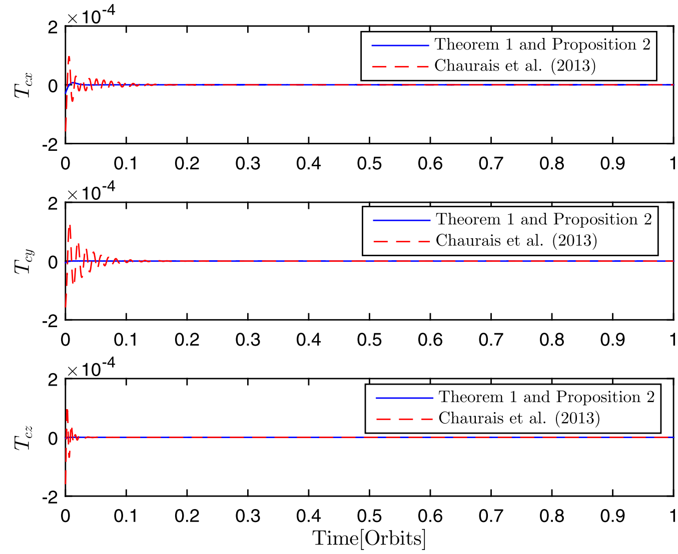

Fig. 4. The control torques time histories with 10° and 0.01°/s initial errors on all three axes with controllers (7) and (30).

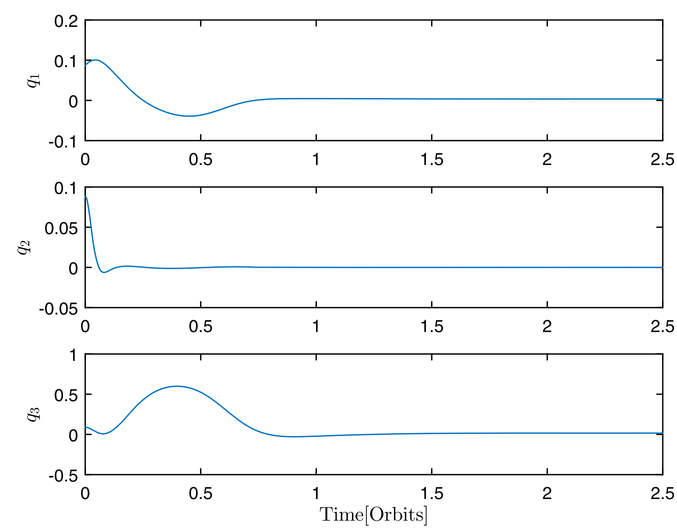

Fig. 5. The attitude quaternion time histories with 10° and 0.01°/s initial errors on all three axes with controllers (20) and (30).

For the comparison purpose, the following control law in [3] is also used to stabilize the attitude control system

$$T_c = -\frac{1}{\bar{b}_0}\left(\left(\left(q_4 I_3 - q_v^{\times}\right) K_{p_1} + k_{p_2}\left(1 - q_4\right) I_3\right) q_v + K_d \omega\right), \tag{34}$$

where $K\left(q_v\right) = \left(q_4 I_3 - q_v^{\times}\right) K_{p_1} + k_{p_2}\left(1 - q_4\right) I_3, \bar{b}_0$ and k_{p_2} are positive constants, K_{p_1} and K_d are positive definite matrices with dimensions 3×3. By choosing the suitable parameters

$$K_{p_1} = \eta \begin{bmatrix} 1.8721 & -0.1969 & -0.1969 \\ -0.1969 & 1.8721 & -0.1969 \\ -0.1969 & -0.1969 & 1.8721 \end{bmatrix},$$

$$K_d = \eta \begin{bmatrix} 1.8261 & 1.0435 & 1.0435 \\ 1.0435 & 1.8261 & 1.0435 \\ 1.0435 & 1.0435 & 1.8261 \end{bmatrix},$$

$$k_{p_2} = 1.3787\eta,\ \bar{b}_0 = 16.1222,\ \eta = 0.02,$$

the state responses with the same initial conditions as the above are also shown in Figs. 2–3. We mention that, when the parame-

