VI. Conclusion

We have studied the problem of static competitive routing to parallel queues with polynomial link holding costs. We have established the uniqueness of the NE for a general topology with the BPR cost [22] and have obtained a simple relationship with the globally optimal solution. We have further obtained some explicit results for the special case of affine link costs.

The results of this note should prove useful for the analysis of networks with source-determined routing, when the link cost functions can be approximated by polynomial (and in particular affine) costs of the type considered here. The fact that the NE was shown to be efficient under a class of nonlinear costs can be used *as a starting point* for designing pricing mechanisms so as to obtain a socially optimal use of the network.

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A New Approach to State Observation of Nonlinear Systems With Delayed Output

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***Abstract*—This note presents a new approach for the construction of a state observer for nonlinear systems when the output measurements are available for computations after a non negligible time delay. The proposed observer consists of a chain of observation algorithms reconstructing the system state at different delayed time instants (chain observer). Conditions are given ensuring global exponential convergence to zero of the observation error for any given the delay in the measurements. The implementation of the observer is simple and computer simulations demonstrate its effectiveness.**

***Index Terms*—Delay systems, nonlinear systems, state observation, state prediction.**

I. Introduction

In many engineering applications a process to be controlled, or simply monitored, is located far from the computing unit and the measured data are transmitted through a low-rate communication system (e.g., in aerospace applications). In the above cases the measured outputs are available for computations after a non negligible time delay. In some applications (e.g., in biochemical reactors) the measurement process intrinsically provides an out-of-date output. In both cases the reconstruction of the present system state using past measurements may be significant. This is a classical *state prediction* problem. An important engineering application of state prediction occurs when the control variable can be applied to the system with a non negligible delay after its computation. In this case it is clear that a state feedback control law can be used only if computed on the predicted state. In the case of linear systems, such a control problem is solved by the so-called *Smith Predictor* [18], which is not exactly a predictor: it is a *predictive model-based control scheme* requiring state-prediction. Many other algorithms for predictive control of systems with input delay have been proposed in the literature (see e.g., [3], [5], [17], and [19]), and all of them include a state predictor. However, in such schemes little attention is devoted to the predictor implementation, often realized in open-loop, under the assumption of stability of the process. In [7], different implementations of the

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state predictor inside a Smith controller have been discussed and a closed-loop implementation is proposed. In [14], [16] the Smith approach is extended for closed-loop control of nonlinear systems with delayed input. As in the case of linear systems the state prediction is obtained by an open-loop algorithm, so that the accuracy of the predicted state is not guaranteed for unstable systems.

The issue of state reconstruction of the present state in the presence of time delays in the system equation and/or in the measurement process is receiving increasing attention. This problem is not only interesting for the control, but also for the supervision and real-time monitoring of systems. In [1], [2], and [12], the problem of state observation for systems with delay only in the state equation is considered. In [15], the same problem is solved for nonlinear systems with delays also in the output which are linearizable by additive output injection. In [10] and [11] some results on the state prediction for nonlinear systems with small output delay are reported.

This note presents a solution for the problem of state observation in nonlinear systems with delayed output measurements. The proposed observation algorithm, which has an interesting chain structure (Chain Observer), is based on the theory of state observers for systems without output delay presented in [6]. It is shown that, under suitable assumptions, for delays of any size there exists an observer of suitable dimension achieving exponential error decay. It should be stressed that the class of nonlinear systems considered in this note does not allow, in general, a linear representation.

II. Preliminaries

This note considers nonlinear systems of the type

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad t \geq -\Delta, \quad x(-\Delta) = \bar{x} \tag{1}$$

$$\bar{y}(t) = h(x(t-\Delta)) \quad t \geq 0 \tag{2}$$

where $\Delta > 0$ is the measurement delay, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}$, the vector functions f, g and h are C^∞. The output $\bar{y}(t) \in \mathbb{R}$ is a function of the state x at time $t-\Delta$. Let $y(t) = h(x(t))$ denote the undelayed output. It is useful to define the square map $z = \Phi(x)$ where

$$\Phi(x) = \begin{bmatrix} h(x) & L_f h(x) & \cdots & L_f^{n-1} h(x) \end{bmatrix}^T \tag{3}$$

($L_f^k h(x)$ denotes the k-th order repeated Lie derivative of the function h along f; see [13]).

Definition 1: The system (1)–(2) is said to be *globally drift-observable* if the function $z = \Phi(x)$ is a diffeomorphism in all $\mathbb{R}^n$. •

The main assumption needed in the note for the derivation of the observer is the following.

H_1) System (1)–(2) is globally drift-observable, and the diffeomorphism $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$ are globally uniformly Lipschitz in $\mathbb{R}^n$, i.e.,

$$\begin{aligned} \|\Phi(x_1) - \Phi(x_2)\| &\leq \gamma_\Phi \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n \\ \|\Phi^{-1}(z_1) - \Phi^{-1}(z_2)\| &\leq \gamma_{\Phi^{-1}} \|z_1 - z_2\|, \quad \forall z_1, z_2 \in \mathbb{R}^n. \end{aligned} \tag{4}$$

Under the assumption H_1 the Jacobian of the map $\Phi(x)$, denoted $Q(x)$, and the Jacobian of the inverse map are nonsingular in all $\mathbb{R}^n$:

$$Q(x) = \frac{\partial \Phi(x)}{\partial x}, \qquad \left.\frac{\partial \Phi^{-1}(z)}{\partial z}\right|_{z=\Phi(x)} = Q^{-1}(x). \tag{5}$$

From definition (3) of Φ, the following properties are obtained:

$$Q(x)f(x) = A_n \Phi(x) + B_n L_f^n h(x), \quad h(x) = C_n \Phi(x) \tag{6}$$

where matrices (A_n, B_n, C_n) define a Brunowski triple

$$\begin{aligned} A_n &= \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \quad B_n = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \\ C_n &= \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}. \end{aligned} \tag{7}$$

Under assumption H_1 the map $z = \Phi(x)$ defines a global change of coordinates. Differentiating $z(t) = \Phi(x(t))$ w.r.t. time and using properties (6) the system equations in z-coordinates are obtained

$$\begin{aligned} \dot{z}(t) &= A_n z(t) + \tilde{H}(z(t), u(t)), \quad t \geq -\Delta, \quad z(-\Delta) = \Phi(\bar{x}) \\ \bar{y}(t) &= C_n z(t-\Delta), \quad t \geq 0 \end{aligned} \tag{8}$$

where

$$\begin{aligned} \tilde{H}(z,u) &= H(x,u)\big|_{x=\Phi^{-1}(z)} \\ H(x,u) &= B_n L_f^n h(x) + Q(x)g(x)u. \end{aligned} \tag{9}$$

Also, the following assumptions will be needed in this note.

H_2) The vector function $\tilde{H}(z,u)$ defined in (9) is globally uniformly Lipschitz with respect to z, and the Lipschitz coefficient $\gamma_{\tilde{H}}$ is a non decreasing function of $|u|$, i.e.,

$$\|\tilde{H}(z_1,u) - \tilde{H}(z_2,u)\| \leq \gamma_{\tilde{H}}(|u|)\|z_1 - z_2\|, \quad \forall z_1, z_2 \in \mathbb{R}^n. \tag{10}$$

H_3) The triple $(f(x), g(x), h(x))$ of system (1)–(2) has uniform observation relative degree equal to n, according to the definition given in [6], i.e., is such that

$$\begin{aligned} \forall x \in \mathbb{R}^n \quad & L_g L_f^k h(x) = 0, \quad k = 0, 1, \ldots, n-2, \\ \exists x \in \mathbb{R}^n: \quad & L_g L_f^{n-1} h(x) \neq 0. \end{aligned} \tag{11}$$

The following Lemma is needed (the proof is reported in Appendix).

Lemma 2: Consider a function $s(t) \geq 0, t \in [-\delta, +\infty)$, with $\delta > 0$, such that

$$\int_{-\delta}^{0} s(\tau)\,d\tau < +\infty, \quad s(t) \leq \mu e^{-\bar{\alpha}t} + \gamma \int_{t-\delta}^{t} s(\tau)\,d\tau, \quad t \geq 0 \tag{12}$$

where $\bar{\alpha}, \gamma, \mu$ are positive real.

If $\gamma\delta < 1$ then there exist a positive $\alpha \leq \bar{\alpha}$ such that

$$s(t) \leq \bar{\mu} e^{-\alpha t}, \quad t \geq 0 \tag{13}$$

where

$$\bar{\mu} = \frac{e^{\alpha\delta}}{1-c}\left(\mu + \gamma \int_{-\delta}^{0} s(\tau)\,d\tau\right), \quad c = \frac{\gamma}{\alpha}(e^{\alpha\delta} - 1) < 1. \tag{14}$$

III. The Chain Observer

In [6], the authors presented the following observer for undelayed nonlinear system:

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + Q^{-1}(\hat{x}(t))K(y(t) - h(\hat{x}(t))). \tag{15}$$

The exponential convergence to zero of the observation error is expressed as

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t}\|x(0) - \hat{x}(0)\|, \quad t \geq 0 \tag{16}$$

and is guaranteed by the following theorems (see [6]).

Theorem 3: Consider system (1) with undelayed output $y(t) = h(x(t))$ under assumptions H_1, H_2. Then, for any positive α there exist a gain vector K for the observer (15) and positive constants μ and u_M such that if $|u(t)| \le u_M$ for $t \ge 0$, then (16) holds.

Theorem 4: Consider system (1) with undelayed output $y(t) = h(x(t))$, under assumptions H_1, H_2, H_3. Assume that there exists u_M such that $|u(t)| \le u_M$, for $t \ge 0$. Then, for any positive α there exists a gain vector K for the observer (15) and a constant μ such that (16) holds.