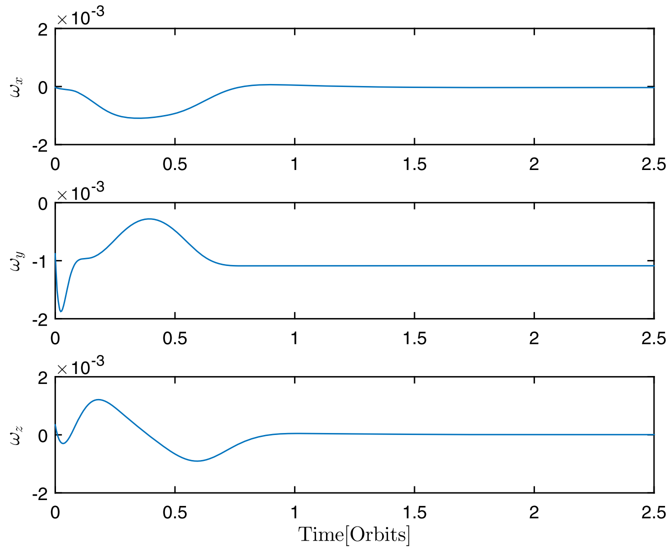

Fig. 6. The attitude angular rate time histories with 10° and 0.01°/s initial errors on all three axes with controllers (20) and (30).

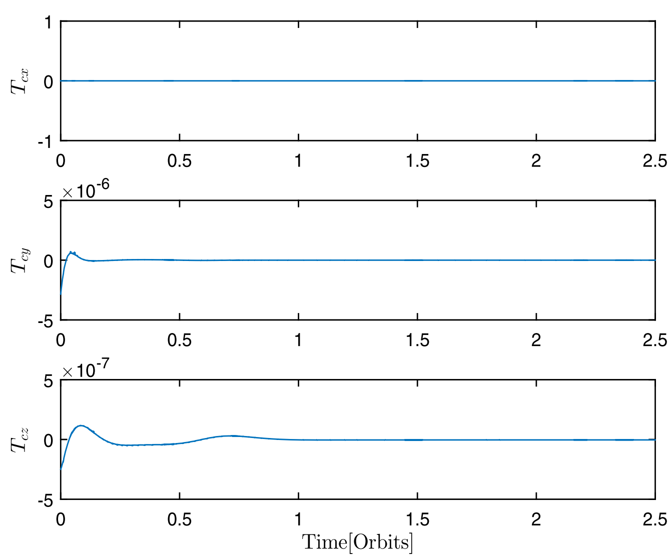

Fig. 7. The control torques time histories with 10° and 0.01°/s initial errors on all three axes with controllers (20) and (30).

ters in the PD-type controller are chosen as $\eta = 10^6$ as in [3], the state responses unfortunately diverge in the presence of input saturation. This simulation shows the effectiveness of the proposed saturated linear controllers in Theorem 1 and Proposition 2.

6. Conclusion

Three-axis attitude stabilization of axisymmetric spacecraft by bounded linear feedback was investigated in this paper. By taking the effect of the gravity-gradient torque into account, linearized attitude equations of motion are used to construct saturated *linear* global stabilizing controllers. It is shown by constructing explicit Lyapunov functions that the proposed controllers guarantee the global asymptotic stability and local exponential stability of the closed-loop system if the parameters in the feedback gain satisfy some explicit conditions. The optimal linear feedback gains for the underactuated attitude control systems were also obtained. The proposed linear controllers were used to stabilize the attitude control system of CubeSat-class nanosatellites and simulations show their effectiveness.

Conflict of interest statement

The authors claim no conflicts of interests.