Remark 5: In Theorems 3 and 4 the convergence of the observer (15) is proven without the assumption of *uniform observability* [8], that is a much stronger assumption than *drift-observability* (in uniformly observable systems all states are distinguishable independently of input). On the other hand some *a priori* limitation on the input is required. Theorem 4 states that if assumptions H_1, H_2, H_3 hold, then, for any given *a priori* bound u_M on the input, an observer with any prescribed exponential convergence rate α can be found. If only assumptions H_1 and H_2 hold, as in Theorem 3, then the existence of an exponential observer is guaranteed if the input amplitude satisfies an upper bound (depending on the prescribed convergence rate α, see [6]). This happens because assumption H_3 (observation relative degree equal to n) implies uniform observability. For systems that do not satisfy assumption H_3 a condition excluding *bad inputs* (those that destroy observability) is needed. The bound on the input amplitude given by Theorem 3 is sufficient to exclude such bad inputs. •

The observer for nonlinear systems with delayed output is based on the observer (15), and is composed by a set of $m+1$ linked systems of delay differential equations (Chain Observer), each one of dimension n, where m is a positive integer to be suitably chosen on the basis of the size of the delay and of the Lipschitz constants of the system. In the Chain Observer, the following notation is used to represent delayed state and input:

$$\begin{aligned} x_j(t) &= x\left(t - \Delta + \tfrac{j}{m}\Delta\right), \quad t \ge -\frac{j}{m}\Delta \\ u_j(t) &= u\left(t - \Delta + \tfrac{j}{m}\Delta\right), \quad j = 0, \dots, m. \end{aligned} \tag{17}$$

The proposed Chain Observer for system (1), (2) has the following structure:

$$\begin{aligned} \dot{\hat{x}}_0(t) &= f(\hat{x}_0(t)) + g(\hat{x}_0(t))u_0(t) \\ &\quad + Q^{-1}(\hat{x}_0(t))K(\bar{y}(t) - h(\hat{x}_0(t))) \\ \dot{\hat{x}}_j(t) &= f(\hat{x}_j(t)) + g(\hat{x}_j(t))u_j(t) + Q^{-1}(\hat{x}_j(t)) \\ &\quad \cdot \Bigg\{ e^{A_n \frac{\Delta}{m} j} K\left(\bar{y}(t) - h(\hat{x}_0(t))\right) + \sum_{i=0}^{j-1} e^{A_n \frac{\Delta}{m}(j-i)} \\ &\quad \cdot \left(H(\hat{x}_i(t), u_i(t)) - H\left(\hat{x}_{i+1}\left(t - \frac{\Delta}{m}\right), u_i(t)\right)\right)\Bigg\}, \\ &\qquad j = 1, \dots, m, \quad t \ge 0. \end{aligned} \tag{18}$$

The matrix A_n is the Brunowsky matrix defined in (7). The initial conditions are

$$\begin{aligned} \hat{x}_0(0) &= \hat{x}(-\Delta) \\ \hat{x}_j(\tau) &= \hat{x}\left(\tau - \Delta + \frac{j}{m}\Delta\right) \\ &\qquad \tau \in \left[-\tfrac{\Delta}{m}, 0\right], \quad j = 1, 2, \dots, m \end{aligned} \tag{19}$$

where $\hat{x}(\tau), \tau \in [-\Delta, 0]$, is any *a priori* estimate of the state. The variable $\hat{x}_j(t)$ is an estimate of the delayed state $x(t - \Delta + j\Delta/m)$, denoted also as $x_j(t)$. Under the assumptions of Theorem 3, or of Theorem 4, it is established that for any α a gain K can be found such that

$$\|x(t-\Delta) - \hat{x}_0(t)\| \le \mu\, e^{-\alpha t} \|x(-\Delta) - \hat{x}_0(0)\| \tag{20}$$

for a suitable μ. Conditions must be found to ensure exponential convergence of the variables $\hat{x}_j(t)$ to the delayed states $x(t - \Delta + j\Delta/m), j = 1, \dots, m-1$, and in particular of $\hat{x}_m(t)$ to the undelayed state $x(t)$.

An expression of the observer (18) in z-coordinates is needed first.

Lemma 6: The observer (18) after the change of coordinates $\hat{z}_j = \Phi(\hat{x}_j)$, for $j = 0, 1, \dots, m$, is as follows:

$$\begin{aligned} \dot{\hat{z}}_0(t) &= A_n \hat{z}_0(t) + \tilde{H}(\hat{z}_0(t), u_0(t)) + K(\bar{y}(t) - C_n \hat{z}_0(t)), \\ &\qquad\qquad t \ge 0 \\ \hat{z}_j(t) &= e^{A_n \frac{\Delta}{m}} \hat{z}_{j-1}(t) \\ &\quad + \int_{t-\frac{\Delta}{m}}^{t} e^{A_n(t-\tau)} \tilde{H}(\hat{z}_j(\tau), u_j(\tau))\, d\tau, \quad j = 1, \dots, m, \\ \hat{z}_0(0) &= \Phi(\hat{x}(-\Delta)), \\ \hat{z}_j(\tau) &= \Phi(\hat{x}(\tau - \Delta + \frac{j}{m}\Delta)), \quad \tau \in \left[-\frac{\Delta}{m}, 0\right]. \\ &\qquad\qquad j = 1, \dots, m \end{aligned} \tag{21}$$

where A_n, C_n are the Brunowsky matrices defined in (7).

Proof: Differentiation of $\hat{z}_0(t) = \Phi(\hat{x}_0(t))$ w.r.t. time gives the first of (21). Differentiation of $\hat{z}_j(t) = \Phi(\hat{x}_j(t))$ for $j = 1, \dots, m$, taking into account the second of (18) in which $\hat{x}_j(t) = \Phi^{-1}(\hat{z}_j(t))$ is substituted, gives

$$\begin{aligned} \dot{\hat{z}}_j(t) &= A_n \hat{z}_j(t) + \tilde{H}(\hat{z}_j(t), u(t)) + e^{A_n \frac{\Delta}{m} j} K(\bar{y}(t) \\ &\quad - C_n \hat{z}_0(t)) + \sum_{i=0}^{j-1} e^{A_n \frac{\Delta}{m}(i-j)} \Bigg(\tilde{H}(\hat{z}_i(t), u_i(t)) \\ &\quad - \tilde{H}\left(\hat{z}_{i+1}\left(t - \frac{\Delta}{m}\right), u_i(t)\right)\Bigg). \end{aligned} \tag{22}$$

Now it is sufficient to show that differentiation of the second of (21), for $j = 1, \dots, m$, gives back the expression (22) for $\dot{\hat{z}}_j(t)$. Differentiation of the second of (21) gives

$$\begin{aligned} \dot{\hat{z}}_j(t) &= e^{A_n \frac{\Delta}{m}} \dot{\hat{z}}_{j-1}(t) \\ &\quad + A_n \int_{t-\frac{\Delta}{m}}^{t} e^{A_n(t-\tau)} \tilde{H}(\hat{z}_j(\tau), u_j(\tau))\, d\tau + \tilde{H}(\hat{z}_j(t), u_j(t)) \\ &\quad - e^{A_n \frac{\Delta}{m}} \tilde{H}\left(\hat{z}_j\left(t - \frac{\Delta}{m}\right), u_j\left(t - \frac{\Delta}{m}\right)\right). \end{aligned} \tag{23}$$

Substitution of the integral in (23) with the difference $\hat{z}_j(t) - e^{A_n \frac{\Delta}{m}} \hat{z}_{j-1}(t)$, gives

$$\begin{aligned} \dot{\hat{z}}_j(t) &= A_n \hat{z}_j(t) + \tilde{H}(\hat{z}_j(t), u_j(t)) \\ &\quad + e^{A_n \frac{\Delta}{m}} (\dot{\hat{z}}_{j-1}(t) - A_n \hat{z}_{j-1}(t)) \\ &\quad - e^{A_n \frac{\Delta}{m}} \tilde{H}\left(\hat{z}_j\left(t - \frac{\Delta}{m}\right), u_j\left(t - \frac{\Delta}{m}\right)\right). \end{aligned} \tag{24}$$

Adding and subtracting $e^{A_n \frac{\Delta}{m}} \tilde{H}(\hat{z}_{j-1}(t), u_{j-1}(t))$ to (24) and rearranging yields

$$\begin{aligned} \dot{\hat{z}}_j(t) &= A_n \hat{z}_j(t) + \tilde{H}(\hat{z}_j(t), u_j(t)) \\ &\quad + e^{A_n \frac{\Delta}{m}} (\dot{\hat{z}}_{j-1}(t) - A_n \hat{z}_{j-1}(t) - \tilde{H}(\hat{z}_{j-1}(t), u_{j-1}(t))) \\ &\quad + e^{A_n \frac{\Delta}{m}} \Bigg(\tilde{H}(\hat{z}_{j-1}(t), u_{j-1}(t)) \\ &\quad - \tilde{H}\left(\hat{z}_j\left(t - \frac{\Delta}{m}\right), u_j\left(t - \frac{\Delta}{m}\right)\right)\Bigg). \end{aligned} \tag{25}$$

Now, defining the variable $s_j(t)$, for $j = 0, 1, \dots, m$, as

$$s_j(t) = \dot{\hat{z}}_j(t) - A_n \hat{z}_j(t) - \tilde{H}(\hat{z}_j(t), u_j(t)). \tag{26}$$

Equation (25) can be rewritten in a difference form, for $j = 1, \ldots, m$, as

$$s_j(t) = e^{A_n \frac{\Delta}{m}} s_{j-1}(t) + e^{A_n \frac{\Delta}{m}} \cdot \left(\tilde{H}(\hat{z}_{j-1}(t), u_{j-1}(t)) - \tilde{H}\left(\hat{z}_j\left(t - \frac{\Delta}{m}\right), u_j\left(t - \frac{\Delta}{m}\right)\right)\right). \quad (27)$$