References

[1] V. Andriano, Global feedback stabilization of the angular velocity of a symmetric rigid body, Syst. Control Lett. 20 (5) (1993) 361–364.
[2] G. Candini, F. Piergentili, F. Santoni, Designing, manufacturing, and testing a self-contained and autonomous nanospacecraft attitude control system, J. Aerosp. Eng. 27 (6) (2014) 04014033.
[3] J.R. Chaurais, H.C. Ferreira, J.Y. Ishihara, R.A. Borges, A high precision attitude determination and control system for the UYS-1 nanosatellite, IEEE Aerosp. Conf. Proc. 1 (12) (2013).
[4] J.R. Chaurais, H.C. Ferreira, J.Y. Ishihara, R.A. Borges, Attitude control of an underactuated satellite using two reaction wheels, J. Guid. Control Dyn. 38 (10) (2015) 2010–2018.
[5] H. Ehrpais, J. Kütt, I. Sünter, E. Kulu, A. Slavinskis, M. Noorma, Nanosatellite spin-up using magnetic actuators: ESTCube-1 flight results, Acta Astronaut. 128 (2016) 210–216.
[6] H. Gui, L. Jin, S. Xu, Simple finite-time attitude stabilization laws for rigid spacecraft with bounded inputs, Aerosp. Sci. Technol. 42 (2015) 176–186.
[7] H. Gui, G. Vukovich, Robust adaptive spin-axis stabilization of a symmetric spacecraft using two bounded torques, Adv. Space Res. 56 (2015) 2495–2507.
[8] H. Gui, G. Vukovich, Global finite-time attitude tracking via quaternion feedback, Syst. Control Lett. 97 (2016) 176–183.
[9] E. Hand, Startup liftoff, Science 348 (6231) (2015) 172–177.
[10] H.K. Hhalil, Nonlinear Systems, third edition, Prentice Hall, 2002.
[11] N.M. Horri, P. Palmer, S. Hodgart, Practical implementation of attitude-control algorithms for an underactuated satellite, J. Guid. Control Dyn. 35 (1) (2012) 40–45.
[12] J. Hu, H. Zhang, Bounded output feedback of rigid-body attitude via angular velocity observers, J. Guid. Control Dyn. 36 (4) (2013) 1240–1247.
[13] Q. Hu, Robust adaptive sliding mode attitude maneuvering and vibration damping of three-axis-stabilized flexible spacecraft with actuator saturation limits, Nonlinear Dyn. 55 (4) (2009) 301–321.
[14] Q. Hu, X. Tan, Unified attitude control for spacecraft under velocity and control constraints, Aerosp. Sci. Technol. 67 (2017) 257–264.
[15] T. Inamori, R. Hamaguchi, K. Ozawa, P. Saisutjarit, N. Sako, S. Nakasuka, Online magnetometer calibration in consideration of geomagnetic anomalies using Kalman filters in nanosatellites and microsatellites, J. Aerosp. Eng. 29 (6) (2016) 04016046.
[16] Z. Ismail, R. Varatharajoo, A study of reaction wheel configurations for a 3-axis satellite attitude control, Adv. Space Res. 45 (6) (2010) 750–759.
[17] M. Li, M. Hou, C. Yin, Constrained dual-loop attitude control design for spacecraft, Aerosp. Sci. Technol. 55 (2016) 170–176.
[18] J. Li, M. Post, T. Wright, R. Le, Design of attitude control systems for CubeSat-class nanosatellite, J. Control Sci. Eng. (2013) 657182.
[19] W. Luo, B. Zhou, Magnetic attitude control of bias momentum spacecraft by bounded linear feedback, Aerosp. Sci. Technol. 70 (2017) 419–427.
[20] M.L. Psiaki, Magnetic torquer attitude control via asymptotic periodic linear quadratic regulation, J. Guid. Control Dyn. 24 (2) (2001) 386–394.
[21] M.L. Psiaki, Nanosatellite attitude stabilization using passive aerodynamics and active magnetic torquing, J. Guid. Control Dyn. 27 (3) (2004) 347–355.
[22] A. Saberi, A.A. Stoorvogel, P. Sannuti, Internal and External Stabilization of Linear Systems with Constraints, Springer Science & Business Media, 2012.
[23] M.J. Sidi, Spacecraft Dynamics and Control: A Practical Engineering Approach, Cambridge University Press, Cambridge, MA, 1997.
[24] Y. Su, Y. Zheng, Globally asymptotic stabilization of spacecraft with simple saturated proportional-derivative control, J. Guid. Control Dyn. 34 (6) (2011) 1932–1935.
[25] H.J. Sussmann, E.D. Sontag, Y. Yang, A general result on the stabilization of linear systems using bounded controls, IEEE Trans. Autom. Control 39 (12) (1994) 2411–2425.
[26] S. Tarbouriech, G. Garcia, J.M.G. da Silva, I. Queinnec, Stability and Stabilization of Linear Systems with Saturating Actuators, Springer, London, U.K., 2011.
[27] P. Tsiotras, M. Corless, J.M. Longuski, A new approach the attitude control of axisymmetric spacecraft, Automatica 31 (8) (1995) 1099–1112.
[28] P. Tsiotras, Optimal regulation and passivity results for axisymmetric rigid bodies using two controls, J. Guid. Control Dyn. 20 (3) (1997) 457–463.
[29] P. Tsiotras, J. Luo, Control of underactuated spacecraft with bounded inputs, Automatica 36 (2000) 1153–1169.
[30] F. Tyan, D.S. Bernstein, Global stabilization of systems containing a double integrator using a saturated linear controller, Int. J. Robust Nonlinear Control 9 (15) (1999) 1143–1156.