From the first of (21), the following expression for $s_0(t)$ is obtained:

$$s_0(t) = K(\bar{y}(t) - C_n \hat{z}_0(t)). \quad (28)$$

Using a standard equation for discrete time systems, the following is obtained:

$$s_j(t) = e^{A_n \frac{\Delta}{m} j} s_0(t) + \sum_{i=0}^{j-1} e^{A_n \frac{\Delta}{m}(j-i-1)} e^{A_n \frac{\Delta}{m}} \cdot \left(\tilde{H}(\hat{z}_i(t), u_i(t)) - \tilde{H}\left(\hat{z}_{i+1}\left(t - \frac{\Delta}{m}\right), u_{i+1}\left(t - \frac{\Delta}{m}\right)\right)\right). \quad (29)$$

Substituting the expressions of s_j and s_0 in (29) yields

$$\dot{\hat{z}}_j(t) = A_n \hat{z}_j(t) + \tilde{H}(\hat{z}_j(t), u_j(t)) + e^{A_n \frac{\Delta}{m} j} K(\bar{y}(t) - C\hat{z}_0(t)) + \sum_{i=0}^{j-1} e^{A_n \frac{\Delta}{m}(j-i)} \cdot \left(\tilde{H}(\hat{z}_i(t), u_i(t)) - \tilde{H}\left(\hat{z}_{i+1}\left(t - \frac{\Delta}{m}\right), u_{i+1}\left(t - \frac{\Delta}{m}\right)\right)\right). \quad (30)$$

Recalling that $u_i(t) = u_{i+1}(t - (\Delta/m))$, equality between expressions (30) and (22) is proven. ■

Theorem 7: For system (1), (2), assume that hypotheses H_1, H_2 are satisfied. Take a positive-real $\tilde{u}_M$ and an integer m such that the Lipschitz coefficient of the function $\tilde{H}(z, u)$ defined in (9) and the delay Δ are such that

$$\gamma_{\tilde{H}}(\tilde{u}_M) \left\| e^{A_n \frac{\Delta}{m}} \right\| \frac{\Delta}{m} < 1. \quad (31)$$

Then there exist a positive α, a positive $u_M \leq \tilde{u}_M$, and a gain vector K for the observer (18) such that if $|u(t)| \leq u_M$ for $t \geq -\Delta$, then

$$\|x(t) - \hat{x}_m(t)\| \leq \nu e^{-\alpha t} \quad (32)$$

where ν depends on the estimation error in $[-\Delta, 0]$ as follows:

$$\nu = \nu_1 \|x(-\Delta) - \hat{x}(-\Delta)\| + \nu_2 \int_{-\Delta}^{0} \|x(\tau) - \hat{x}(\tau)\| \, d\tau \quad (33)$$

in which ν_1 and ν_2 are suitable positive constants.

If also assumption H_3 holds, then the bound u_M on $|u(t)|$ can be chosen equal to $\tilde{u}_M$, given by (31).

Proof: Assumption (31) and Lemma 2 allow to choose a positive α that solves

$$\gamma_{\tilde{H}}(\tilde{u}_M) \left\| e^{A_n \frac{\Delta}{m}} \right\| \frac{e^{\alpha \frac{\Delta}{m}} - 1}{\alpha} < 1. \quad (34)$$

Denote the observation errors in z-coordinates as

$$e_{z,j}(t) = z_j(t) - \hat{z}_j(t), \quad j = 0, \ldots, m. \quad (35)$$

Under assumptions H_1, H_2, Theorem 3 guarantees that for the chosen α there exists a constant $\bar{u}_M$ and a gain K to be put in (18) such that, if $|u(t)| \leq \bar{u}_M$, then for a suitable μ_0 we have

$$\|e_{z,0}(t)\| \leq \mu_0 e^{-\alpha t} \|e_{z,0}(0)\|, \quad t \geq 0. \quad (36)$$

In order to also satisfy (31), let $u_M = \min\{\bar{u}_M, \tilde{u}_M\}$. (If, besides assumptions H_1, H_2, also assumption H_3 is satisfied, then taking $u_M = \tilde{u}_M$ a suitable choice for K in (18) allows to satisfy (36) for the given α.)

From (8) and (21), it follows, for $j = 1, \ldots, m$

$$e_{z,j}(t) = e^{A_n \frac{\Delta}{m}} e_{z,j-1}(t) + \int_{t-\frac{\Delta}{m}}^{t} e^{A_n(t-\tau)} \left(\tilde{H}\left(z_j(\tau), u_j(\tau)\right) - \tilde{H}(\hat{z}_j(\tau), u_j(\tau))\right) d\tau. \quad (37)$$

By assumption H_2 and since $|u(t)| \leq u_M$, we have $\|\tilde{H}(z_j, u) - \tilde{H}(\hat{z}_j, u)\| \leq \gamma_{\tilde{H}}(u_M) \|e_{z,j}\|$, so that

$$\begin{aligned} \|e_{z,j}(t)\| &\leq \left\| e^{A_n \frac{\Delta}{m}} \right\| \cdot \|e_{z,j-1}(t)\| + \gamma_{\tilde{H}}(u_M) \int_{t-\frac{\Delta}{m}}^{t} \left\| e^{A_n(t-\tau)} \right\| \cdot \|e_{z,j}(\tau)\| \, d\tau \\ &\leq \left\| e^{A_n \frac{\Delta}{m}} \right\| \cdot \|e_{z,j-1}(t)\| + \gamma_{\tilde{H}}(u_M) \left\| e^{A_n \frac{\Delta}{m}} \right\| \int_{t-\frac{\Delta}{m}}^{t} \|e_{z,j}(\tau)\| \, d\tau. \end{aligned} \quad (38)$$

Now, the following implication is proven:

$$(\exists \tilde{\mu}_{j-1} > 0 : \|e_{z,j-1}(t)\| \leq \tilde{\mu}_{j-1} e^{-\alpha t}) \implies (\exists \tilde{\mu}_j > 0 : \|e_{z,j}(t)\| \leq \tilde{\mu}_j e^{-\alpha t}). \quad (39)$$

This result is obtained by considering inequality (38) that, if the first term of implication (39) holds, becomes

$$\|e_{z,j}(t)\| \leq \left\| e^{A_n \frac{\Delta}{m}} \right\| \cdot \tilde{\mu}_{j-1} e^{-\alpha t} + \gamma_{\tilde{H}}(u_M) \left\| e^{A_n \frac{\Delta}{m}} \right\| \int_{t-\frac{\Delta}{m}}^{t} \|e_{z,j}(\tau)\| \, d\tau. \quad (40)$$

The second term of implication (39) holds thanks to Lemma 2, with

$$\tilde{\mu}_j = \frac{e^{\alpha \frac{\Delta}{m}}}{1 - c} \left\| e^{A_n \frac{\Delta}{m}} \right\| \left(\tilde{\mu}_{j-1} + \gamma_{\tilde{H}}(u_M) \times \int_{-\frac{\Delta}{m}}^{0} \|e_{z,j}(\tau)\| \, d\tau \right), \quad \text{where}$$
$$c = \gamma_{\tilde{H}}(\tilde{u}_M) \left\| e^{A_n \frac{\Delta}{m}} \right\| \frac{e^{\alpha \frac{\Delta}{m}} - 1}{\alpha} < 1. \quad (41)$$

Considering that, from (36)

$$\|e_{z,0}(t)\| \leq \tilde{\mu}_0 e^{-\alpha t} \quad (42)$$

where $\tilde{\mu}_0 = \mu_0 \|e_{z,0}(0)\|$, from implication (39) it follows that:

$$\|e_{z,m}(t)\| \leq \tilde{\mu}_m e^{-\alpha t}. \quad (43)$$

Using (43) the following expression for $\tilde{\mu}_m$ is obtained:

$$\tilde{\mu}_m = \lambda^m \mu_0 \|e_{z,0}(0)\| + \sum_{j=0}^{m-1} \lambda^{m-j} \gamma_{\tilde{H}}(u_M) \int_{-\frac{\Delta}{m}}^{0} \|e_{z,j}(\tau)\| \, d\tau,$$
$$\text{where } \lambda = \frac{e^{\alpha \frac{\Delta}{m}}}{1 - c} \left\| e^{A_n \frac{\Delta}{m}} \right\| > 1. \quad (44)$$

Inequality (43) proves exponential convergence to zero of the observation error in z-coordinates. From assumption H_1, it follows:

$$\|x(t) - \hat{x}_m(t)\| \leq \tilde{\nu} e^{-\alpha t} \quad (45)$$

with

$$\tilde{\nu} = \gamma_{\Phi}\gamma_{\Phi^{-1}} \left(\lambda^m \mu_0 \|x(-\Delta) - \hat{x}_0(t)\| + \gamma_{\tilde{H}}(u_M) \sum_{j=1}^{m} \lambda^{m-j} \int_{-\frac{\Delta}{m}}^{0} \|x_j(\tau) - \hat{x}_j(\tau)\| \, d\tau \right). \quad (46)$$

Now define

$$\nu = \gamma_{\Phi}\gamma_{\Phi^{-1}} \lambda^m \left(\mu_0 \|x(-\Delta) - \hat{x}(-\Delta)\| + \gamma_{\tilde{H}}(u_M) \int_{-\Delta}^{0} \|x(\tau) - \hat{x}(\tau)\| \, d\tau \right). \quad (47)$$

Recalling the observer initialization (19) and the definition $x_j(t) = x(t - \Delta + (j/m)\Delta)$, noting also that since $\lambda > 1$ then also $\lambda^m > \lambda^{m-j}$, it follows that $\nu > \tilde{\nu}$. From this and from (46), inequality (32) follows with:

$$\nu_1 = \gamma_{\Phi}\gamma_{\Phi^{-1}} \lambda^m \mu_0, \quad \nu_2 = \gamma_{\Phi}\gamma_{\Phi^{-1}} \lambda^m \gamma_{\tilde{H}}(u_M) \quad (48)$$