[31] Q. Wang, B. Zhou, G-R. Duan, Robust gain scheduled control of spacecraft rendezvous system subject to input saturation, Aerosp. Sci. Technol. 42 (2015) 442–450.
[32] J. Wen, K. Kreutz-Delgado, The attitude control problem, IEEE Trans. Autom. Control 36 (10) (1991) 1148–1162.
[33] J. Wertz, Spacecraft Attitude Determination and Control, D. Reidel, Dordrecht, The Netherlands, 1978.
[34] B. Wie, Space Vehicle Dynamics and Control, AIAA, Reston, VA, 1998.
[35] M. Wood, W.H. Chen, Attitude control of magnetically actuated satellites with an uneven inertia distribution, Aerosp. Sci. Technol. 25 (2013) 29–39.
[36] H. Yan, S. Tan, Y. He, A small-gain method for integrated guidance and control in terminal phase of reentry, Acta Astronaut. 132 (2017) 282–292.
[37] H. Yoon, K.M. Riesing, K. Cahoy, Kalman filtering for attitude and parameter estimation of nanosatellites without gyroscopes, J. Guid. Control Dyn. (2017), https://doi.org/10.2514/1.54075.
[38] B. Zhou, Q. Wang, Z. Lin, G.R. Duan, Gain scheduled control of linear systems subject to actuator saturation with application to spacecraft rendezvous, IEEE Trans. Control Syst. Technol. 22 (5) (2014) 2031–2038.
[39] B. Zhou, Global stabilization of periodic linear systems by bounded controls with applications to spacecraft magnetic attitude control, Automatica 60 (2015) 145–154.
[40] B. Zhou, J. Lam, Global stabilization of linearized spacecraft rendezvous system by saturated linear feedback, IEEE Trans. Control Syst. Technol. 25 (6) (2017) 2185–2193.
[41] Q. Zheng, F. Wu, Nonlinear H_∞ control designs with axisymmetric spacecraft control, J. Guid. Control Dyn. 32 (3) (2009) 850–859.