and the thesis is proven. ■

Remark 8: The choice of m depends essentially on the product $\gamma_{\tilde{H}}(\tilde{u}_M)\Delta$, as it is seen in condition (31) of Theorem 7, which can always be satisfied for a sufficiently large integer m. This means that for a given system, with a given *a priori* bound on the input amplitude (i.e., $\gamma_{\tilde{H}}(\tilde{u}_M)$ is given), the greater is the size of the delay Δ, the larger must be the number of rings in the Chain Observer (18) that ensure exponential convergence. It can be easily shown that although the size of the delay can be reduced by a suitable time-scaling, this reduction occurs at the expenses of the Lipschitz constant $\gamma_{\tilde{H}}(\tilde{u}_M)$: the product $\gamma_{\tilde{H}}(\tilde{u}_M)\Delta$ can be proven to be invariant with respect to time-scaling. The meaning of the product $\gamma_{\tilde{H}}(\tilde{u}_M)\Delta$ is intuitively understood looking at inequality (38), with $m = 1$: such a product gives a gross bound on the propagation of the prediction error in z-coordinates from time $t - \Delta$ to time t.

IV. Example

As a simple example, consider the following nonlinear system with delayed measurements:

$$\begin{aligned} \dot{x}_1(t) &= c_1 x_2(t) \\ \dot{x}_2(t) &= c_2 x_1(t) + c_3 x_1(t) x_2(t) + c_4 x_1(t) u(t) \\ \bar{y}(t) &= x_1(t - \Delta) \end{aligned} \quad (49)$$

with all $c_i \neq 0$. In this example the observation relative degree is $r = 2$. The map Φ defined in (3) and the observability matrix are given by

$$z = \Phi(x) = \begin{bmatrix} x_1 \\ c_1 x_2 \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 1 & 0 \\ 0 & c_1 \end{bmatrix}.$$

The simulations here reported are made with $c_1 = c_3 = c_4 = 1, c_2 = -2$. Both eigenvalues of matrix $A_2 - KC_2$ have been chosen equal to -1. The input is $u(t) = 0.1\sin(0.1t)$. First, the observer (18) is applied with $m = 1$. The initial conditions for the system and for the observer have been chosen as

$$x(-\Delta) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \hat{x}_0(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hat{x}(\tau) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tau \in [-\Delta, 0]. \quad (50)$$

Simulations have been performed increasing the size of the output delay. As a result, the observation algorithm performs well when $\Delta \leq 1.3$. Fig. 1 reports the true states and the observed states in a simulation with $\Delta = 1.3$. For larger delays the observer output does not converge to the true state: it is necessary to increase m. Simulations have shown that with $m = 2$ the Chain Observer performs well for delays up to $\Delta = 1.7$. Fig. 2 reports the true and observed states for

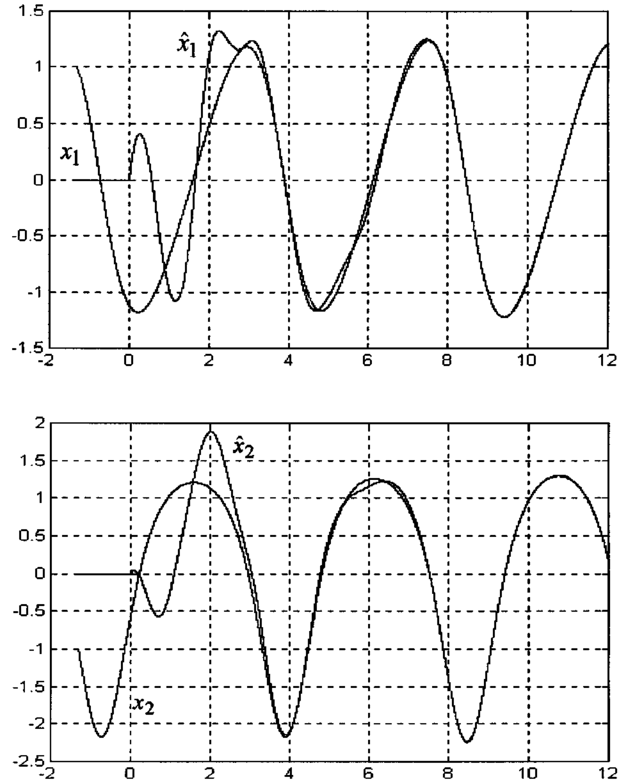

Fig. 1. True and estimated states for $\Delta = 1.3$ and $m = 1$.

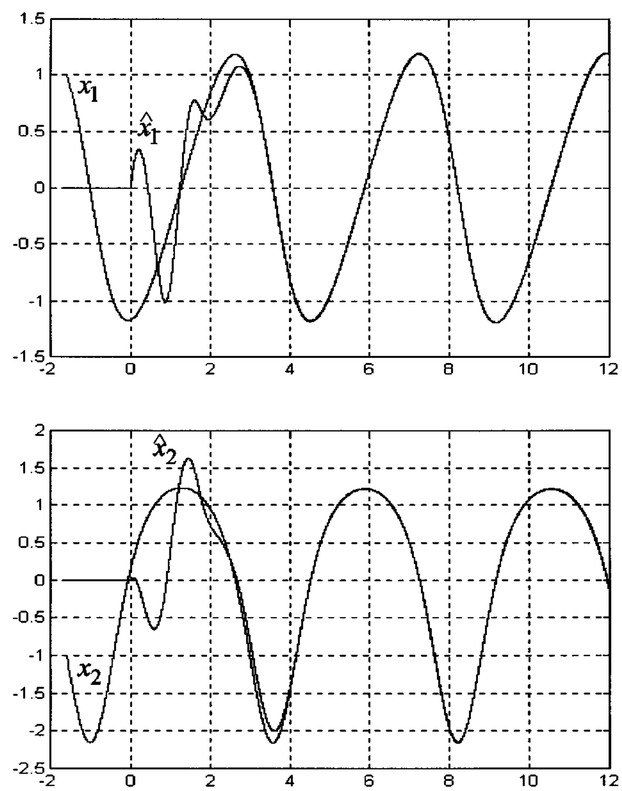

Fig. 2. True and estimated states for $\Delta = 1.7$ and $m = 2$.

$\Delta = 1.7$. Larger delays require a larger m, that means a longer chain of observers.

V. Conclusion

This note presents an algorithm that solves the problem of state reconstruction for nonlinear systems when the output measurements are available after a nonnegligible delay. The algorithm is composed by m observers in a chained form, each one estimating the state at a given fraction of the output delay. The last observer of the chain estimates the present state. Exponential convergence of the estimate is ensured if the integer m is sufficiently large. Computer simulations show a good performance of the observer.

Appendix

Proof of Lemma 2: The assumption $\gamma\delta < 1$ guarantees the existence of a real $\alpha \in (0, \bar{\alpha}]$ such that the constant c defined in (14) is strictly less than 1 (this happens because the function $(e^{\alpha\delta} - 1)/(\alpha\delta)$ is monotone and increases from 1 to ∞ as α goes from 0 to ∞).

Since $\alpha \leq \bar{\alpha}$ the second of inequalities (12) can be written with the substitution of $\bar{\alpha}$ with α, and from this the following inequality can be derived:

$$s(t) \leq \tilde{\mu} e^{-\alpha t} + \gamma \int_{\max\{0,(t-\delta)\}}^{t} s(\tau)\, d\tau, \quad t \geq 0 \tag{A.1}$$

where

$$\tilde{\mu} = e^{\alpha\delta} \left(\mu + \gamma \int_{-\delta}^{0} s(\tau)\, d\tau \right). \tag{A.2}$$

Now consider inequality (A.1) and substitute $s(\tau)$ with the inequality itself. We obtain the following:

$$\begin{aligned} s(t) \leq{}& \tilde{\mu} e^{-\alpha t} + \tilde{\mu}\gamma \left[\frac{e^{-\alpha t_1}}{-\alpha} \right]_{\max\{0,(t-\delta)\}}^{t} \\ &+ \gamma^2 \int_{\max\{0,(t-\delta)\}}^{t} \int_{\max\{0,(t_1-\delta)\}}^{t_1} s(t_2)\, dt_2\, dt_1 \end{aligned} \tag{A.3}$$

and from this

$$\begin{aligned} s(t) \leq{}& \tilde{\mu} e^{-\alpha t} + \tilde{\mu}\gamma \left(\frac{e^{\alpha\delta} - 1}{\alpha} \right) e^{-\alpha t} \\ &+ \gamma^2 \int_{\max\{0,(t-\delta)\}}^{t} \int_{\max\{0,(t_1-\delta)\}}^{t_1} s(t_2)\, dt_2\, dt_1. \end{aligned} \tag{A.4}$$

Recall that $c = \gamma(e^{\alpha\delta} - 1)/\alpha$. Iterated substitution of (A.1) in (A.4) gives the sequence of inequalities

$$\begin{aligned} s(t) \leq{}& \sum_{h=0}^{k} c^h \tilde{\mu} e^{-\alpha t} + \gamma^{k+1} \\ &\cdot \int_{\max\{0,(t-\delta)\}}^{t} \cdots \int_{\max\{0,(t_k-\delta)\}}^{t_k} s(t_{k+1}) dt_{k+1} \ldots dt_1. \end{aligned} \tag{A.5}$$

From (A.1), using Gronwall's lemma, the bound $s(t) \leq \tilde{\mu} e^{\gamma t}$, $t \geq 0$, is obtained. Using this bound in (A.5) it follows:

$$s(t) \leq \sum_{h=0}^{k} c^h \tilde{\mu} e^{-\alpha t} + \frac{(\gamma t)^{k+1}}{(k+1)!} e^{\gamma t} \tilde{\mu}, \quad k = 0, 1, \ldots. \tag{A.6}$$

Since $c < 1$, the limit for $k \to \infty$ of the right-hand term of (A.6) exists and is given by

$$s(t) \leq \frac{1}{1-c} \tilde{\mu} e^{-\alpha t}. \tag{A.7}$$

Recalling the expression (A.2) for $\tilde{\mu}$, the thesis (13) follows. ■

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